Third Edition

A FIRST COURSE IN CONTINUUM MECHANICS

for Physical and Biological Engineers and Scientists

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Dedicated to

students who would share
my enthusiasm
for the application
of mechanics,

and to

Luna, Conrad, and Brenda.

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1.23 Figure P1.23 is a classic from a book by Giovanni Alphonso Borelli (1608–1679) entitled "De Motu Animalium" (On the Movement of Animals), published in 1680 (first part) and 1681 (second part), recently translated by P. Maquet, Springer-Verlag, New York, 1989. The figure shows a person carrying a heavy load. Several parts are cut open to show how bones and muscles work in this effort. Further clarification can be obtained, of course, by use of more detailed free-body diagrams. Use them to estimate how large is the load acting on the hip joint when a 70 kg person walks carrying a 30 kg globe on the shoulder.

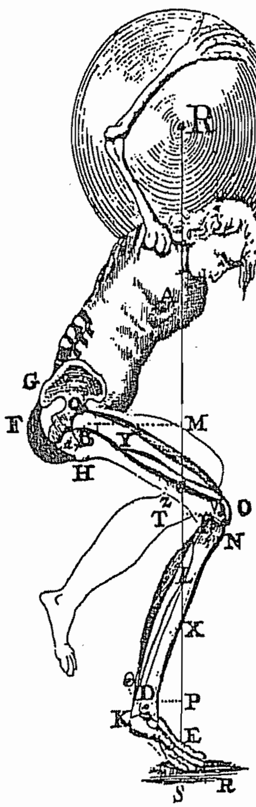

Figure P1.23 A figure from Table VI, Fig. 1 of Borelli's book.

2 VECTORS AND TENSORS

A beautiful story needs a beautiful language to tell. Tensor is the language of mechanics.

2.1 VECTORS

A vector in a three-dimensional Euclidean space is defined as a directed line segment with a given magnitude and a given direction. We shall denote vectors by $\overrightarrow{AB}$, $\overrightarrow{PQ}$, . . . , or by boldface letters, $\mathbf{u}$, $\mathbf{v}$, $\mathbf{F}$, $\mathbf{T}$, ...

Two vectors are *equal* if they have the same direction and same magnitude. A *unit vector* is a vector of magnitude 1. The *zero vector*, denoted by $\mathbf{0}$, is a vector of zero magnitude. We use the symbols $|\overrightarrow{AB}|$, $|\mathbf{u}|$, and v to represent the magnitudes of $\overrightarrow{AB}$, $\mathbf{u}$, and $\mathbf{v}$, respectively.

The sum of two vectors is another vector obtained by the "parallelogram law," and we write, for example, $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$. Vector addition is commutative and associative.

A vector multiplied by a number yields another vector. If k is a positive real number, $k\mathbf{a}$ represents a vector having the same direction as $\mathbf{a}$ and a magnitude k times as large. If k is negative, $k\mathbf{a}$ is a vector whose magnitude is $|k|$ times as large and whose direction is opposite to $\mathbf{a}$. If $k = 0$, we have $0 \cdot \mathbf{a} = \mathbf{0}$.

The subtraction of vectors can be defined by

$$\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b}).$$

If we let $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ be the unit vectors in the directions of the positive x_1, x_2, x_3 axes, respectively, we can show that every vector in a three-dimensional Euclidean space with coordinate axes x_1, x_2, x_3 may be represented by a linear combination of $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$. Furthermore, if the vector $\mathbf{u}$ is represented by the linear combination

$$\mathbf{u} = u_1\mathbf{e}_1 + u_2\mathbf{e}_2 + u_3\mathbf{e}_3, \tag{2.1–1}$$

then u_1, u_2, u_3 are the components of $\mathbf{u}$, and $\mathbf{u}$ can be represented by a matrix (u_1, u_2, u_3).

The magnitude $|\mathbf{u}|$ is then given by

$$|\mathbf{u}| = \sqrt{u_1^2 + u_2^2 + u_3^2}, \tag{2.1–2}$$

and therefore $\mathbf{u} = \mathbf{0}$ if and only if $u_1 = u_2 = u_3 = 0$.

The *scalar* (or *dot*) *product* of $\mathbf{u}$ and $\mathbf{v}$, denoted by $\mathbf{u}\cdot\mathbf{v}$, is defined by the formula

$$\mathbf{u}\cdot\mathbf{v} = |\mathbf{u}||\mathbf{v}|\cos\theta \qquad (0 \le \theta \le \pi), \tag{2.1–3}$$

where θ is the angle between the given vectors. This represents the product of the magnitude of one vector and the component of the second vector in the direction of the first; that is,

$$\mathbf{u}\cdot\mathbf{v} = (\text{magnitude of } \mathbf{u})(\text{component of } \mathbf{v} \text{ along } \mathbf{u}). \tag{2.1–4}$$

If

$$\mathbf{u} = u_1\mathbf{e}_1 + u_2\mathbf{e}_2 + u_3\mathbf{e}_3 \qquad \mathbf{v} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + v_3\mathbf{e}_3$$

the scalar product of these two vectors can also be expressed in terms of the components:

$$\mathbf{u}\cdot\mathbf{v} = u_1v_1 + u_2v_2 + u_3v_3. \tag{2.1–5}$$

Whereas the scalar product of two vectors is a scalar quantity, the *vector* (or *cross*) *product* of two vectors $\mathbf{u}$ and $\mathbf{v}$ produces another vector $\mathbf{w}$; and we write $\mathbf{w} = \mathbf{u} \times \mathbf{v}$. The magnitude of $\mathbf{w}$ is defined as

$$|\mathbf{w}| = |\mathbf{u}||\mathbf{v}|\sin\theta \qquad (0 \le \theta \le \pi), \tag{2.1–6}$$

where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$, and the direction of $\mathbf{w}$ is defined as perpendicular to the plane determined by $\mathbf{u}$ and $\mathbf{v}$, in such a way that $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ form a right-handed system. Vector products satisfy the following relations:

$$\begin{aligned}
&\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})\\
&\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}\\
&\mathbf{u} \times \mathbf{u} = 0\\
&\mathbf{e}_1 \times \mathbf{e}_1 = \mathbf{e}_2 \times \mathbf{e}_2 = \mathbf{e}_3 \times \mathbf{e}_3 = \mathbf{0}\\
&\mathbf{e}_1 \times \mathbf{e}_2 = \mathbf{e}_3 \qquad \mathbf{e}_2 \times \mathbf{e}_3 = \mathbf{e}_1 \qquad \mathbf{e}_3 \times \mathbf{e}_1 = \mathbf{e}_2\\
&k\mathbf{u} \times \mathbf{v} = \mathbf{u} \times k\mathbf{v} = k(\mathbf{u} \times \mathbf{v}).
\end{aligned} \tag{2.1–7}$$

Using these relations, the vector product can be expressed in terms of the components as follows:

$$\mathbf{u} \times \mathbf{v}\ (u_2v_3 - u_3v_2)\mathbf{e}_1 + (u_3v_1 - u_1v_3)\mathbf{e}_2 + (u_1v_2 - u_2v_1)\mathbf{e}_3. \tag{2.1–8}$$

PROBLEMS

2.1 Given vector $\mathbf{u} = -3\mathbf{e}_1 + 4\mathbf{e}_2 + 5\mathbf{e}_3$, find a unit vector in the direction of $\mathbf{u}$.

Answer: $(\sqrt{2}/10)\mathbf{u}$.

2.2 If $\overrightarrow{AB} = -2\mathbf{e}_1 + 3\mathbf{e}_2$, and the midpoint of the segment $\overrightarrow{AB}$ has coordinates $(-4, 2)$, find the coordinates of A and B.

Answer: $(-3, \frac{1}{2}), (-5, \frac{7}{2})$.

2.3 Prove that, for any two vectors $\mathbf{u}$, $\mathbf{v}$, $|\mathbf{u} - \mathbf{v}|^2 + |\mathbf{u} + \mathbf{v}|^2 = 2(|\mathbf{u}|^2 + |\mathbf{v}|^2)$.

2.4 Find the magnitude and direction of the resultant force of three coplanar forces of 10 lb each acting outward on a body at the origin and making angles of 60°, 120°, and 270°, respectively, with the x-axis.

Answer: $10(\sqrt{3} - 1)$, $\perp x$.

2.5 Find the angles between $\mathbf{u} = 6\mathbf{e}_1 + 2\mathbf{e}_2 - 3\mathbf{e}_3$ and $\mathbf{v} = -\mathbf{e}_1 + 8\mathbf{e}_2 + 4\mathbf{e}_3$.

Answer: $\cos^{-1}(-\frac{2}{63})$.

2.6 Given $\mathbf{u} = 3\mathbf{e}_1 + 4\mathbf{e}_2 - \mathbf{e}_3$, $\mathbf{v} = 2\mathbf{e}_1 + 5\mathbf{e}_3$, find the value of α so that $\mathbf{u} + \alpha\mathbf{v}$ is orthogonal to $\mathbf{v}$.

Answer: $-\frac{1}{29}$.

2.7 Given $\mathbf{u} = 2\mathbf{e}_1 + 3\mathbf{e}_2$, $\mathbf{v} = \mathbf{e}_1 - \mathbf{e}_2 + 2\mathbf{e}_3$, $\mathbf{w} = \mathbf{e}_1 - 2\mathbf{e}_3$, evaluate $\mathbf{u}\cdot(\mathbf{v} \times \mathbf{w})$ and $(\mathbf{u} \times \mathbf{v})\cdot\mathbf{w}$.

Answer: 16.

2.8 $(\mathbf{u} \times \mathbf{v})\cdot\mathbf{w}$ is called the *scalar triple product* of $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$. Show that $(\mathbf{u} \times \mathbf{v})\cdot\mathbf{w} = \mathbf{u}\cdot(\mathbf{v} \times \mathbf{w})$.

2.9 Find the equation of the plane through $A(1, 0, 2)$, $B(0, 1, -1)$, and $C(2, 2, 3)$.

Answer: $7x - 2y - 3z - 1 = 0$.

2.10 Find the area of ΔABC in Prob. 2.9.

Answer: $\sqrt{62}/2$.

2.11 Find a vector perpendicular to both $\mathbf{u} = 2\mathbf{e}_1 + 3\mathbf{e}_2 - \mathbf{e}_3$ and $\mathbf{v} = \mathbf{e}_1 - 2\mathbf{e}_2 + 3\mathbf{k}$.

Answer: $7\mathbf{e}_1 - 7\mathbf{e}_2 - 7\mathbf{e}_3$.

2.2 VECTOR EQUATIONS

The spirit of vector analysis is to use symbols to represent physical or geometric quantities and to express a physical relationship or a geometric fact by an equation. For example, if we have a particle on which the forces $\mathbf{F}^{(1)}, \mathbf{F}^{(2)}, \ldots, \mathbf{F}^{(n)}$ act, then we say that the condition of equilibrium for this particle is

$$\mathbf{F}^{(1)} + \mathbf{F}^{(2)} + \cdots + \mathbf{F}^{(n)} = \mathbf{0}. \tag{2.2–1}$$

As another example, we say that the following equation for the vector $\mathbf{r}$ represents a plane if $\mathbf{n}$ is a unit vector and p is a constant:

$$\mathbf{r} \cdot \mathbf{n} = p. \tag{2.2–2}$$

By this statement, we mean that the locus of the end point of a radius vector $\mathbf{r}$ satisfying the preceding equation is a plane. The geometric meaning is again clear. The vector $\mathbf{n}$, called the *unit normal vector* of the plane, is specified. The scalar product $\mathbf{r} \cdot \mathbf{n}$ represents the scalar projection of $\mathbf{r}$ on $\mathbf{n}$. Equation (2.2–2) then states that if we consider all radius vectors $\mathbf{r}$ whose component on $\mathbf{n}$ is a constant p, we shall obtain a plane. (See Fig. 2.1.)

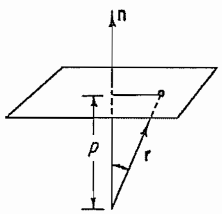

Figure 2.1 Equation of a plane, $\mathbf{r} \cdot \mathbf{n} = p$.

On the other hand, elegant as they are, vector equations are not always convenient. Indeed, when Descartes introduced analytic geometry in which vectors are expressed by their components with respect to a fixed frame of reference, it was a great contribution. Thus, with reference to a set of rectangular Cartesian coordinate axes O-xyz, Eqs. (2.2–1) and (2.2–2) may be written, respectively, as

$$\sum_{i=1}^{n} F_x^{(i)} = 0, \qquad \sum_{i=1}^{n} F_y^{(i)} = 0, \qquad \sum_{i=1}^{n} F_z^{(i)} = 0, \tag{2.2–3}$$

$$ax + by + cz = p, \tag{2.2–4}$$

where $F_x^{(i)}, F_y^{(i)}, F_z^{(i)}$ represent the components of the vector $\mathbf{F}^{(i)}$ with respect to the frame of reference O-xyz; x, y, z represent the components of $\mathbf{r}$; and a, b, c represent those of the unit normal vector $\mathbf{n}$.

Why is the analytic form preferred? Why are we willing to sacrifice the elegance of the vector notation? The answer is compelling: We like to express physical quantities in numbers. To specify a radius vector, it *is* convenient to specify a triple of numbers (x, y, z). To specify a force $\mathbf{F}$, it *is* convenient to define the three components F_x, F_y, F_z. In fact, in practical calculations, we use Eqs. (2.2–3) and (2.2–4) much more frequently than Eqs. (2.2–1) and (2.2–2).

PROBLEMS

2.12 Express the basic laws of elementary physics—e.g., Newton's law of motion, Coulomb's law for the attraction or repulsion between electric charges, and Maxwell's equation for the electromagnetic field—in the form of vector equations.

For example, to express Newton's law of gravitation in vector form, let m_1 and m_2 be the masses of two particles. Let the position vector from particle 1 to particle 2 be $\mathbf{r}_{12}$. Then the force produced on particle 1 due to the gravitational attraction between 1 and 2 is

$$\mathbf{F}_{12} = G \frac{m_1 m_2}{|\mathbf{r}_{12}|^2} \frac{\mathbf{r}_{12}}{|\mathbf{r}_{12}|}$$

where G is the gravitational constant.

2.13 Consider a particle constrained to move in a circular orbit at a constant speed. Let $\mathbf{v}$ be the velocity at any instant. What is the acceleration of the particle; i.e., what is the vector $d\mathbf{v}/dt$?

Answer. The velocity vector $\mathbf{v}$ may be represented in polar coordinates as follows. Let $\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}, \hat{\mathbf{z}}$, be, respectively, the unit vectors with origin at P in the directions of the radius, the tangent, and the polar axis perpendicular to the plane of the orbit. (See Fig. P2.13.) Then $\mathbf{v} = v\hat{\boldsymbol{\theta}}$, where v is the absolute value of $\mathbf{v}$. Hence, by differentiation,

$$\frac{d\mathbf{v}}{dt} = v\frac{d\hat{\boldsymbol{\theta}}}{dt} + \frac{dv}{dt}\hat{\boldsymbol{\theta}}.$$

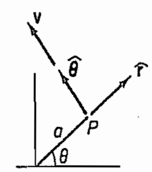

Figure P2.13 Velocity vector of a particle moving in a circular orbit.

The last term vanishes because v is a constant. To evaluate $d\hat{\boldsymbol{\theta}}/dt$, we note that $\hat{\boldsymbol{\theta}}$ is a unit vector; hence, it can only change direction. $d\hat{\boldsymbol{\theta}}/dt$ is, therefore, perpendicular to the vector $\hat{\boldsymbol{\theta}}$, i.e., parallel to $\hat{\mathbf{r}}$. Let ω be the angular velocity of the particle about the center of the orbit. Obviously, $\hat{\boldsymbol{\theta}}$ is turning at a rate of $\omega = v/a$. Hence, $d\hat{\boldsymbol{\theta}}/dt = -(v/a)\hat{\mathbf{r}}$, and $d\mathbf{v}/dt = -(v^2/a)\hat{\mathbf{r}}$.

2.14 A particle is constrained to move along a circular helix of radius a and pitch h at a constant speed v. What is the acceleration of the particle? If the particle is located at P, as shown in Fig. P2.14, express the velocity and acceleration vectors in terms of

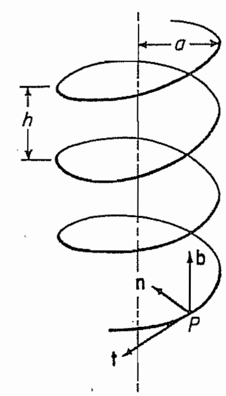

Figure P2.14 A helical orbit.

unit vectors $\mathbf{t}$, $\mathbf{n}$, and $\mathbf{b}$ that are, respectively, tangent, normal, and binormal to the helix at P.

Answer. The velocity vector is parallel to $\mathbf{t}$ and has a magnitude v. Hence, $\mathbf{v} = v\mathbf{t}$. By differentiation, and noting that v is a constant, we have $d\mathbf{v}/dt = v\, d\mathbf{t}/dt$. But since $\mathbf{t}$ has a constant length of unity, $d\mathbf{t}/dt$ must be perpendicular to $\mathbf{t}$ and, hence, must be a combination of $\mathbf{n}$ and $\mathbf{b}$. That is,

$$\frac{d\mathbf{t}}{dt} = \kappa\mathbf{n} + \tau\mathbf{b}$$

where κ and τ are constants. If the particle moves with unit velocity, the constants κ and τ are called the *curvature* and the *torsion* of the space curve, respectively.

It is convenient to use polar coordinates for this problem. Let the unit vectors in the direction of the radial, circumferential, and axial directions be $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\theta}}$, and $\hat{\mathbf{z}}$, respectively. Then

$$\mathbf{v} = u\hat{\boldsymbol{\theta}} + w\hat{\mathbf{z}}$$

where u and w are the circumferential and axial velocities, respectively. Hence, $d\mathbf{v}/dt = (du/dt)\hat{\boldsymbol{\theta}} + u\, d\hat{\boldsymbol{\theta}}/dt + (dw/dt)\hat{\mathbf{z}} + w(d\hat{\mathbf{z}}/dt) = u\, d\hat{\boldsymbol{\theta}}/dt = -(u^2/a)\hat{\mathbf{r}}$. The velocities u and w are related to v as follows: In the time interval $\Delta t = 2\pi a/u$, the axial position z is changed by h. Hence, $w = h/\Delta t = hu/2\pi a$, and $v = u[1 + h^2/(4\pi^2 a^2)]^{1/2}$.

2.3 THE SUMMATION CONVENTION

For further development, an important matter of notation must be mastered.

A set of n variables $x_1, x_2, \ldots, x_n$ is usually denoted as x_i, $i = 1, \ldots, n$. When written singly, the symbol x_i stands for *any one* of the variables $x_1, x_2, \ldots, x_n$. The *range* of i must be indicated in every case; the simplest way is to write, as illustrated here, $i = 1, 2, \ldots, n$. The symbol i is an *index*. An index may be either a subscript or a superscript. A system of notations using indices is said to be an *indicial notation.*

Consider an equation describing a plane in a three-dimensional space referred to a rectangular Cartesian frame of reference with axes x_1, x_2, x_3, i.e.,

$$a_1x_1 + a_2x_2 + a_3x_3 = p, \tag{2.3–1}$$

where a_i and p are constants. This equation can be written as

$$\sum_{i=1}^{3} a_ix_i = p. \tag{2.3–2}$$

However, we shall introduce the *summation convention* and write the preceding equation in the simple form

$$a_ix_i = p. \tag{2.3–3}$$

The convention is as follows: *The repetition of an index in a term will denote a summation with respect to that index over its range.* The *range* of an index i is the set of n integers 1 to n. An index that is summed over is called a *dummy index*; one that is not summed is called a *free index.*

Since a dummy index indicates summation, it is immaterial which symbol is used. Thus, a_ix_i is the same as a_jx_j, etc. This is analogous to the dummy variable in an integral, e.g.,

$$\int_a^b f(x)\,dx = \int_a^b f(y)\,dy.$$

Examples

The use of the index and summation convention may be illustrated by other examples. Consider a unit vector $\boldsymbol{\nu}$ in a three-dimensional Euclidean space with rectangular Cartesian coordinates x, y, and z. Let the direction cosines α_i be defined as

$$\alpha_1 = \cos(\boldsymbol{\nu}, x), \qquad \alpha_2 = \cos(\boldsymbol{\nu}, y), \qquad \alpha_3 = \cos(\boldsymbol{\nu}, z),$$

where $(\boldsymbol{\nu}, x)$ denotes the angle between $\boldsymbol{\nu}$ and the x-axis, and so forth. The set of numbers $\alpha_i (i = 1, 2, 3)$ represents the components of the unit vector on the coordinate axes. The fact that the length of the vector is unity is expressed by the equation

$$(\alpha_1)^2 + (\alpha_2)^2 + (\alpha_3)^2 = 1,$$

or, simply,

$$\alpha_i\alpha_i = 1. \tag{2.3–4}$$

As another illustration, consider a line element with components dx, dy, dz in a three-dimensional Euclidean space with rectangular Cartesian coordinates x, y, and z. The square of the length of the line element is

$$ds^2 = dx^2 + dy^2 + dz^2. \tag{2.3–5}$$

If we define

$$dx_1 = dx, \qquad dx_2 = dy, \qquad dx_3 = dz, \tag{2.3–6}$$

and

$$\begin{aligned} &\delta_{11} = \delta_{22} = \delta_{33} = 1, \\ &\delta_{12} = \delta_{21} = \delta_{13} = \delta_{31} = \delta_{23} = \delta_{32} = 0, \end{aligned} \tag{2.3–7}$$

then Eq. (2.3–5) may be written as

$$ds^2 = \delta_{ij}dx_idx_j, \qquad \blacktriangle \tag{2.3–8}$$

with the understanding that the range of the indices i and j is 1 to 3. Note that there are two summations in this expression, one over i and one over j. The symbol δ_{ij}, as defined in Eq. (2.3–7), is called the *Kronecker delta*.

Matrices and Determinants

The rules of matrix algebra and the evaluation of determinants can be expressed more simply with the summation convention. An $m \times n$ *matrix* $\mathbf{A}$ is an ordered rectangular array of mn elements. We denote

$$\mathbf{A} = (a_{ij}) = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \tag{2.3–9}$$

so that a_{ij} is the element in the ith row and jth column of the matrix $\mathbf{A}$. The index i takes the values $1, 2, \ldots, m$, and the index j takes the values $1, 2, \ldots, n$. A transpose of $\mathbf{A}$ is another matrix, denoted by $\mathbf{A}^T$, whose elements are the same as those of $\mathbf{A}$, except that the row numbers and column numbers are interchanged. Thus,

$$\mathbf{A}^T = (a_{ij})^T = \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{pmatrix} \tag{2.3–10}$$

The product of two 3×3 matrices $\mathbf{A} = (a_{ij})$, $\mathbf{B} = (b_{ij})$ is a 3×3 square matrix defined as

$$\begin{aligned}\mathbf{A}\cdot\mathbf{B} &= \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} \\ &= \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & \cdots \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} & \cdots \\ a_{31}b_{11} + a_{32}b_{21} + a_{33}b_{31} & \cdots \end{pmatrix}\end{aligned} \tag{2.3–11}$$

whose element in the ith row and jth column can be written, with the summation convention, as

$$(\mathbf{A}\cdot\mathbf{B})_{ij} = (a_{ik}b_{kj}) \tag{2.3–12}$$

A vector $\mathbf{u}$ may be represented by a row matrix (u_i), and Eq. (2.1–2) can be written

$$|\mathbf{u}|^2 = (u_i)\cdot(u_j)^T = u_1^2 + u_2^2 + u_3^2 = u_iu_i. \tag{2.3–13}$$

By this rule, the scalar product of two vectors $\mathbf{u}\cdot\mathbf{v}$, Eq. (2.1–3), can be written as

$$\mathbf{u}\cdot\mathbf{v} = (u_i)(v_i)^T = u_1v_1 + u_2v_2 + u_3v_3 = u_iv_i. \tag{2.3–14}$$

The *determinant* of a square matrix is a number that is the sum of all the products of the elements of the matrix, taken one from each row and one from each column, and no two or more from any row or column, and with sign specified by a rule given shortly. For example, the determinant of a 3×3 matrix $\mathbf{A}$ is written as det $\mathbf{A}$ and is defined as

$$\begin{aligned}\det \mathbf{A} = \det (a_{ij}) &= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \\ &= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\ &\quad - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}\end{aligned} \tag{2.3–15}$$

The special rule of signs is as follows: Arrange the first index in the order 1, 2, 3. Then check the order of the second index. If they permute as $1, 2, 3, 1, 2, 3, \ldots$, then the sign is positive; otherwise the sign is negative.

Let us introduce a special symbol, ϵ_{rst}, called the *permutation symbol* and defined by the equations

$$\begin{aligned}&\epsilon_{111} = \epsilon_{222} = \epsilon_{333} = \epsilon_{112} = \epsilon_{121} = \epsilon_{211} = \epsilon_{221} = \epsilon_{331} = \cdots = 0, \\ &\epsilon_{123} = \epsilon_{231} = \epsilon_{312} = 1, \\ &\epsilon_{213} = \epsilon_{321} = \epsilon_{132} = -1.\end{aligned} \tag{2.3–16}$$

In other words, ϵ_{ijk} vanishes whenever the values of any two indices coincide; $\epsilon_{ijk} = 1$ when the subscripts permute as 1, 2, 3; and $\epsilon_{ijk} = -1$ otherwise. Then the determinant of the matrix (a_{ij}) can be written as

$$\det (a_{ij}) = \epsilon_{rst}a_{r1}a_{s2}a_{t3} \tag{2.3–17}$$

Using the symbol ϵ_{rst}, we can write Eq. (2.1–8) defining the vector product $\mathbf{u} \times \mathbf{v}$ as

$$\mathbf{u} \times \mathbf{v} = \epsilon_{rst}u_sv_t\mathbf{e}_r \tag{2.3–18}$$

The ϵ-δ Identity

The Kronecker delta and the permutation symbol are very important quantities that will appear again and again in this book. They are connected by the identity

$$\epsilon_{ijk}\epsilon_{ist} = \delta_{js}\delta_{kt} - \delta_{jt}\delta_{ks}. \qquad \blacktriangle \tag{2.3–19}$$

This ϵ-δ identity is used frequently enough to warrant special attention here. It can be verified by actual trial.

Differentiation

Finally, we shall extend the summation convention to differentiation formulas. Let $f(x_1, x_2, \ldots, x_n)$ be a function of n variables $x_1, x_2, \ldots, x_n$. Then its differential shall be written as

$$df = \frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2 + \cdots + \frac{\partial f}{\partial x_n} dx_n = \frac{\partial f}{\partial x_i} dx_i. \tag{2.3–20}$$

PROBLEMS

2.15 Write Eq. (2.2–1) or (2.2–3) in the index form. Let the components of $\mathbf{F}^{(i)}$ be written as $F_k^{(i)}$, $k = 1, 2, 3$; i.e., $F_x = F_1$, etc.

Answer. $\sum_{i=1}^{n} F_k^{(i)} = 0.$

2.16 Show that
(a) $\delta_{ii} = 3$
(b) $\delta_{ij}\delta_{ij} = 3$
(c) $\epsilon_{ijk}\epsilon_{jki} = 6$
(d) $\epsilon_{ijk} A_j A_k = 0$
(e) $\delta_{ij}\delta_{jk} = \delta_{ik}$
(f) $\delta_{ij}\epsilon_{ijk} = 0$

2.17 Write Eqs. (2.1–1) and (2.1–5) in the index form, e.g., $\mathbf{u}\cdot\mathbf{v} = u_i v_i$.

Note. For Eq. (2.1–1), we may do the following: Define three unit vectors $\boldsymbol{\nu}^{(1)} = \mathbf{e}_1$, $\boldsymbol{\nu}^{(2)} = \mathbf{e}_2$, $\boldsymbol{\nu}^{(3)} = \mathbf{e}_3$; then $\mathbf{u} = u_i \cdot \boldsymbol{\nu}^{(i)}$.

2.18 Use the index form of vector equations to solve Probs. 2.5 through 2.9.

2.19 The vector product of two vectors $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ is the vector $\mathbf{w} = \mathbf{u} \times \mathbf{v}$ whose components are

$$w_1 = u_2v_3 - u_3v_2, \qquad w_2 = u_3v_1 - u_1v_3, \qquad w_3 = u_1v_2 - u_2v_1.$$

Show that this can be shortened by writing

$$w_i = \epsilon_{ijk} u_j v_k.$$

2.20 Express Eqs. (2.1–7) in the index form.

2.21 Derive the vector identity connecting three arbitrary vectors $\mathbf{A}, \mathbf{B}, \mathbf{C}$ by the method of vector analysis:

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A}\cdot\mathbf{C})\mathbf{B} - (\mathbf{A}\cdot\mathbf{B})\mathbf{C}.$$

Solution. Since $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$ is perpendicular to $\mathbf{B} \times \mathbf{C}$, it must lie in the plane of $\mathbf{B}$ and $\mathbf{C}$. Hence, we may write $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = a\mathbf{B} + b\mathbf{C}$, where a, b are scalar quantities. But $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$ is a linear function of $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$; hence, a must be a linear scalar combination of $\mathbf{A}$ and $\mathbf{C}$, and b must be a linear scalar combination of $\mathbf{A}$ and $\mathbf{B}$. Accordingly, a, b are proportional to $\mathbf{A}\cdot\mathbf{C}$ and $\mathbf{A}\cdot\mathbf{B}$, respectively, and we may write

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \lambda(\mathbf{A}\cdot\mathbf{C})\mathbf{B} + \mu(\mathbf{A}\cdot\mathbf{B})\mathbf{C}$$

where λ, μ are pure numbers, independent of $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$. We can, therefore, evaluate λ, μ by special cases, e.g., if $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are the unit vectors in the directions of the x-, y-, and z-axes (a right-handed rectangular Cartesian coordinate system), respectively, we may put $\mathbf{B} = \mathbf{i}$, $\mathbf{C} = \mathbf{j}$, $\mathbf{A} = \mathbf{i}$ to show that $\mu = -1$; and $\mathbf{B} = \mathbf{i}$, $\mathbf{C} = \mathbf{j}$, $\mathbf{A} = \mathbf{j}$ to show that $\lambda = 1$.

2.22 Write the equation in Prob. 2.21 in the index form, and prove its validity by means of the ϵ-δ identity (2.3–19).

Note. Since the equation in Prob. 2.21 is valid for arbitrary vectors $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, this proof may be regarded as a proof of the ϵ-δ identity.

Solution. $[\mathbf{A} \times (\mathbf{B} \times \mathbf{C})]_l = \epsilon_{lmn} a_m (\mathbf{B} \times \mathbf{C})_n = \epsilon_{lmn} a_m \epsilon_{njk} b_j c_k = \epsilon_{nlm}\epsilon_{njk} a_m b_j c_k$. By the ϵ-δ identity, Eq. (2.3–19), this becomes $(\delta_{lj}\delta_{mk} - \delta_{lk}\delta_{mj}) a_m b_j c_k$. Hence, it is $\delta_{lj} a_m c_m b_j - \delta_{lk} a_m b_m c_k = a_m c_m b_l - a_m b_m c_l = (\mathbf{A}\cdot\mathbf{C})(\mathbf{B})_l - (\mathbf{A}\cdot\mathbf{B})(\mathbf{C})_l$.

2.4 TRANSLATION AND ROTATION OF COORDINATES

Two-Dimensional Space

Consider two sets of rectangular Cartesian frames of reference O-xy and O'-$x'y'$ on a plane. If the frame of reference O'-$x'y'$ is obtained from O-xy by a shift of origin without a change in orientation, then the transformation is a *translation*. If a point P has coordinates (x, y) and (x', y') with respect to the old and new frames of reference, respectively, and if the coordinates of the new origin O' are (h, k) relative to O-xy, then

$$\begin{cases} x = x' + h \\ y = y' + k \end{cases} \quad \text{or} \quad \begin{cases} x' = x - h \\ y' = y - k. \end{cases} \tag{2.4–1}$$

If the origin remains fixed, and the new axes are obtained by rotating Ox and Oy through an angle θ in the counterclockwise direction, then the transformation of axes is a *rotation*. Let P have coordinates (x, y), (x', y') relative to the old and new frames of reference, respectively. Then (see Fig. 2.2),

$$\begin{aligned} x &= x' \cos\theta - y' \sin\theta \\ y &= x' \sin\theta + y' \cos\theta. \end{aligned} \tag{2.4–2}$$

$$\begin{aligned} x' &= x\cos\theta + y\sin\theta \\ y' &= -x\sin\theta + y\cos\theta. \end{aligned} \tag{2.4–3}$$

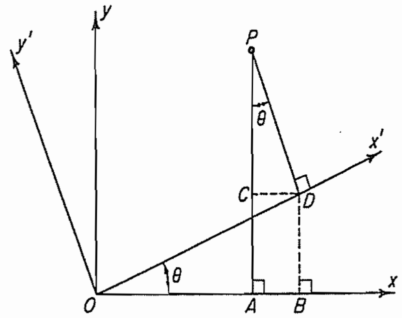

Figure 2.2 Rotation of coordinates.

Using the index notion, we let x_1, x_2 replace x, y and x_1', x_2' replace x', y'. Then obviously, a rotation specified by Eq. (2.4–3) can be represented by the equation

$$x_i' = \beta_{ij}x_j, \qquad (i = 1, 2) \tag{2.4–4}$$

where β_{ij} are elements of the square matrix

$$(\beta_{ij}) = \begin{pmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}. \tag{2.4–5}$$

The inverse transform of Eq. (2.4–4) is

$$x_i = \beta_{ji}x_j', \qquad (i = 1, 2) \tag{2.4–6}$$

where, according to Eq. (2.4–2), β_{ji} is the element in the jth row and ith column of the matrix (β_{ij}). It is clear that the matrix (β_{ji}) is the *transpose* of the matrix (β_{ij}), i.e.,

$$(\beta_{ji}) = (\beta_{ij})^T. \tag{2.4–7}$$

On the other hand, from the point of view of the solution of the set of simultaneous linear equations (2.4–4), the matrix (β_{ji}) in Eq. (2.4–6) must be identified as the *inverse* of the matrix (β_{ij}), i.e.,

$$(\beta_{ji}) = (\beta_{ij})^{-1}. \tag{2.4–8}$$

Thus, we obtain a fundamental property of the transformation matrix (β_{ij}) that defines a rotation of rectangular Cartesian coordinates:

$$(\beta_{ij})^T = (\beta_{ij})^{-1}. \tag{2.4–9}$$

A matrix (β_{ij}), $i, j = 1, 2, \ldots, n$, that satisfies Eq. (2.4–9) is called an *orthogonal* matrix. A transformation is said to be orthogonal if the associated matrix is orthogonal. The matrix of Eq. (2.4–5) defining a rotation of coordinates is orthogonal.

For an orthogonal matrix, we have

$$(\beta_{ij})(\beta_{ij})^T = (\beta_{ij})(\beta_{ij})^{-1} = (\delta_{ij}),$$

where δ_{ij} is the Kronecker delta. Hence,

$$\beta_{ik}\beta_{jk} = \delta_{ij}. \tag{2.4–10}$$

To clarify the geometric meaning of this important equation, we rederive it directly for the rotation transformation as follows. A unit vector issued from the origin along the x_i'-axis has direction cosines β_{i1}, β_{i2} with respect to the x_1-, x_2-axes, respectively. The fact that its length is unity is expressed by the equation

$$(\beta_{i1})^2 + (\beta_{i2})^2 = 1, \qquad (i = 1, 2). \tag{2.4–11}$$

The fact that a unit vector along the x_i'-axis is perpendicular to a unit vector along the x_j'-axis if $j \neq i$ is expressed by the equation

$$\beta_{i1}\beta_{j1} + \beta_{i2}\beta_{j2} = 0, \qquad (i \neq j). \tag{2.4–12}$$

Combining Eqs. (2.4–11) and (2.4–12), we obtain Eq. (2.4–10).

Note: Alternatively, since we know what the β_{ij}'s are from Eq. (2.4–5), we can verify Eq. (2.4–10) by direct computation.

Three-Dimensional Space

Obviously, the preceding discussion can be extended to three dimensions without much ado. The range of indices i, j can be extended to 1, 2, 3. Thus, consider two right-handed rectangular Cartesian coordinate systems x_1, x_2, x_3 and x_1', x_2', x_3', with the same origin O. Let $\mathbf{x}$ denote the position vector of a point P with components x_1, x_2, x_3 or x_1', x_2', x_3'. Let $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ be unit vectors in the directions of the positive x_1, x_2, x_3-axes. They are called *base vectors* of the x_1, x_2, x_3 coordinate system. Let $\mathbf{e}_1', \mathbf{e}_2', \mathbf{e}_3'$ be the base vectors of the x_1', x_2', x_3' coordinate system. Note that since the coordinates are orthogonal, we have

$$\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}, \qquad \mathbf{e}_i' \cdot \mathbf{e}_j' = \delta_{ij}. \tag{2.4–13}$$

In terms of the base vectors, the vector $\mathbf{x}$ may be expressed as follows:

$$\mathbf{x} = x_j\mathbf{e}_j = x_j'\mathbf{e}_j'. \tag{2.4–14}$$

A scalar product of both sides of Eq. (2.4–14) with $\mathbf{e}_i$ gives

$$x_j(\mathbf{e}_j \cdot \mathbf{e}_i) = x_j'(\mathbf{e}_j' \cdot \mathbf{e}_i). \tag{2.4–15}$$

But

$$x_j(\mathbf{e}_j \cdot \mathbf{e}_i) = x_j\delta_{ij} = x_i;$$

therefore,

$$x_i = (\mathbf{e}_j' \cdot \mathbf{e}_i)x_j'. \tag{2.4–16}$$

Now, define

$$(\mathbf{e}_j' \cdot \mathbf{e}_i) \equiv \beta_{ji}; \tag{2.4–17}$$

then,

$$x_i = \beta_{ji}x_j', \qquad (j = 1, 2, 3). \tag{2.4–18}$$

Next, dot both sides of Eq. (2.4–14) with $\mathbf{e}_i'$. This gives

$$x_j(\mathbf{e}_j \cdot \mathbf{e}_i') = x_j'(\mathbf{e}_j' \cdot \mathbf{e}_i').$$

But $(\mathbf{e}_j' \cdot \mathbf{e}_i') = \delta_{ij}$ and $(\mathbf{e}_j \cdot \mathbf{e}_i') = \beta_{ij}$; therefore, we obtain

$$x_i' = \beta_{ij}x_j, \qquad (i = 1, 2, 3). \tag{2.4–19}$$

Equations (2.4–18) and (2.4–19) are generalizations of Eqs. (2.4–4) and (2.4–6) to the three-dimensional case.

Equation (2.4–17) shows the geometric meaning of the coefficient β_{ij}. That Eqs. (2.4–7) and (2.4–8) hold for $i, j = 1, 2, 3$ is clear because Eqs. (2.4–18) and (2.4–19) are inverse transformations of each other. Then, Eqs. (2.4–9) and (2.4–10) follow.

Now, the numbers x_1, x_2, x_3 that represent the coordinates of the point P in Fig. 2.3 are also the components of the radius vector $\mathbf{A}$. A recognition of this fact gives us immediately the law of transformation of the components of a vector in rectangular Cartesian coordinates:

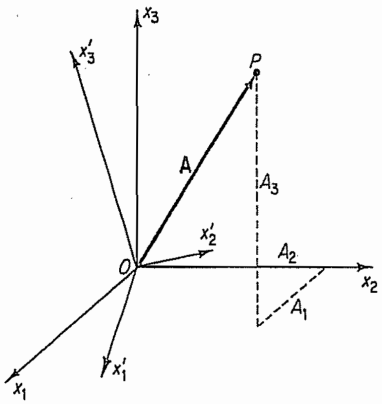

Figure 2.3 Radius vector and coordinates.

$$A_i' = \beta_{ij}A_j, \qquad A_i = \beta_{ji}A_j', \tag{2.4–20}$$

in which β_{ij} represents the cosine of the angle between the axes Ox_i' and Ox_j.

Finally, let us point out that the three unit vectors along x_1', x_2', x_3' form the edges of a cube with volume 1. The volume of a parallelepiped having any three vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ as edges is given either by the triple product $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$ or by its negative; the sign is determined by whether the three vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$, in this order, form a right-handed screw system or not. If they are right handed, then the volume is equal to the determinant of their components:

$$\text{Volume} = (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}. \tag{2.4–21}$$

Let us assume that $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ and $\mathbf{x}_1', \mathbf{x}_2', \mathbf{x}_3'$ are right handed. Then it is clear that the determinant of β_{ij} represents the volume of a unit cube and hence has the value 1:

$$|\beta_{ij}| \equiv \begin{vmatrix} \beta_{11} & \beta_{12} & \beta_{13} \\ \beta_{21} & \beta_{22} & \beta_{23} \\ \beta_{31} & \beta_{32} & \beta_{33} \end{vmatrix} = 1. \tag{2.4–22}$$

PROBLEMS

2.23 Write out Eq. (2.4–10) in extenso, and interpret the geometric meaning of the six resulting equations; $i = 1, 2, 3$.

Solution. Let the index i stand for 1, 2, 3.

$$\text{If } i = 1, j = 1\text{: then } \beta_{11}\beta_{11} + \beta_{12}\beta_{12} + \beta_{13}\beta_{13} = 1. \tag{1}$$

$$\text{If } i = 1, j = 2\text{: then } \beta_{11}\beta_{21} + \beta_{12}\beta_{22} + \beta_{13}\beta_{23} = 0. \tag{2}$$

Equation (1) means that the length of the vector $(\beta_{11}, \beta_{12}, \beta_{13})$ is 1. Equation (2) means that the vectors $(\beta_{11}, \beta_{21}, \beta_{13})$, $(\beta_{21}, \beta_{22}, \beta_{23})$ are orthogonal to each other.

Other combinations of i, j are similar.

2.24 Derive Eq. (2.4–10) by the following alternative procedure. Differentiate both sides of Eq. (2.4–4) with respect to x_j'. Then use Eq. (2.4–6) and the fact that $\partial x_i/\partial x_j = \delta_{ij}$ to simplify the results.

Solution. Differentiating Eq. (2.4–4) with respect to x_j', we obtain $\delta_{ij} = \beta_{ik}\partial x_k/\partial x_j'$. But $x_i = \beta_{ji}x_j'$. On changing the index i to k and differentiating, we have $\partial x_k/\partial x_j' = \beta_{jk}$. Combining these results yields $\delta_{ij} = \beta_{ik}\beta_{jk}$.

2.5 COORDINATE TRANSFORMATION IN GENERAL

A set of independent variables x_1, x_2, x_3 specifies the coordinates of a point in a frame of reference. A set of equations

$$\bar{x}_i = f_i(x_1, x_2, x_3), \qquad (i = 1, 2, 3) \tag{2.5–1}$$

describes a transformation from x_1, x_2, x_3 to a set of new variables $\bar{x}_1, \bar{x}_2, \bar{x}_3$. The inverse transformation

$$x_i = g_i(\bar{x}_1, \bar{x}_2, \bar{x}_3), \qquad (i = 1, 2, 3) \tag{2.5–2}$$

proceeds in the reverse direction. In order to ensure that such a reversible transformation exists and is in one-to-one correspondence in a certain region R of the variables (x_1, x_2, x_3)—i.e., in order that each set of numbers $(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ defines a unique set of numbers (x_1, x_2, x_3), for (x_1, x_2, x_3) in the region R, and vice versa—it is sufficient that

(1) The functions f_i are single valued, are continuous, and possess continuous first partial derivatives in the region R.

(2) The *Jacobian determinant* $J = \det(\partial\bar{x}_i/\partial x_j)$ does not vanish at any point of the region R. That is,

$$J = \det\left(\frac{\partial \bar{x}_i}{\partial x_j}\right) \equiv \begin{vmatrix} \dfrac{\partial \bar{x}_1}{\partial x_1} & \dfrac{\partial \bar{x}_1}{\partial x_2} & \dfrac{\partial \bar{x}_1}{\partial x_3} \\ \dfrac{\partial \bar{x}_2}{\partial x_1} & \dfrac{\partial \bar{x}_2}{\partial x_2} & \dfrac{\partial \bar{x}_2}{\partial x_3} \\ \dfrac{\partial \bar{x}_3}{\partial x_1} & \dfrac{\partial \bar{x}_3}{\partial x_2} & \dfrac{\partial \bar{x}_3}{\partial x_3} \end{vmatrix} \neq 0. \qquad (2.5\text{–}3)$$

Coordinate transformations with the properties 1 and 2 are called *admissible transformations*. If the Jacobian is positive everywhere, then a right-hand set of coordinates is transformed into another right-hand set, and the transformation is said to be *proper*. If the Jacobian is negative everywhere, a right-hand set of coordinates is transformed into a left-hand one, and the transformation is said to be *improper. In this book, we shall* tacitly assume *that our transformations are admissible and proper*.

Significance of the Jacobian Determinant

To appreciate the significance of the *Jacobian determinant*, let us assume that we have found that (x_1^0, x_2^0, x_3^0) corresponds to $(\bar{x}_1^0, \bar{x}_2^0, \bar{x}_3^0)$, i.e., they satisfy Eq. (2.5–1), and ask whether we can find an inverse transformation in a small neighborhood of this point. We differentiate Eq. (2.5–1) to obtain

$$d\bar{x}_i = \frac{\partial f_i}{\partial x_j}\, dx_j \qquad (i = 1, 2, 3) \qquad (2.5\text{–}4)$$

and evaluate the partial derivatives $\partial f_i/\partial x_j$ at the point (x_1^0, x_2^0, x_3^0). The Eq. (2.5–4) defines a linear transformation of the vector dx_j to a vector $d\bar{x}_i$. If we solve the set of linear equations (2.5–4) for dx_j, we know that the solution exists only if the determinant of the coefficients does not vanish:

$$\det\left(\frac{\partial f_i}{\partial x_j}\right) \neq 0. \qquad (2.5\text{–}5)$$

Thus, an inverse exists in the neighborhood of (x_1^0, x_2^0, x_3^0) only if Eq. (2.5–3) is valid. Further, when $J \neq 0$, Eq. (2.5–4) can be solved to obtain

$$dx_i = c_{ij}\, d\bar{x}_j \qquad (2.5\text{–}6)$$

where c_{ij} are constants. Hence, a small neighborhood of the known point, an inverse transformation [an approximation of Eq. (2.5–2)] can be found in a small neighborhood of the known point. Thus, conditions 1 and 2 stated earlier are sufficient conditions for the existence of an inverse in a small region around the known point. By repeated application of this argument to new known points away from the initial known point, one can extend and find the region R in which a one-to-one inverse transformation given by Eq. (2.5–2) exists.

PROBLEM

2.25 (a) Review the methods of solving linear simultaneous equations. One of the methods uses determinants. Use that method to solve Eq. (2.5–4) for dx_1, dx_2, dx_3. Use the permutation symbol ϵ_{rst}, defined in Eq. (2.3–16), to express the final result.
(b) R is a region in and on a circle of unit radius on a plane. The equation of the circle is $r = 1$ in polar coordinates and $x^2 + y^2 = 1$ in rectangular Cartesian coordinates. Show that the Jacobian J is equal to r and that the area of the circle is

$$\iint_R J\, dr\, d\theta = \iint_R dx\, dy,$$

or

$$\int_0^1\!\!\int_0^{2\pi} r\, dr\, d\theta = \int_0^1\!\!\int_0^{\sqrt{1-x^2}} dx\, dy.$$

Here, an integration of the Jacobian multiplied by the product of the differentials $dr\, d\theta$ gives the area.

2.6 ANALYTICAL DEFINITIONS OF SCALARS, VECTORS, AND CARTESIAN TENSORS

Let (x_1, x_2, x_3) and $(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ be two fixed sets of rectangular Cartesian frames of reference related by the transformation law

$$\bar{x}_i = \beta_{ij} x_j \qquad (2.6\text{–}1)$$

where β_{ij} is the direction cosine of the angle between unit vectors along the coordinate axes $\bar{x}_i$ and x_j. Thus,

$$\beta_{21} = \cos(\bar{x}_2, x_1), \qquad (2.6\text{–}2)$$

and so forth. The inverse transform is

$$x_i = \beta_{ji} \bar{x}_j. \qquad (2.6\text{–}3)$$

A system of quantities is called a *scalar*, a *vector*, or a *tensor*, depending upon how the components of the system are defined in the variables x_1, x_2, x_3 and how they are transformed when the variables x_1, x_2, x_3 are changed to $\bar{x}_1, \bar{x}_2, \bar{x}_3$.

A system is called a *scalar* if it has only a single component Φ in the variables x_i and a single component $\bar{\Phi}$ in the variables $\bar{x}_i$ and if Φ and $\bar{\Phi}$ are numerically equal at the corresponding points,

$$\Phi(x_1, x_2, x_3) = \bar{\Phi}(\bar{x}_1, \bar{x}_2, \bar{x}_3). \tag{2.6–4}$$

A system is called a *vector field* or a *tensor field of rank 1* if it has three components ξ_i in the variables x_i and three components $\bar{\xi}_i$ in the variables $\bar{x}_i$ and if the components are related by the characteristic law

$$\begin{aligned} \bar{\xi}_i(\bar{x}_1, \bar{x}_2, \bar{x}_3) &= \xi_k(x_1, x_2, x_3)\beta_{ik}, \\ \xi_i(x_1, x_2, x_3) &= \bar{\xi}_k(\bar{x}_1, \bar{x}_2, \bar{x}_3)\beta_{ki}. \end{aligned} \tag{2.6–5}$$

Generalizing these definitions to a system that has nine components when i and j range over 1, 2, 3, we define a *tensor field of rank 2* if it is a system that has nine components t_{ij} in the variables x_1, x_2, x_3 and nine components $\bar{t}_{ij}$ in the variables $\bar{x}_1, \bar{x}_2, \bar{x}_3$ and if the components are related by the characteristic law

$$\begin{aligned} \bar{t}_{ij}(\bar{x}_1, \bar{x}_2, \bar{x}_3) &= t_{mn}(x_1, x_2, x_3)\beta_{im}\beta_{jn}, \\ t_{ij}(x_1, x_2, x_3) &= \bar{t}_{mn}(\bar{x}_1, \bar{x}_2, \bar{x}_3)\beta_{mi}\beta_{nj}. \end{aligned} \tag{2.6–6}$$

Further generalization to tensor fields of higher ranks is immediate. These definitions can obviously be modified to two dimensions if the indices range over 1, 2, or to n dimensions if the range of the indices is $1, 2, \ldots, n$. Since our definitions are based on transformations from one rectangular Cartesian frame of reference to another, the systems so defined are called *Cartesian tensors*. For simplicity, only Cartesian tensor equations will be used in this book.

Elaboration on Why Vectors and Tensors Are Defined in This Manner

The analytical definition of vectors is designed to follow the idea of a radius vector. We all know that the radius vector, a vector joining the origin (0, 0, 0) to a point (x_1, x_2, x_3), embodies our idea of a vector and expresses it numerically in terms of the components (x_1-0, x_2-0, x_3-0), i.e., (x_1, x_2, x_3). When this vector is viewed from another frame of reference, the components referred to the new frame can be computed from the old according to Eq. (2.6–1), which is the *law of transformation of the components of a radius vector. Our generalization of Eq. (2.6–1) into Eq. (2.6–5), which defines all vectors, is equivalent to saying that we can call an entity a vector if it behaves like a radius vector, namely, if it has a fixed direction and a fixed magnitude.*

These remarks are intended to differentiate a matrix from a vector. We can list the components of a vector in the form of a column matrix; but not all column matrices are vectors. For example, to identify myself, I can list my age, social security number, street address, and zip code in a column matrix. What can you say about this matrix? Nothing very interesting! It is certainly not a vector.

The mathematical steps we took in generalizing the definition given in Eq. (2.6–5) for a vector to Eq. (2.6–6) for a tensor are natural enough. These equations are so similar that if we call a vector a tensor of rank 1, we cannot help but call the others tensors of rank 2 or 3, etc. What is the physical significance of these higher order tensors? The most effective way to answer this question is to consider some concrete examples, such as the stress tensor. However, before we turn our attention to specific examples to discuss the significance of tensor equations, consider the following problems:

PROBLEMS

2.26 Show that, *if all components of a Cartesian tensor vanish in one coordinate system, then they vanish in all other Cartesian coordinate systems*. This is perhaps the most important property of tensor fields.

Proof. The property follows immediately from Eq. (2.6–6). If every component of t_{mn} vanishes, then the right-hand side vanishes and $\bar{t}_{ij} = 0$ for all i, j.

2.27 Prove the following theorem: *The sum or difference of two Cartesian tensors of the same rank is again a tensor of the same rank.* Thus, any linear combination of tensors of the same rank is again a tensor of the same rank.

Proof. Let A_{ij}, B_{ij} be two tensors. Under the coordinate transformation given by Eq. (2.6–1), we have the new components

$$\bar{A}_{ij} = A_{mn}\beta_{im}\beta_{jn}, \qquad \bar{B}_{ij} = B_{mn}\beta_{im}\beta_{jn}.$$

Adding or subtracting, we obtain

$$\bar{A}_{ij} \pm \bar{B}_{ij} = \beta_{im}\beta_{jn}(A_{mn} \pm B_{mn})$$

and the theorem is proved.

2.28 Prove the following theorem: *Let $A_{\alpha_1\ldots\alpha_r}$, $B_{\alpha_1\ldots\alpha_r}$ be tensors. Then the equation*

$$A_{\alpha_1\ldots\alpha_r}(x_1, x_2, \ldots, x_n) = B_{\alpha_1\ldots\alpha_r}(x_1, x_2, \ldots, x_n)$$

is a tensor equation; i.e., if this equation is true in one Cartesian coordinate system, then it is true in all Cartesian coordinate systems.

Proof. Multiplying both sides of the equation by

$$\beta_{i\alpha_1}\beta_{j\alpha_2}\cdots\beta_{k\alpha_r}$$

and summing over the repeated indices yields the equation

$$\bar{A}_{ij\ldots k}(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n) = \bar{B}_{ij\ldots k}(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n).$$

Alternatively, write the equation as $\mathbf{A} - \mathbf{B} = 0$. Then every component of $\mathbf{A} - \mathbf{B}$ vanishes. Then apply the results of Probs. 2.27 and 2.26, in turn.

2.7 THE SIGNIFICANCE OF TENSOR EQUATIONS

The theorems stated in the problems at the end of the previous section contain the most important property of tensor fields: *If all the components of a tensor field vanish in one coordinate system, they vanish likewise in all coordinate systems that can be obtained by admissible transformations*. Since the sum and difference of tensor fields of a given type are tensors of the same type, we deduce that *if a tensor equation can be established in one coordinate system, then it must hold for all coordinate systems obtained by admissible transformations*.

Thus, the importance of tensor analysis may be summarized by the following statement: The form of an equation can have general validity with respect to any frame of reference only if every term in the equation has the same tensor characteristics. If this condition is not satisfied, a simple change of the system of reference will destroy the form of the relationship, and the form would, therefore, be merely fortuitous.

We see that tensor analysis is as important as dimensional analysis in any formulation of physical relations. In dimensional analysis, we study the changes a physical quantity undergoes with particular choices of fundamental units. Two physical quantities cannot be equal unless they have the same dimensions. An equation describing a physical relation cannot be correct unless it is invariant with respect to a change of fundamental units.

Because of the design of the tensor transformation laws, the tensorial equations are in harmony with physics.

2.8 NOTATIONS FOR VECTORS AND TENSORS: BOLDFACE OR INDICES?

In continuum mechanics we are concerned with vectors describing displacements, velocities, forces, etc., and with tensors describing stress, strain, constitutive equations, etc. For vectors, the usual notation of boldface letters or an arrow, such as **u** or $\vec{u}$, is agreeable to all; but for tensors, there are differences of opinion. A tensor of rank 2 may be printed as a boldface letter or with a double arrow or with a pair of braces. Thus, if T is a tensor of rank 2, it may be printed as **T**, $\overset{\Rightarrow}{T}$ or $\{T\}$. The first notation is the simplest, but then you have to remember what the symbol represents; it may be a vector or it may be a tensor. The other notations are cumbersome. More important objections to the simple notation arise when several vectors and tensors are associated together. In vector analysis, we have to distinguish scalar products from vector products. How about tensors? Shall we define many kinds of tensor products? We have to, because there is a variety of ways tensors can be associated. The matter becomes complicated. For this reason, in most theoretical works that require extensive use of tensors, an index notation is

used. In this notation, vectors and tensors are resolved into their components with respect to a frame of reference and denoted by symbols such as u_i, u_{ij}, etc. These components are real numbers. Mathematical operations on them follow the usual rules of arithmetic. No special rules of combination need to be introduced. Thus, we gain a measure of simplicity. Furthermore, the index notation exhibits the rank and the range of a tensor clearly. It displays the role of the frame of reference explicitly.

The last-mentioned advantage of the index notation, however, is also a weakness: It draws the attention of the reader away from the physical entity. Hence, one has to be adaptive and familiarize oneself with both systems.

2.9 QUOTIENT RULE

Consider a set of n^3 functions $A(1, 1, 1)$, $A(1, 1, 2)$, $A(1, 2, 3)$, etc., or $A(i, j, k)$ for short, with each of the indices i, j, k ranging over $1, 2, \ldots, n$. Although the set of functions $A(i, j, k)$ has the right number of components, we do not know whether it is a tensor. Now suppose we know something about the nature of the product of $A(i, j, k)$ with an arbitrary tensor. Then there is a method that enables us to establish whether $A(i, j, k)$ is a tensor without going to the trouble of determining the law of transformation directly.

For example, let $\xi_i(x)$ be a vector. Let us suppose that the product $A(i, j, k)\xi_i$ (summation convention used over i) is known to yield a tensor of the type $A_{jk}(x)$, i.e.,

$$A(i, j, k)\xi_i = A_{jk}. \tag{2.9–1}$$

Then we can prove that $A(i, j, k)$ is a tensor of the type $A_{ijk}(x)$.

The proof is very simple. Since $A(i, j, k)\xi_i$ is of the type A_{jk}, it is transformed into $\bar{x}$-coordinates as

$$\bar{A}(i, j, k)\bar{\xi}_i = \bar{A}_{jk} = \beta_{jr}\beta_{ks}A_{rs} = \beta_{jr}\beta_{ks}[A(m, r, s)\xi_m]. \tag{2.9–2}$$

But $\xi_m = \beta_{im}\bar{\xi}_i$. Inserting this in the right-hand side of Eq. (2.9–2) and transposing all terms to one side of the equation, we obtain

$$[\bar{A}(i, j, k) - \beta_{jr}\beta_{ks}\beta_{im}A(m, r, s)]\bar{\xi}_i = 0. \tag{2.9–3}$$

Now $\bar{\xi}_i$ is an arbitrary vector. Hence, the quantity within the brackets must vanish, and we have

$$\bar{A}(i, j, k) = \beta_{im}\beta_{jr}\beta_{ks}A(m, r, s), \tag{2.9–4}$$

which is precisely the law of transformation of the tensor of the type A_{ijk}.

The pattern of the preceding example can be generalized to higher order tensors.

2.10 PARTIAL DERIVATIVES

When only Cartesian coordinates are considered, the partial derivatives of any tensor field behave like the components of a Cartesian tensor. To show this, let us consider two sets of Cartesian coordinates (x_1, x_2, x_3) and $(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ related by

$$\bar{x}_i = \beta_{ij}x_j + \alpha_i, \tag{2.10–1}$$

where β_{ij} and α_i are constants.

Now, if $\xi_i(x_1, x_2, x_3)$ is a tensor, so that

$$\bar{\xi}_i(\bar{x}_1, \bar{x}_2, \bar{x}_3) = \xi_k(x_1, x_2, x_3)\beta_{ik}, \tag{2.10–2}$$

then, on differentiating both sides of this equation, one obtains

$$\frac{\partial \bar{\xi}_i}{\partial \bar{x}_j} = \beta_{ik}\frac{\partial \xi_k}{\partial x_m}\frac{\partial x_m}{\partial \bar{x}_j} = \beta_{ik}\beta_{jm}\frac{\partial \xi_k}{\partial x_m} \tag{2.10–3}$$

which verifies the statement.

It is a common practice to *use a comma to denote partial differentiation*. Thus,

$$\xi_{i,j} \equiv \frac{\partial \xi_i}{\partial x_j}, \qquad \Phi_{,i} \equiv \frac{\partial \Phi}{\partial x_i}, \qquad \sigma_{ij,k} \equiv \frac{\partial \sigma_{ij}}{\partial x_k}.$$

When we restrict ourselves to Cartesian coordinates, $\Phi_{,i}$, $\xi_{i,j}$, and $\sigma_{ij,k}$ are tensors of rank 1, 2, and 3, respectively, provided that Φ, ξ, and σ_{ij} are tensors.

PROBLEMS

2.29 In any tensor $A_{ijk\dots m}$, equating two indices and summing over that index is called a *contraction*. Thus, for a tensor A_{ijk}, a contraction over i and $j(i, j = 1, 2, 3)$ results in a vector $A_{iik} = A_{11k} + A_{22k} + A_{33k}$. Prove that the contraction of any two indices in a Cartesian tensor of rank n results in a tensor of rank $n - 2$.

Solution. The only significant part of the statement is that the result of contraction is a *tensor*. Let $A_{ijk\dots n}$ be a tensor of rank n. Then $A_{iik\dots n}$ has only $(n - 2)$ indices. To show that it is a tensor, consider the definition

$$\bar{A}_{ijk\dots n} = A_{\alpha_1\alpha_2\alpha_3\dots\alpha_n}\beta_{i\alpha_1}\beta_{j\alpha_2}\beta_{k\alpha_3}\cdots\beta_{n\alpha_n}.$$

A contraction over i and j yields

$$\bar{A}_{iik\dots n} = A_{\alpha_1\alpha_2\alpha_3\dots\alpha_n}\beta_{i\alpha_1}\beta_{i\alpha_2}\beta_{k\alpha_3}\cdots\beta_{n\alpha_n}.$$

But we know from Eq. (2.4–10) that

$$\beta_{i\alpha_1}\beta_{i\alpha_2} = \delta_{\alpha_1\alpha_2}.$$

Hence,

$$\begin{aligned}\bar{A}_{iik\dots n} &= A_{\alpha_1\alpha_2\alpha_3\dots\alpha_n}\delta_{\alpha_1\alpha_2}\beta_{k\alpha_3}\cdots\beta_{n\alpha_n} \\ &= A_{\alpha_1\alpha_1\alpha_3\dots\alpha_n}\beta_{k\alpha_3}\cdots\beta_{n\alpha_n}.\end{aligned}$$

Thus, $A_{\alpha_1\alpha_1\alpha_3\dots\alpha_n}$ obeys the transformation law for a tensor of rank $(n - 2)$, and we have proved the statement.

2.30 If A_{ij} is a Cartesian tensor of rank 2, show that A_{ii} is a scalar.

Solution. From Prob. 2.29, A_{ii} is a tensor of rank 0 and hence is scalar. More directly, we have

$$\bar{A}_{ij} = A_{mn}\beta_{im}\beta_{jn}$$

$$\bar{A}_{ii} = A_{mn}\beta_{im}\beta_{in} = \delta_{mn}A_{mn} = A_{mm},$$

which obeys the definition of a scalar, Eq. (2.6–4).

2.31 Use the index notation and summation convention to prove the following relations (see the table of notations below):

(a) $\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}$

(b) $(\mathbf{s} \times \mathbf{t})\cdot(\mathbf{u} \times \mathbf{v}) = (\mathbf{s}\cdot\mathbf{u})(\mathbf{t}\cdot\mathbf{v}) - (\mathbf{s}\cdot\mathbf{v})(\mathbf{t}\cdot\mathbf{u})$

(c) curl curl $\mathbf{v}$ = grad div $\mathbf{v} - \Delta\mathbf{v}$

Example of solution.

(c) $$\begin{aligned}\text{curl curl } \mathbf{v} &= \epsilon_{ijk}\frac{\partial}{\partial x_j}\left(\epsilon_{klm}\frac{\partial v_m}{\partial x_l}\right) \\ &= \epsilon_{ijk}\epsilon_{lmk}\frac{\partial^2 v_m}{\partial x_j \partial x_l} \\ &= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\frac{\partial^2 v_m}{\partial x_j \partial x_l} \\ &= \frac{\partial^2 v_j}{\partial x_j \partial x_i} - \frac{\partial^2 v_i}{\partial x_j \partial x_j} = \frac{\partial}{\partial x_i}\left(\frac{\partial v_j}{\partial x_j}\right) - \frac{\partial}{\partial x_j}\left(\frac{\partial v_i}{\partial x_j}\right) \\ &= \nabla(\nabla\cdot\mathbf{v}) - \nabla\cdot\nabla\mathbf{v} = \text{grad div } \mathbf{v} - \Delta\mathbf{v}.\end{aligned}$$

2.32 Let $\mathbf{r}$ be the radius vector of a typical point in a field and r be the magnitude of $\mathbf{r}$. Prove that, with the notations defined in the following table,

Vector Notation	Index Notation	Rank of Tensor
$\mathbf{v}$ (vector)	v_i	1
$\lambda = \mathbf{u}\cdot\mathbf{v}$ (dot, scalar, or inner product)	$\lambda = u_i v_i$	0
$\mathbf{w} = \mathbf{u} \times \mathbf{v}$ (cross or vector product)	$w_i = \epsilon_{ijk}u_j v_k$	1
grad $\phi = \nabla\phi$ (gradient of scalar field)	$\dfrac{\partial\phi}{\partial x_i}$	1
grad $\mathbf{v} = \nabla\mathbf{v}$ (vector gradient)	$\dfrac{\partial v_i}{\partial x_j}$	2
div $\mathbf{v} = \nabla\cdot\mathbf{v}$ (divergence)	$\dfrac{\partial v_i}{\partial x_i}$	0
curl $\mathbf{v} = \nabla \times \mathbf{v}$ (curl)	$\epsilon_{ijk}\dfrac{\partial v_k}{\partial x_j}$	1
$\nabla^2\mathbf{v} = \nabla\cdot\nabla\mathbf{v} = \Delta\mathbf{v}$ (Laplacian)	$\dfrac{\partial}{\partial x_i}\left(\dfrac{\partial v_j}{\partial x_i}\right) = \dfrac{\partial^2 v_j}{\partial x_i \partial x_i}$	1

(a) div $(r^n\mathbf{r}) = (n+3)r^n$
(b) curl $(r^n\mathbf{r}) = 0$
(c) $\Delta(r^n) = n(n+1)r^{n-2}$

Example of solution.
(a) Let the components of $\mathbf{r}$ be x_i $(i = 1, 2, 3)$.

$$\text{div}\,\mathbf{r} = \nabla\cdot\mathbf{r} = \frac{\partial x_i}{\partial x_i} = 3$$

$$r^2 = x_ix_i; \qquad r\frac{\partial r}{\partial x_i} = x_i; \qquad \therefore \frac{\partial r}{\partial x_i} = \frac{x_i}{r}$$

$$\text{div}\,(r^n\mathbf{r}) = \nabla\cdot(r^n\mathbf{r}) = \frac{\partial}{\partial x_i}(r^nx_i) = r^n\frac{\partial x_i}{\partial x_i} + x_i\frac{\partial r^n}{\partial x_i}$$

$$= 3r^n + n_i\left(nr^{n-1}\frac{\partial r}{\partial x_i}\right) = 3r^n + nr^{n-2}x_ix_i = (n+3)r^n.$$

2.33 A matrix-valued quantity $a_{ij}(i, j = 1, 2, 3)$ is given as follows:

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 2 \\ 0 & 2 & 3 \end{pmatrix}$$

What are the values of (a) a_{ii}, (b) $a_{ij}a_{ij}$, (c) $a_{ij}a_{jk}$ when $i = 1$; $k = 1$ and $i = 1$; and $k = 2$.

Answer. 6, 24, 2, 3.

2.34 It is well known that rigid-body rotation is noncommutative. For example, take a book, and fix a frame of reference with x-, y-, z-axes directed along the edges of the book. First rotate the book 90° about y; then rotate it 90° about z. We obtain a certain configuration. But a reversal of the order of rotation yields a different result.

The rotation of coordinates is also noncommutative; i.e., the transformation matrices (β_{ij}) are noncommutative. Demonstrate this in a special case that is analogous to the rigid-body rotation of the book just considered. First transform x, y, z to x', y', z' by a rotation of 90° about the y-axis. Then transform x', y', z' to x'', y'', z'' by a 90° rotation about z'. Thus,

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}, \qquad \begin{pmatrix} x'' \\ y'' \\ z'' \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}.$$

Derive the transformation matrix from x, y, z to x'', y'', z''. Now, reverse the order of rotation. Show that a different result is obtained.

2.35 Infinitesimal rotations, however, are commutative. Demonstrate this by considering an infinitesimal rotation by an angle θ about y, followed by another infinitesimal

rotation ψ about z. Compare the results with the case in which the order of rotations is reversed.

2.36 Express the following set of equations in a single equation using index notation:

$$\epsilon_{xx} = \frac{1}{E}[\sigma_{xx} - \nu(\sigma_{yy} + \sigma_{zz})], \qquad \epsilon_{xy} = \frac{1+\nu}{E}\sigma_{xy}$$

$$\epsilon_{yy} = \frac{1}{E}[\sigma_{yy} - \nu(\sigma_{xx} + \sigma_{zz})], \qquad \epsilon_{yz} = \frac{1+\nu}{E}\sigma_{yz}$$

$$\epsilon_{zz} = \frac{1}{E}[\sigma_{zz} - \nu(\sigma_{xx} + \sigma_{yy})], \qquad \epsilon_{xz} = \frac{1+\nu}{E}\sigma_{xz}$$

2.37 Write out in longhand, in unabridged form, the following equation:

$$G\left(u_{i,kk} + \frac{1}{1-2\nu}u_{k,ki}\right) + X_i = \rho\frac{\partial^2 u_i}{\partial t^2}.$$

Let

$$x_1 = x, x_2 = y, x_3 = z; \qquad u_1 = u, u_2 = v, u_3 = w.$$

2.38 Show that $\epsilon_{ijk}\sigma_{jk} = 0$, where ϵ_{ijk} is the permutation symbol and σ_{jk} is a symmetric tensor, i.e., $\sigma_{jk} = \sigma_{kj}$.

2.39 Write down a full set of basic laws of physics in tensor notation, using the indicial system. Take a good physics book and go through it from beginning to end.