PARALLEL FINITE ELEMENT MODEL FOR SURFACE AND SUBSURFACE HYDROLOGY

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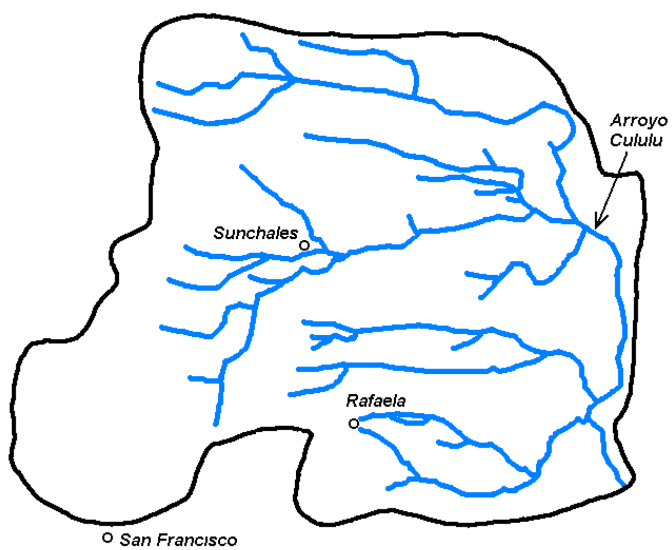
Parallel Solution for Large Scale Hydrological Systems.

- **State of development of Large Scale Model for Surface and Subsurface water flow in multi-aquifer systems.**
- **Study Domain: Santa Fe province in the argentinian Litoral (130.000 km^2, Fig. 1).**



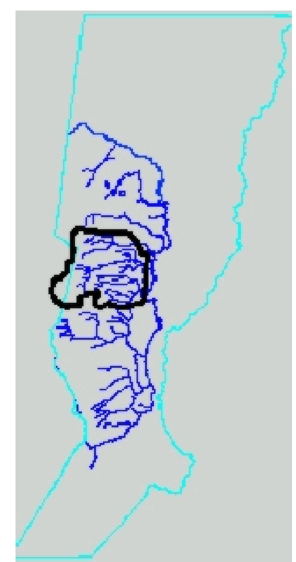


Figure 1: Drainagqe network and system location in province of Santa Fe.

- **Surface and subsurface water flow problems (i.e. multi-aquifer/river systems) deal with coupled nonlinear and linear differential equations in complex domains and dissimilar time scales.**
- **The problem is highly dependent of the terrain topology where water flows.**
- **Construction of a Delaunay tesellation from satellite data images and the DTM data (Digital Terrain Model obtained from 88 topographical maps, scale 1:50.000), which represents the geometry of the stream/aquifer system and therefore the mesh for the parallel finite element model.**
- **Interpolation of any hydrological physical properties via the Natural Neighbor Method.**
- **Mathematical treatment of ground-water flows follows the confined aquifer theory (Classical Dupuit approximation) for unconfined aquifers whereas surface-water flows are treated with the kinematic wave approximation (KWM model) or Shallow-Water theory for unsteady open channel flow.**

- **A mathematical expression similar to Ohm's law is used to simulate the interacting term among the two hydrological components.**
- **Both Manning and Chèzy friction models are available for the streams.**
- **Due to the the spread in time (slow time scale for ground-water flow and fast time scale for surface flow) and length scales, an important degree of refinement is required in the drainage network neighborhoods.**
- **In a 2D multi-aquifer model, the number of unknowns per surface node is, at least, equal to the number of aquifers and aquitards; due to this fact, it is expected to have a very high demand of CPU computation time, calling for parallel processing techniques.**
- **A C++ FEM code based on the PETSc-FEM and PETSc libraries was developed.**

The Hydrological Model.

- **The implemented module solves the problem of subsurface flow in a free aquifer, coupled with a surface net of 1D streams for KW model or a 2D triangulation for Shallow-Water model (Fig. 2 and Fig. 3).**

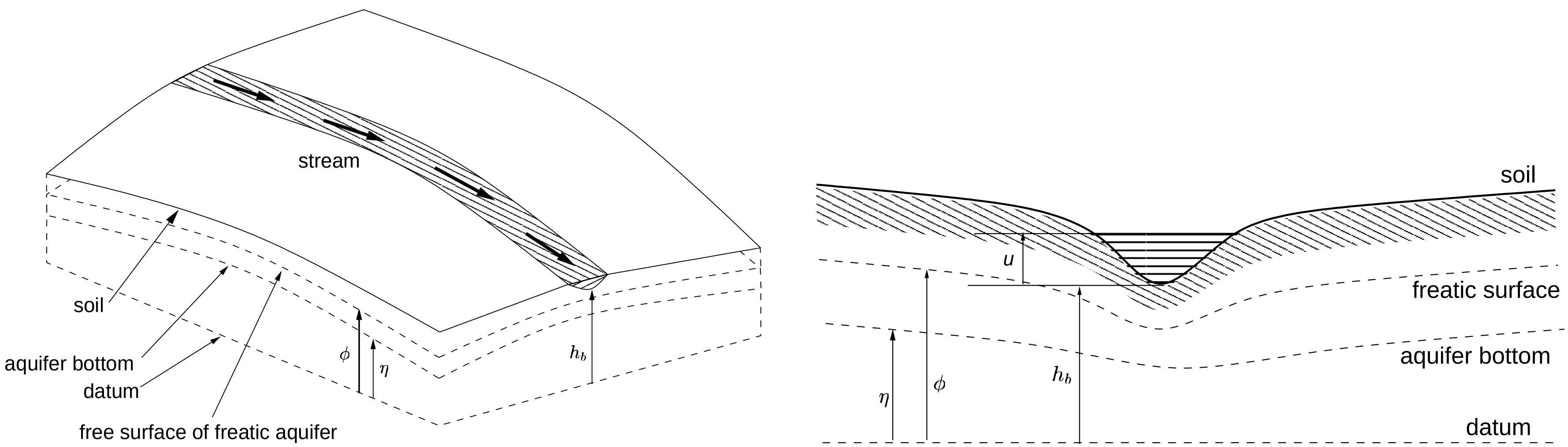


Figure 2: Aquifer/stream system.

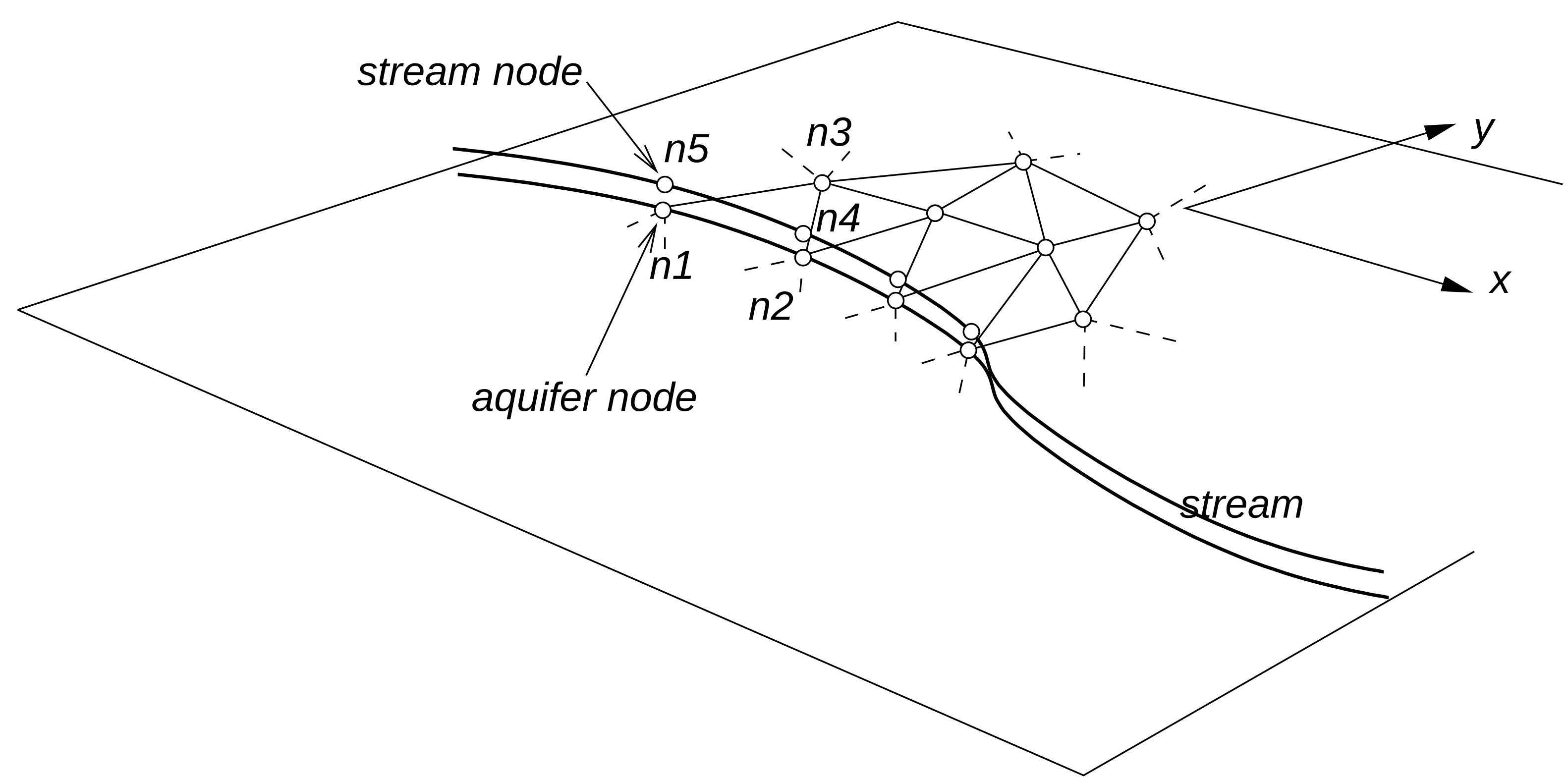


Figure 3: Aquifer/stream system. Discretization.

- **The equation for the aquifer integrated in the vertical direction is:**

$$\frac{\partial}{\partial t}\left(S(\phi-\eta)\phi\right) = \mathrm{div}\big(K(\phi-\eta)\nabla\phi\big) + \sum G_a,$$

where S is the storativity, K is the hydraulic conductivity and G is a source term, due to rain, losses from streams or other aquifers.

- ***“Kinematic Wave Model” KWM* approach [Whitham1974],**

$$\frac{\partial A(u)}{\partial t} + \frac{\partial Q(A(u))}{\partial s} = G_s,$$

where u is the height of the water, A is the transverse cross section of the stream. Q is the flow rate and that is a function of A through the friction law under the KWM mode.

- ***2D Shallow-Water Stream Model.***

$$\frac{\partial U}{\partial t} + \frac{\partial E}{\partial x} + \frac{\partial G}{\partial y} = S,$$

where

$$\mathbf{U} = \begin{bmatrix} h \\ hw \\ hv \end{bmatrix}; \mathbf{E} = \begin{bmatrix} hw \\ hw^2 + g\frac{h^2}{2} \\ hwv \end{bmatrix}; \mathbf{G} = \begin{bmatrix} hv \\ hwv \\ hv^2 + g\frac{h^2}{2} \end{bmatrix};$$

$$\mathbf{S} = \begin{bmatrix} 0 \\ gh(S_{0x} - S_{fx}) \\ gh(S_{0y} - S_{fy}) \end{bmatrix}$$

where ***h*** is the flow depth,

$\bar{u}$=(w,v) is the velocity vector,

g is the gravitational constant,

$\boldsymbol{S}_0$ is the bottom slope and

$\boldsymbol{S}_f$ is the slope friction.

- **Friction Laws**
  - ***KW Model.*** **($Q = \gamma A^m$)**

$$\gamma = C_h\, S_0^{1/2}\, P^{-1}, \qquad m = 3/2 \qquad \textbf{Chèzy model.}$$

$$\gamma = \bar{a}\, n^{-1}\, S^{1/2}\, P^{-2/3}, \qquad m = 5/3 \qquad \textbf{Manning model.}$$

  - ***Shallow-Water Model.***

$$S_{fx} = \frac{1}{C_h h} w|\bar{u}|, \qquad S_{fy} = \frac{1}{C_h h} v|\bar{u}| \qquad \textbf{Chèzy model.}$$

$$S_{fx} = \frac{n^2}{h^{4/3}} w|\bar{u}|, \qquad S_{fy} = \frac{n^2}{h^{4/3}} v|\bar{u}| \qquad \textbf{Manning model.}$$

where P is the wetted perimeter, and C_h, $\bar{a}$ and n (the Manning roughness) are model constants.

- **Coupling Term**

$$G_s = P/R_f(\phi - h_b - u), \qquad \textbf{KW model.}$$

$$G_s = 1/R_f(\phi - h_b - h), \qquad \textbf{S-W model.}$$

where G_s represent the gain or loss of the stream, and the main component is the loss to the aquifer and R_f is the resistivity factor per unit arc length of the perimeter in the *KW* model and the resistivity factor per surface unit in the Shallow-Water model .

The corresponding gain to the aquifer is

$$G_a = -G_s\,\delta_{\Gamma_s}, \tag{4}$$

where Γ_s represents the planar curve of the stream and δ_{Γ_s} is a Dirac's delta distribution with a unit intensity per unit length, i.e.

$$\int f(\mathbf{x})\,\delta_{\Gamma_s}\,\mathrm{d}\Sigma = \int f(\mathbf{x}(s))\,\mathrm{d}s. \tag{5}$$

Natural Neighbor Interpolation.

- **A powerful set of C^∞ (except at the nodes where they are C^0) shape functions for smooth interpolation of physical properties.**

$$u^h(\mathbf{x}) = \sum_{i=1}^{N} \phi_i(\mathbf{x}) u_i, \quad i = 1, 2 \ldots$$ **a vector-valued function.**

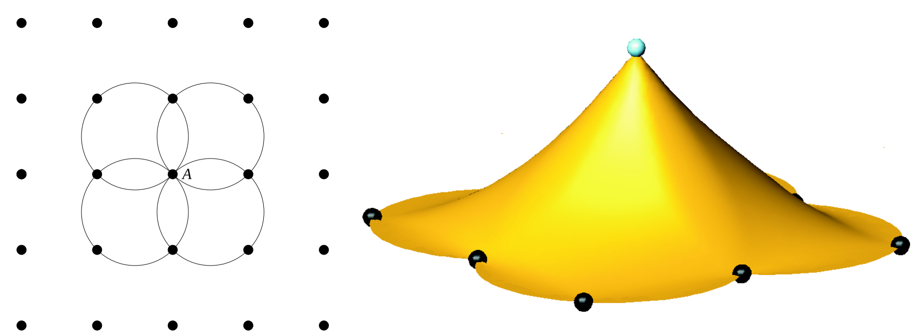


Figure 4: Support and Natural Neighbor shape function.

$$\phi_i(\mathbf{x}) = \frac{A_i(\mathbf{x})}{A(\mathbf{x})}$$

Figure 5: Shape functions.

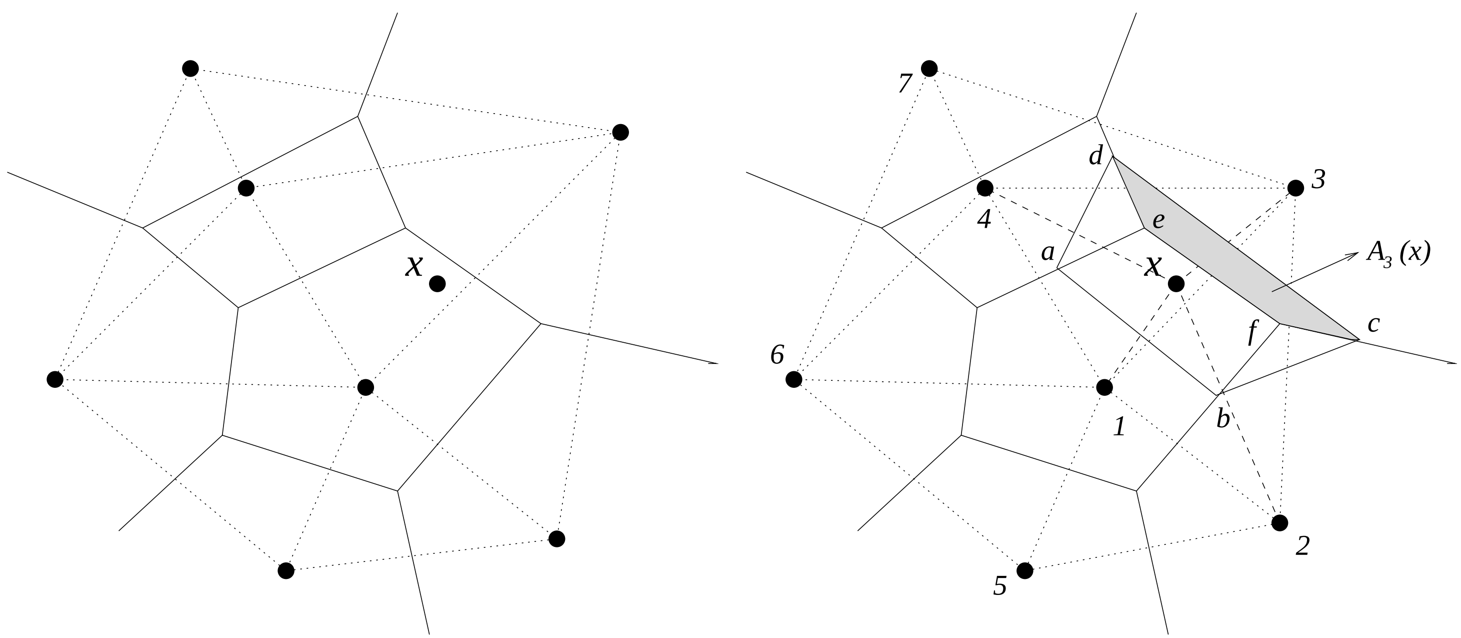
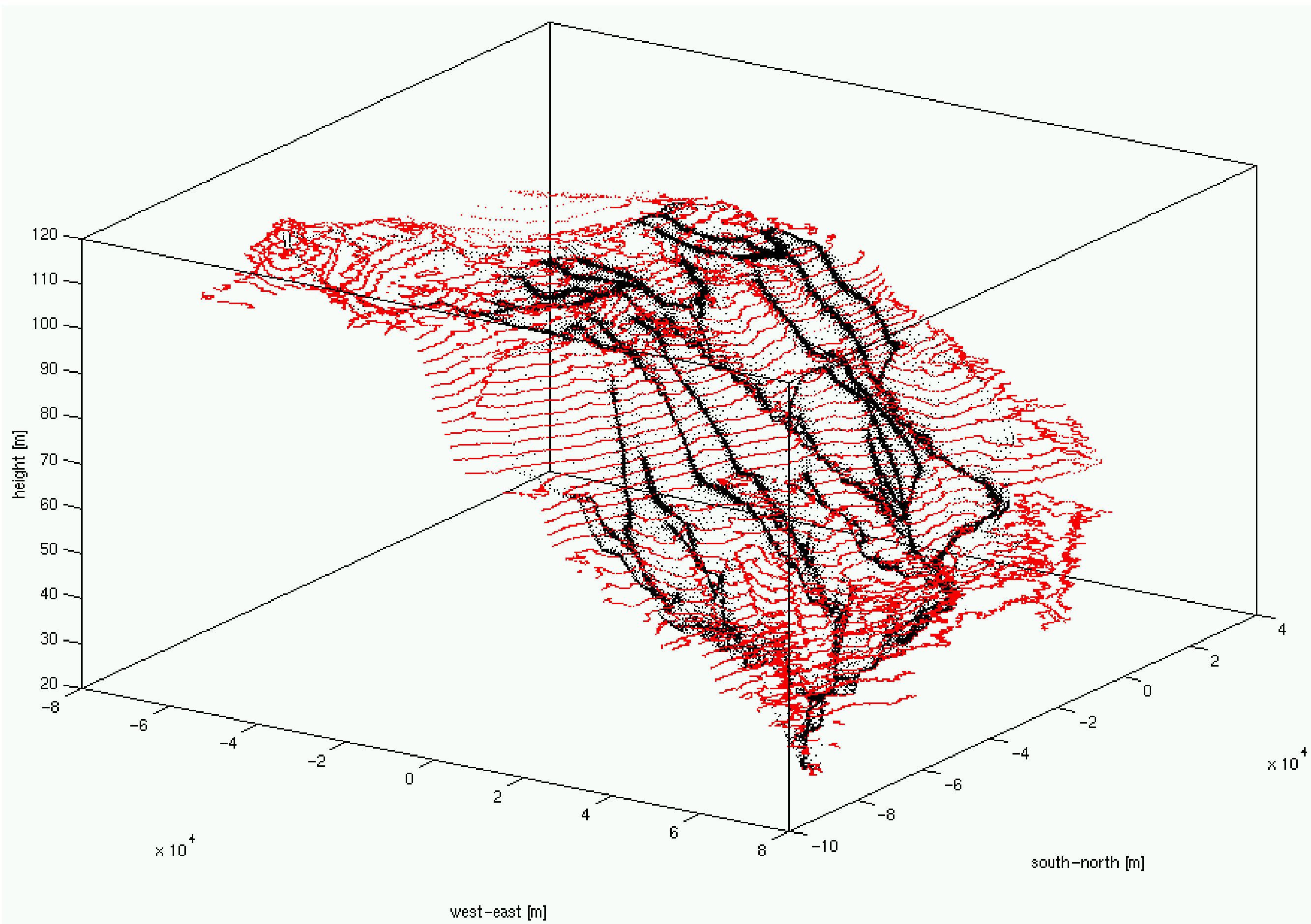


Figure 6: $2^{nd} order$ voronoi diagram for $\mathbf{x}$.

- **An implemented algorithm as a C++ function using CGAL geometrical library.**

1. **Construct a Delaunay tessellation with nodes where values are knowns**
2. **For each node n_i where we wish know the vector-value $u^h(\mathbf{x})$**
   - **Find the face F_{n_i} such as $n_i \in F_{n_i}$**
   - **Construct a secondary local Delaunay tessellation with nodes in F_{n_i} and node n_i**
   - **Compute $A_i(\mathbf{x})$, $(i = 1, 2, ..., N)$**
   - **Compute $u^h(\mathbf{x})$ with $u^h(\mathbf{x}) = \sum_{i=1}^{N} \phi_i(\mathbf{x}) u_i$.**

Stream and Aquifer Height Interpolation



Fem Computations

- **The large time and length scales presents in surface-water and ground-water flow problems entails to a large demand of computational resources.**
- **Considering multi-aquifer system where several layers are added the number of unknowns will be 10 to 20 unknowns per node.**
- **A rough estimate suggests that triangulations in the order of 100,000 to 1,000,000 triangles should be used, therefore, the expected number of unknowns lie between 10^6 and 10^7.**
- **System solution strategy: Domain Decomposition Method, i.e. a mixture of iterative and direct solvers. We iterate a GMRES method on the interface unknowns and solve with a LU factorization in the subdomain interiors.**

Numerical Results

- **KWM runs.**

  The first example is periodic case with wet and dry seasons. A wet season of 200 days with a precipitation rate of 1,000 mm/year (the annual average in last years) is introduced.

  A mesh of 96,131 triangular elements and 48,452 nodal points is used. At time $t = 0$ $\phi = 30\ m$ above the aquifer bottom and $u = 10\ m$ above the streambed. The physical constants are: $\mathbf{K} = 2 \cdot 10^{-3}$, $\mathbf{S} = 2.5 \cdot 10^{-2}$, $\mathbf{R_f} = 3 \cdot 10^{-3}$. The time step adopted is $Dt = 1$ day.

  With 7 processor Pentium IV 1.4-1.7 Ghz and 512 Mb RAM (Rambus) connected through a switch Fast Ethernet (100 Mbit/sec, latency=$O(100)$) was 3.6 seconds in average. The second example is a case with no raindrop and a constant recharge of 5 meters in upstream boundaries.

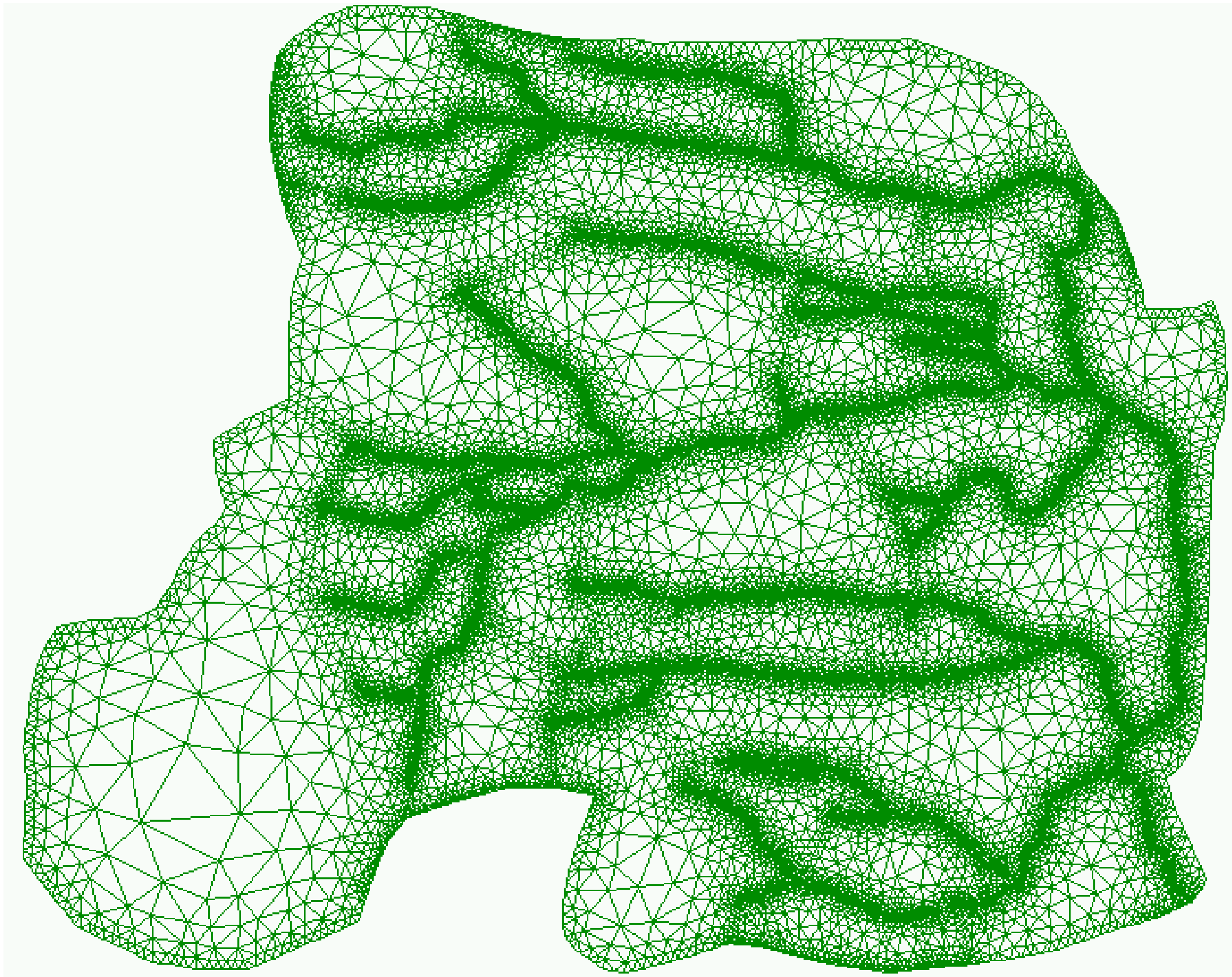


Figure 7: Aquifer mesh.

- **Shallow-Water runs**

  1. **An impermeable river with a parabolic bump in the bottom and a wavetrain perturbation.**
  2. **An small scale stream/aquifer system with long wave perturbation and rainfall periods.**

Conclusions

- **The primary goal of the present work was the development of a large scale hydrological model for surface and subsurface water flow.**
- **More accurate physical data are being obtained through intensive measurements work.**
- **Future work includes multi-layers/multi-aquifer systems, pollutant transport, etc.**

Acknowledgment

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We made extensive use of freely distributed software as *Linux* OS, MPI, PETSc, Newmat, *GMV* **and many others.**