DOUBLING PROPERTY ON HUTCHINSON ORBITS OF SOME FAMILIES OF CONTRACTIVE SIMILITUDES

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Abstract. We are interested in the behavior of the dynamical system generated by successive applications of Hutchinson similitudes starting from a metric measure space. We prove that, in the case of families of similitudes with the same contraction ratio, even when no point of the orbit is a doubling space, a gradual doubling property is taking place and the limit point recovers the homogeneity property.

Introduction

Throughout this paper (X, d) shall always be a compact metric space. Given a finite set $\{\phi_i : X \to X,\ i = 1, \dots, M\}$ of continuous functions on X and a probability sequence $\{p_1, p_2, \dots, p_M\}$ (i.e. $0 < p_i < 1$ and $\sum_{i=1}^{M} p_i = 1$), we define two special mappings. First, given $Y \in \mathcal{K}$, the family of all closed subsets of X, we consider $T_1 Y = \bigcup_{i=1}^{M} \phi_i(Y)$. Second, given $\mu \in \mathcal{P}$, the set of all probability Borel measures on X, we define $T_2\mu = \mu'$ with

$$\mu'(B) = \sum_{i=1}^{M} p_i \mu\left(\phi_i^{-1}(B \cap \phi_i(Y))\right),$$

for every Borel subset B of X. Set $T : \mathcal{K} \times \mathcal{P} \to \mathcal{K} \times \mathcal{P}$ to denote the application $T(Y, \mu) = (T_1 Y, T_2 \mu)$. Following the standard notation (see [6],[5]) we shall say that $\{\phi_1, \dots, \phi_M\}$ is an *iterated function system* (IFS) when each ϕ_i is a contraction on X, i.e. when there exist $a_1, a_2, \dots, a_M > 1$ such that

$$d(\phi_i(x), \phi_i(y)) \le \frac{1}{a_i} d(x, y)$$

for every $x, y \in X$.

The results in [8] show that under the open set condition for the IFS the limit set (attractor) equipped with the invariant measure and the usual Euclidean distance, is a normal space of homogeneous type. In other words, the measure of a ball of radius r less than the diameter of the attractor, is comparable to a power of r.

The invariant measure can be obtained also as the limit of the iteration of T_2 starting at any measure in $\mathcal{P}$. In [2] some sufficient conditions, in terms of separation properties of the IFS, are given in order to have the uniform doubling along the whole orbit starting at any doubling space. Also in [2] the authors show examples proving that the doubling property is generally not preserved by the iteration of T. In other words, it may happen that no point of the orbit generated by a contraction

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is a space of homogeneous type, even when the starting point, and the limit point, have both the doubling property.

Since under the assumptions in [8] the limit space is doubling, no matter wether or not the initial point is a space of homogeneous type, the question of how suddenly appears the doubling property of the limit seems natural.

In this note we prove that in a precise sense, and in a particular case of similitudes on metric space, the elements of Hutchinson orbits become more and more doubling as the step of the iteration grows.

The proof of our main result, Theorem 3.3, is based on the construction given in Section 2 of discrete approximations to the attractor. There we prove that typically the orbit starting at $(\{x_0\}, \delta_{x_0})$ is a sequence of uniformly normal, and hence doubling, metric measure spaces.

In the first section we introduce the basic notation and definitions. We also state as a lemma some elementary properties of IFS. In Section 2 we consider the orbits starting from a mass point space, defined by a finite family of contractive similitudes, and we prove the uniform normality, and hence doubling, for the whole orbit. In Section 3 we search for gradual improvement for the doubling property of the orbit as the iteration step increases.

1. Notation and basic results

As we said in the introduction, (X, d) is a given compact metric space. We shall use $B_d(x, r)$ to denote the ball $\{y \in X : d(x, y) < r\}$, $r > 0$.

Let $\mathcal{K} = \{K \subseteq X : K \neq \emptyset,\ K \text{ compact}\}$. With $[A]_\varepsilon$ we shall denote the ε-enlargement of the set $A \subset X$; i.e. $[A]_\varepsilon = \bigcup_{x\in A} B_d(x, \varepsilon) = \{y \in X : d(y, A) < \varepsilon\}$. Here $d(x, A) = \inf\{d(x, y) : y \in A\}$. Given A and B two sets in $\mathcal{K}$ the Hausdorff distance from A to B is given by

$$d_H(A, B) = \inf\{\varepsilon > 0 : A \subseteq [B]_\varepsilon \text{ and } \mathrm{B} \subseteq [\mathrm{A}]_\varepsilon\}.$$

Let us now introduce the Kantorovich-Hutchinson distance on the set of all Borel regular probability measures on the quasi-metric space (X, d). Let

$$\mathcal{P} = \{\mu : \mu \text{ is a positive Borel measure on } X \text{ and } \mu(X) = 1\},$$

and let $\mathcal{C}(X)$ be the space of continuous real valued functions on X. Let Lip_1 be the space of all Lipschitz continuous functions defined on X with Lipschitz constant less than or equal to one, i.e. $f \in \mathrm{Lip}_1$ if and only if $|f(x) - f(y)| \leq d(x, y)$ for every x and $y \in X$.

Since (X, d) is compact, $d_K(\mu, \nu) = \sup\left\{\left|\int f\, d\mu - \int f\, d\nu\right| : f \in \mathrm{Lip}_1\right\}$ gives a distance on $\mathcal{P}$ such that the d_K-convergence of a sequence is equivalent to its weak star convergence to the same limit (see [5]).

We are now in position to describe the metric on $\mathscr{X} = \mathcal{K} \times \mathcal{P}$. Given two elements (Y_i, μ_i) of $\mathscr{X}$, $i = 1, 2$, define

$$\delta\left((Y_1, \mu_1), (Y_2, \mu_2)\right) = d_H(Y_1, Y_2) + d_K(\mu_1, \mu_2)\,,$$

so that $(\mathscr{X}, \delta)$ becomes a complete metric space. Let

$$\mathcal{E} = \{(Y, \mu) \in \mathscr{X} : \mathrm{supp}(\mu) \subseteq Y\}$$

In [1] the authors prove that $(\mathcal{E}, \delta)$ is a complete metric space.

Next, following Hutchinson ([6]), we introduce the basic facts regarding iterated function systems acting on a metric space. An iterated function system (IFS) Φ is a finite family $\Phi = \{\phi_1, \ldots, \phi_M\}$ of contractions on (X, d). This means that there exist $a_1, a_2, \ldots, a_M > 1$ such that

$$d(\phi_i(x), \phi_i(y)) \leq \frac{1}{a_i} d(x, y)$$

for every $x, y \in X$. Set $a_{\min} := \min_i a_i$.

Given an IFS $\Phi = \{\phi_1, \ldots, \phi_M\}$, let $T : \mathscr{X} \to \mathscr{X}$ the application defined by $T(Y, \mu) = (T_1 Y, T_2 \mu) = (Y', \mu')$, where

$$Y' = \bigcup_{i=1}^{M} \phi_i(Y),$$

and

$$\mu'(B) = \sum_{i=1}^{M} p_i \mu \left(\phi_i^{-1}(B \cap \phi_i(Y))\right),$$

for a given probabilistic sequence $\{p_i : i = 1, \ldots, M\}$ (i.e. $0 < p_i < 1$ and $\sum_{i=1}^{M} p_i = 1$) and every Borel subset B of Y'. This transformation T is called the mapping induced by Φ associated with the probabilities $\{p_i\}$. As usual, it is easy to see that T is a contractive on $(\mathscr{X}, \delta)$ with contraction ratio $1/a_{\min}$, and that $\mathcal{E}$ is invariant under T.

With our notation we have that T_1 is a contraction on $\mathcal{K}$ and from the completeness of $(\mathcal{E}, \delta)$ and the Banach fixed point theorem we obtain, as in Theorem 2.6 in [5], a fixed point Y_∞ for T_1 which is the only compact set in X satisfying

$$Y_\infty = \bigcup_{i=1}^{M} \phi_i(Y_\infty).$$

On the other hand T_2 is a contraction on $\mathcal{P}$ and we obtain, as in Theorem 2.8 in [5], a fixed point μ_∞ for T_2 which is the only probability Borel measure supported in Y_∞ such that

$$\mu_\infty(B) = \sum_{i=1}^{M} p_i \mu_\infty \left(\phi_i^{-1}(B \cap \phi_i(Y))\right),$$

for every Borel set B.

Given an IFS $\Phi = \{\phi_1, \ldots, \phi_M\}$ and $\boldsymbol{i} = (i_1, i_2, \ldots, i_k) \in \{1, \ldots, M\}^k$, we denote with $\boldsymbol{\phi_i}$ the composition $\phi_{i_k} \circ \phi_{i_{k-1}} \circ \cdots \circ \phi_{i_2} \circ \phi_{i_1}$. Then for any subset E of X we have

$$\boldsymbol{\phi_i}(E) = \left(\phi_{i_k} \circ \phi_{i_{k-1}} \circ \cdots \circ \phi_{i_2} \circ \phi_{i_1}\right)(E),$$

and

$$\boldsymbol{\phi_i}^{-1}(E) = \left(\phi_{i_1}^{-1}\left(\phi_{i_2}^{-1}\left(\ldots\left(\phi_{i_{k-1}}^{-1}\left(\phi_{i_k}^{-1}(E)\right)\right)\right)\right)\right).$$

We say that an IFS $\{\phi_1, \dots, \phi_M\}$ satisfies the *open set condition* (OSC) if there exists a non-empty bounded open set $U \subset X$ such that

$$\bigcup_{i=1}^{M} \phi_i(U) \subseteq U,$$

and $\phi_i(U) \cap \phi_j(U) = \emptyset$ if $i \neq j$. We shall say that U is a *set for the OSC* for Φ.

Now we shall state, and for the shake of completeness, prove some basic results about IFS which we shall need later.

Lemma 1.1. *Let $\Phi = \{\phi_1, \dots, \phi_M\}$ be an IFS with the OSC and such that each ϕ_i is one to one. Then, with U an open set for the OSC, we have*

(a) *if $\boldsymbol{i} = (i_0, \boldsymbol{i'})$, $\boldsymbol{\phi_i}(U) \subseteq \boldsymbol{\phi_{i'}}(U)$;*
(b) *if $\boldsymbol{i}$, $\boldsymbol{j} \in \{1, 2, \dots, M\}^k$ and $\boldsymbol{i} \neq \boldsymbol{j}$, $\boldsymbol{\phi_i}(U) \cap \boldsymbol{\phi_j}(U) = \emptyset$;*
(c) *if $\boldsymbol{i} \neq (i_0, \boldsymbol{i'})$, $\boldsymbol{\phi_i}(U) \cap \boldsymbol{\phi_{i'}}(U) = \emptyset$;*
(d) *for any fixed $x_0 \in U$ and each positive integer n, we have that*

$$\operatorname{card}(\boldsymbol{\phi_\ell}(U) \cap X_n) = M^{n-k}$$

for every $k \leq n$ and every $\boldsymbol{\ell} \in \{1, 2, \dots, M\}^k$, where

$$X_n = \left\{ \boldsymbol{\phi_j}(x_0) : \boldsymbol{j} \in \{1, 2, \dots, M\}^n \right\}.$$

Proof. To prove (a), let $\boldsymbol{i'} = (i_1, i_2, \dots, i_k)$ and $\boldsymbol{i} = (i_0, \boldsymbol{i'})$. Since $\phi_{i_0}(U) \subseteq U$, we have that

$$\boldsymbol{\phi_i}(U) = (\phi_{i_k} \circ \cdots \circ \phi_{i_1})(\phi_{i_0}(U)) \subseteq (\phi_{i_k} \circ \cdots \circ \phi_{i_1})(U) = \boldsymbol{\phi_{i'}}(U).$$

In order to prove (b), fix $\boldsymbol{i} = (i_1, i_2, \dots, i_k)$ and $\boldsymbol{j} = (j_1, j_2, \dots, j_k)$ such that $\boldsymbol{i} \neq \boldsymbol{j}$. Let ℓ be the largest index satisfying $j_\ell \neq i_\ell$. So that $j_m = i_m$ for every $m > \ell$, and then

$$\boldsymbol{\phi_i}(U) = (\varphi \circ \phi_{i_\ell} \circ \cdots \circ \phi_{i_1})(U),$$
$$\boldsymbol{\phi_j}(U) = (\varphi \circ \phi_{j_\ell} \circ \cdots \circ \phi_{j_1})(U),$$

where $\varphi = \phi_{i_k} \circ \cdots \circ \phi_{i_{\ell+1}} = \phi_{j_k} \circ \cdots \circ \phi_{j_{\ell+1}}$. From the OSC we have

$$(\phi_{i_{\ell-1}} \circ \cdots \circ \phi_{i_1})(U) \subseteq U \quad \text{and} \quad (\phi_{j_{\ell-1}} \circ \cdots \circ \phi_{j_1})(U) \subseteq U.$$

Hence

$$\boldsymbol{\phi_i}(U) \subseteq \varphi(\phi_{i_\ell}(U)),$$
$$\boldsymbol{\phi_j}(U) \subseteq \varphi(\phi_{j_\ell}(U)).$$

Since $\phi_{i_\ell}(U) \cap \phi_{j_\ell}(U) = \emptyset$ and φ is one to one, we have $\varphi(\phi_{i_\ell}(U)) \cap \varphi(\phi_{j_\ell}(U)) = \emptyset$, which implies (b).

To see (c) let $\boldsymbol{i} = (i_0, i_1, \dots, i_k)$ and $\boldsymbol{i'} = (j_1, j_2, \dots, j_k)$ such that $\boldsymbol{i'} \neq (i_1, i_2, \dots, i_k)$. Since

$$\boldsymbol{\phi_i}(U) = (\phi_{i_k} \circ \cdots \circ \phi_{i_1} \circ \phi_{i_0})(U) \subseteq (\phi_{i_k} \circ \cdots \circ \phi_{i_1})(U),$$

from (b) we have that $\boldsymbol{\phi_i}(U) \cap \boldsymbol{\phi_{i'}}(U) = \emptyset$.

Finally let us fix two positive integers n and k with $k \leq n$, and let $\boldsymbol{\ell} = (\ell_1, \ell_2, \dots, \ell_k) \in \{1, 2, \dots, M\}^k$. If $x \in \boldsymbol{\phi_\ell}(U) \cap X_n$, (c) implies that $x = \boldsymbol{\phi_i}(x_0)$ for some $\boldsymbol{i} = (i_1, i_2, \dots, i_{n-k}, \boldsymbol{\ell})$. Then $\operatorname{card}(\boldsymbol{\phi_\ell}(U) \cap X_n) \leq M^{n-k}$. On the other hand, from (a) we have that if $\boldsymbol{j}$ is any n-tuple of the type $(j_1, j_2, \dots, j_{n-k}, \boldsymbol{\ell})$, then $\boldsymbol{\phi_j}(x_0) \in \boldsymbol{\phi_j}(U) \cap X_n$. Also we have that $\boldsymbol{\phi_i}(x_0) \neq \boldsymbol{\phi_j}(x_0)$ for every $\boldsymbol{i} = (\boldsymbol{i'}, \ell)$, $\boldsymbol{j} = (\boldsymbol{j'}, \ell)$, with $\boldsymbol{i'}, \boldsymbol{j'} \in \{1, 2, \dots, M\}^{n-k}$, $\boldsymbol{i'} \neq \boldsymbol{j'}$. Then $\operatorname{card}(\boldsymbol{\phi_\ell}(U) \cap X_n) \geq M^{n-k}$. $\square$

In the last part of this section we introduce the notions of doubling and normality of measures on a metric space.

Given $(Y,\mu) \in \mathcal{E}$, we say that (Y,μ) is a *space of homogeneous type* (s.h.t.), or that μ is a *doubling measure* on Y if there exists a constant $A \geq 1$ such that the inequalities

$$0 < \mu(B_d(y,2r)) \leq A\mu(B_d(y,r))$$

hold for every $y \in Y$ and $r > 0$. We shall write $(Y,\mu) \in \mathcal{D}(A)$ to keep record of the quantitative parameter of the doubling property.

Given a metric space (Y,d), a measure μ on Y and a constant $\beta > 0$, the space (Y,μ) is said to be ***β**-normal* provided that there exist positive and finite constants C_1 and C_2, and constants $0 < K_1 \leq 1 \leq K_2 < \infty$ such that if $K_1\mu(\{y\}) < r^\beta < K_2\mu(Y)$, then $C_1 r^\beta \leq \mu(B_d(y,r)) \leq C_2 r^\beta$. We shall write $\mathcal{N}(\beta)$ to denote the set of all couples $(Y,\mu) \in \mathcal{E}$ which are β-normal spaces.

The relationship between normal spaces and spaces of homogeneous type is considered for the first time in [7]. There Macías and Segovia give an explicit construction of a quasi-metric on a space of homogeneous type in such a way that the new structure becomes a 1-normal space with the same topology as the original. It is also a known fact that each β-normal space is a space of homogeneous type and that the doubling constant A depends only on K_1, K_2, C_1, C_2 and β.

Let us observe that a measure can be doubling but not normal. The examples can even be obtained in the interval $[0,1]$ for measures that are absolutely continuous with respect to Lebesgue measure. In fact Lebesgue measure is 1-normal on the interval $[0,1]$ and, $d\mu(x) = w(x)dx$ with $w(x) = x^{-1/2}$ is a doubling measure, but μ is not β-normal for any $\beta > 0$. This is a consequence of the fact that, for small $\varepsilon > 0$, $\int_0^\varepsilon w\,dx \simeq \sqrt{\varepsilon}$ while $\int_{1-\varepsilon}^1 w\,dx \simeq \varepsilon$.

2. Uniform normality on orbits starting at a mass points

Let $\Phi = \{\phi_1, \ldots, \phi_M\}$ be a given IFS on (X,d), and let T be the induced mapping. With $\mathcal{O}_T(Y_0,\mu_0)$ we shall denote the *orbit* $\{T^n(Y_0,\mu_0) : n \in \mathbb{N}_0\}$ generated by successive application of T to the initial space (Y_0,μ_0).

A metric space (X,d) has *finite metric (or Assouad) dimension* if there exists a constant $N \in \mathbb{N}$, called a constant for the Assouad dimension of X, such that no ball of radius $2r$ contains more than N points of any r-disperse subset of X. By r-disperse we mean that the distance between two different points of the set is larger than or equal to $r > 0$. Then every r-disperse subset of X has at most N^m points in each ball of radius $2^m r$, with m a positive integer (see [4] and [3]).

The next result shows that the orbit under T of a mass point is uniformly normal for contractive similitudes with the same contraction ratio on a metric space with finite metric dimension.

Theorem 2.1. *Let (X,d) be a metric space with finite metric dimension. Let $\Phi = \{\phi_i : i = 1, \ldots, M\}$ be a family of contractive similitudes on X with the OSC, such that*

$$d(\phi_i(x), \phi_i(y)) = \frac{1}{a} d(x,y)$$

for every $x, y \in X$ and every $1 \leq i \leq M$, where $a > 1$. Let T be the contractive mapping induced by Φ on $\mathscr{X}$, and let $\beta = \log_a M$. If $b > 0$ and U is a set for the OSC for Φ, then $\{(X_n, \nu_n) := T^n(\{x_0\}, \delta_{x_0}); n \in \mathbb{N}\}$ is a uniformly β-normal sequence, for every $x_0 \in U - [\partial U]_b$. This means that there exist positive and finite constants C_1, C_2, K_1 and K_2, which do not depend on n and x_0, such that

$$C_1 r^\beta \leq \nu_n(B_d(x,r)) \leq C_2 r^\beta, \tag{2.1}$$

for every $x \in X_n$, every r with $K_1\nu_n(\{x\}) < r^\beta < K_2$ and every natural number n.

Proof. Fix $b > 0$ such that $U - [\partial U]_b$ is non-empty, and take $x_0 \in U - [\partial U]_b$. Then $b < R := \operatorname{diam}(U)$. For each $n \in \mathbb{N}$, since $(X_n, \nu_n) = T^n(\{x_0\}, \delta_{x_0})$ we have that

$$X_n = \left\{\boldsymbol{\phi}_{\boldsymbol{j}}(x_0) : \boldsymbol{j} \in \{1, 2, \ldots, M\}^n\right\}.$$

Hence X_n has M^n elements and $\nu_n(\{x\}) = M^{-n}$ for every $x \in X_n$. We claim that X_n is ba^{-n}-disperse. In fact, assume that $x = x_{n,\boldsymbol{j}}$ and $y = x_{n,\boldsymbol{i}}$ with $\boldsymbol{j} \neq \boldsymbol{i}$. Since U is an open set, we have that $B_d(x_0, b) \subseteq U$. Then

$$B_d(x_{n,\boldsymbol{j}}, ba^{-n}) = \boldsymbol{\phi}_{\boldsymbol{j}}\left(B_d(x_0, b)\right) \subseteq \boldsymbol{\phi}_{\boldsymbol{j}}(U),$$

$$B_d(x_{n,\boldsymbol{i}}, ba^{-n}) = \boldsymbol{\phi}_{\boldsymbol{i}}\left(B_d(x_0, b)\right) \subseteq \boldsymbol{\phi}_{\boldsymbol{i}}(U),$$

and since $\boldsymbol{\phi}_{\boldsymbol{j}}(U)$ and $\boldsymbol{\phi}_{\boldsymbol{i}}(U)$ are disjoint, we have $B_d(x, ba^{-n}) \cap B_d(y, ba^{-n}) = \emptyset$. This implies that $d(x,y) \geq ba^{-n}$.

Notice also that if $\boldsymbol{\ell} \in \{1, 2, \ldots, M\}^k$ and $k \leq n$, from Lemma 1.1 (d) we have

$$\nu_n(\boldsymbol{\phi}_{\boldsymbol{\ell}}(U)) = M^{-n}\operatorname{card}(\boldsymbol{\phi}_{\boldsymbol{\ell}}(U) \cap X_n) = M^{-k} = a^{-k\beta},$$

where $\beta = \log_a M$.

Let us define $K_1 = b^\beta$ and $K_2 = Ra^\beta$. Fix $n \in \mathbb{N}$, $x \in X_n$ and $r > 0$ such that $K_1\nu_n(\{x\}) < r^\beta < K_2$. Then $r > ba^{-n}$. In order to find constants C_1 and C_2 for which (2.1) holds, we shall consider two cases:

Case 1: $\boldsymbol{ba^{-n} < r \leq Ra^{-n}}$. Notice first that

$$\nu_n(B_d(x,r)) \geq \nu_n(B_d(x, ba^{-n})) \geq M^{-n} = a^{-n\beta} \geq R^{-\beta}r^\beta,$$

and on the other hand,

$$\begin{aligned} \nu_n(B_d(x,r)) &\leq \nu_n(B_d(x, Ra^{-n+1})) \\ &= M^{-n}\operatorname{card}\left(X_n \cap B_d(x, Ra^{-n+1})\right) \\ &\leq N^\ell a^{-n\beta} \\ &\leq N^\ell b^{-\beta} r^\beta, \end{aligned}$$

where ℓ is a positive integer such that $2^\ell \geq Ra/b$, and N is a constant for the Assouad dimension of X.

Caso 2: $\boldsymbol{r > Ra^{-n}}$. Let us fix $j \leq n$ such that $Ra^{-j} < r \leq Ra^{-j+1}$, and define

$$\mathcal{J} = \{\boldsymbol{j} \in \{1, 2, \ldots, M\}^j : B_d(x,r) \cap X_n \cap \boldsymbol{\phi}_{\boldsymbol{j}}(U) \neq \emptyset\}.$$

Since $\{\boldsymbol{\phi}_{\boldsymbol{j}}(U), \boldsymbol{j} \in \{1, 2, \ldots, M\}^j\}$ is a covering of X_n we have that

$$B_d(x,r) \cap X_n \subseteq \bigcup_{\boldsymbol{j} \in \mathcal{J}} \boldsymbol{\phi}_{\boldsymbol{j}}(U).$$

On the other hand, if $x = \boldsymbol{\phi_i}(x_0)$, $\boldsymbol{i} = (i_1, i_2, \ldots, i_n)$ and $\boldsymbol{j} = (i_{n-j+1}, i_{n-j+2}, \ldots, i_n)$, we claim that

$$\boldsymbol{\phi_j}(U) \cap X_n \subseteq B_d(x, r) \cap X_n.$$

In fact, if $y \in \boldsymbol{\phi_j}(U) \cap X_n$ then $y = \boldsymbol{\phi_{i'}}(x_0)$, where

$$\boldsymbol{i'} = (\ell_1, \ell_2, \ldots, \ell_{n-j}, i_{n-j+1}, i_{n-j+2}, \ldots, i_n),$$

for some $\ell_1, \ell_2, \ldots, \ell_{n-j} \in \{1, 2, \ldots, M\}$. Then

$$d(x, y) \leq a^{-j} R < r.$$

Hence

$$\begin{aligned} \nu_n(B_d(x, r)) &\leq \sum_{\boldsymbol{j} \in \mathcal{J}} \nu_n(\boldsymbol{\phi_j}(U)) \\ &= \operatorname{card}(\mathcal{J}) a^{-j\beta} \\ &\leq \operatorname{card}(\mathcal{J}) R^{-\beta} r^{\beta}, \end{aligned}$$

and for every $\boldsymbol{j} \in \mathcal{J}$ we have that

$$\begin{aligned} \nu_n(B_d(x, r)) &\geq \nu_n(\boldsymbol{\phi_j}(U)) \\ &= a^{-j\beta} \\ &\geq (aR)^{-\beta} r^{\beta}. \end{aligned}$$

We only have to show that $\operatorname{card}(\mathcal{J})$ is bounded by a constant which does not depend on x, r and j. In order to prove it, let us identify each $\boldsymbol{j} \in \mathcal{J}$ with the point $\boldsymbol{\phi_j}(x_0) \in \boldsymbol{\phi_j}(U)$, and leu us define the set $A = \{\boldsymbol{\phi_j}(x_0) : \boldsymbol{j} \in \mathcal{J}\}$. Since $\boldsymbol{\phi_j}(U)$ are pairwise disjoint for $\boldsymbol{j}$ ranging on the set of indices with fixed length, we have that

$$\operatorname{card}(\mathcal{J}) = \operatorname{card}\{\boldsymbol{\phi_j}(x_0) : \boldsymbol{j} \in \mathcal{J}\} = \operatorname{card}(A).$$

Notice that $A \subseteq B_d(x, 2r)$. In fact, if $\boldsymbol{j} \in \mathcal{J}$ then there exists $y \in B_d(x, r) \cap X_n \cap \boldsymbol{\phi_j}(U)$, and

$$d(\boldsymbol{\phi_j}(x_0), x) \leq d(\boldsymbol{\phi_j}(x_0), y) + d(y, x) < a^{-j} R + r \leq 2r.$$

Since, being a subset of X_j, the set A is ba^{-j}-disperse, we have that

$$\begin{aligned} \operatorname{card}(A) &\leq \operatorname{card}(B_d(x, 2r) \cap X_j) \\ &\leq \operatorname{card}(B_d(x, 2Ra^{-j+1}) \cap X_j) \\ &\leq N^{\ell+1}, \end{aligned}$$

with ℓ and N as before.

Hence (2.1) follows with $C_1 = R^{-\beta}$ and $C_2 = N^{\ell+1} b^{-\beta}$. □

As a corollary we have the following result.

Corollary 2.2. *Let (X, d), Φ, T and U be as in Theorem 2.1. Then for each $b > 0$ there exists a constant $A = A(b) \geq 1$ such that for every $x_0 \in U - [\partial U]_b$, the orbit $\{T^n(\{x_0\}, \delta_{x_0}) : n \in \mathbb{N}\}$ is contained in $\mathcal{D}(A)$.*

3. Gradual improvement of the doubling property along the orbits

The results in [2] suggest that there is a deep interplay between the separation properties of the IFS and the behavior of the orbits generated by the iteration from different starting points of the mapping T induced by this IFS. In particular, show that the doubling property may "suddenly appear" for the limit even when no term of the sequence has this property.

In this section we shall prove that, in many cases, the terms of the approximating sequence become more and more doubling in a precise sense, and that the doubling property of the limit, in this sense, is not so "sudden". Given $\varepsilon \geq 0$ and a constant $A \geq 1$, we say that (Y,μ) belongs to $\mathcal{D}^{\varepsilon}(A)$, or that μ is ***ε-doubling*** with constant A, if $(Y,\mu) \in \mathcal{E}$ and the inequalities

$$0 < \mu(B_d(y,2r)) \leq A\mu(B_d(y,r))$$

hold for every $y \in Y$ and every $r > \varepsilon$. When $\varepsilon = 0$ we have that $\mathcal{D}^0(A) = \mathcal{D}(A)$, and in this case (Y,μ) is a space of homogeneous type.

The next result shows that if a metric measure space is the limit in the metric δ of a sequence of uniformly ε_n-doubling spaces with $\varepsilon_n \to 0$, then it is a space of homogeneous type.

Proposition 3.1. *Let (Y_n,μ_n) be a sequence in $\mathcal{E}$ such that $(Y_n,\mu_n) \in \mathcal{D}^{\varepsilon_n}(A)$, with $\varepsilon_n \to 0$ when $n \to \infty$. If $(Y_n,\mu_n) \xrightarrow{\delta} (Y,\mu)$ then $(Y,\mu) \in \mathcal{D}(A^4)$.*

Proof. Take $y \in Y$ and $r > 0$. Let φ be the continuous function defined on $\mathbb{R}_0^+$ as $\varphi \equiv 1$ on $[0,1]$, $\varphi \equiv 0$ on $[2,\infty)$ which is linear in the interval $[1,2]$. Since $Y_n \xrightarrow{d_H} Y$, we can choose $y_n \in Y_n$ such that $d(y_n,y) \to 0$ when $n \to \infty$. Then, since there exists n_0 such that $y_n \in B_d(y,r/16)$ and $\varepsilon_n < 5r/16$ for every $n \geq n_0$, we have that

$$\begin{aligned}
\mu(B_d(y,2r)) &\leq \int \varphi\left(\frac{d(x,y)}{2r}\right) d\mu(x) \\
&= \lim_{n\to\infty} \int \varphi\left(\frac{d(x,y)}{2r}\right) d\mu_n(x) \\
&\leq \liminf_{n\to\infty} \mu_n\left(B_d\left(y,4r\right)\right) \\
&\leq \liminf_{n\to\infty} \mu_n\left(B_d\left(y_n,5r\right)\right) \\
&\leq \liminf_{n\to\infty} A^4\mu_n\left(B_d\left(y_n,\frac{5r}{16}\right)\right) \\
&\leq A^4 \liminf_{n\to\infty} \mu_n\left(B_d\left(y,\frac{r}{2}\right)\right) \\
&\leq A^4 \lim_{n\to\infty} \int \varphi\left(\frac{2d(x,y)}{r}\right) d\mu_n(x) \\
&= A^4 \int \varphi\left(\frac{2d(x,y)}{r}\right) d\mu(x) \\
&\leq A^4\mu(B_d(y,r)).
\end{aligned}$$

□

The main result of this section, contained in Theorem 3.3, is that under the conditions of Theorem 2.1, for any $(Y_0,\mu_0) \in \mathcal{E}$ with $Y_0 \subset U$, the sequence $\{T^n(Y_0,\mu_0)\}$

becomes, in a uniform way, more and more doubling as n grows. This means that there exists $A \geq 1$ such that for every $\varepsilon > 0$ there exists $N = N(\varepsilon, Y_0, \mu_0)$ such that for $n \geq N$ we have that $T^n(Y_0, \mu_0) \in \mathcal{D}^\varepsilon(A)$ for every $n \geq N$. This fact is a consequence of Corollary 2.2 and of the next result, which we state in the following general context.

Given a compact metric space (X, d), let (Y_n, μ_n) be any sequence in $\mathcal{E}$ with the following structure: each Y_n can be written as a disjoint union

$$Y_n = \bigcup_{m=1}^{M_n} Y_n^m$$

of M_n Borel pieces Y_n^m, such that $\mu_n(Y_n^m) = M_n^{-1}$ and

$$d_n := \sup_{m=1,\ldots,M_n} \operatorname{diam}(Y_n^m)$$

tends to zero when $n \to \infty$.

With the above notation, we shall say that (Y_n, μ_n) satisfies the *uniform gradual doubling* property (UGD) if there exists $A \geq 1$ such that for every $n \in \mathbb{N}$ we have that $(Y_n, \mu_n) \in \mathcal{D}^{5d_n}(A)$.

As we have already mentioned, Theorem 3.3 will be a consequence of the following result, which proves that the UGD property can be deduced from the *discrete uniform doubling control* (DUDC) of (Y_n, μ_n): there exists $A \geq 1$ and for each n there exists a finite set $X_n \subseteq Y_n$ such that $\operatorname{card}(X_n \cap Y_n^m) = 1$ for every $m = 1, \ldots, M_n$, and $\{(X_n, \nu_n) : n \in \mathbb{N}\} \subseteq \mathcal{D}(A)$, where ν_n is the counting measure on X_n normalized to a probability.

Theorem 3.2. *DUDC implies UGD.*

Proof. Fix $n \in \mathbb{N}$, $y \in Y_n$ and $r > 5d_n$. There exists one and only one Y_n^m such that $y \in Y_n^m$. Let us write x_n^m to denote the unique point in $X_n \cap Y_n^m$. Then $d(y, x_n^m) \leq d_n$. For $s > 2d_n$ denote

$$B^i = B_d(x_n^m, s + (i-2)d_n),$$

$i = 0, 1, 2, 3, 4$. Notice that

$$B^1 \subseteq B_d(y, s) \subseteq B^3,$$

and then

$$\mu_n(B^1) \leq \mu_n(B_d(y, s)) \leq \mu_n(B^3).$$

We claim that the comparison of the measure μ_n with the counting measure ν_n on X_n is the following

$$\mu_n(B^1) \geq \nu_n(B^0) \quad \text{and} \quad \mu_n(B^3) \leq \nu_n(B^4). \tag{3.1}$$

If the claim holds, then

$$\nu_n(B^0) \leq \mu_n(B_d(y, s)) \leq \nu_n(B^4)$$

for every $y \in Y_n^m$ and $s > 2d_n$. Let $A \geq 1$ be the constant for the DUDC, i.e. $\{(X_n, \nu_n) : n \in \mathbb{N}\} \subseteq \mathcal{D}(A)$. Then

$$\begin{aligned} \mu_n(B_d(y, 2r)) &\leq \nu_n(B_d(x_n^m, 2r + 2d_n)) \\ &\leq A^2 \nu_n(B_d(x_n^m, (r + d_n))/2) \end{aligned}$$

$$\begin{aligned} &\leq A^2\mu_n(B_d(y,(r+5d_n)/2)) \\ &\leq A^2\mu_n(B_d(y,r)), \end{aligned}$$

and the UGD holds with $A_2 = A^2$. Then it only remains to prove the inequalities contained in (3.1). To show the first one we define the set

$$\mathcal{J} = \{j : Y_n^j \subseteq B^1\}.$$

Notice that if $x_n^j \in B^0 \cap X_n$, then $j \in \mathcal{J}$. In fact, suppose that $d(x_n^j, x_n^m) < s - 2d_n$. To see that $Y_n^j \subseteq B^1$ fix $z \in Y_n^j$. Since $\operatorname{diam}(Y_n^j) \leq d_n$ we have that $d(z, x_n^j) \leq d_n$. Then

$$\begin{aligned} d(z, x_n^m) &\leq d(z, x_n^j) + d(x_n^j, x_n^m) \\ &< d_n + s - 2d_n \\ &= s - d_n, \end{aligned}$$

and hence $Y_n^j \subseteq B^1$. So that

$$\begin{aligned} \mu_n(B^1) &\geq \sum_{j\in\mathcal{J}} \mu_n(Y_n^j) \\ &= \sum_{j\in\mathcal{J}} M_n^{-1} \\ &= \sum_{j\in\mathcal{J}} \nu_n(\{x_n^j\}) \\ &\geq \nu_n(B^0). \end{aligned}$$

To prove the second inequality let us now define the set

$$\mathcal{Q} = \{q : Y_n^q \cap B^3 \neq \emptyset\}.$$

Observe that if $q \in \mathcal{Q}$ then $Y_n^q \subseteq B^4$. In fact, if $q \in \mathcal{Q}$ there exists $z_n^q \in Y_n^q \cap B_d(x_n^m, s + d_n)$. Then for every $z \in Y_n^q$ we have

$$\begin{aligned} d(z, x_n^m) &\leq d(z, z_n^q) + d(z_n^q, x_n^m) \\ &< d_n + s + d_n \\ &= s + 2d_n, \end{aligned}$$

and then $z \in B^4$. Hence

$$\begin{aligned} \mu_n(B^3) &\leq \sum_{q\in\mathcal{Q}} \mu_n(Y_n^q) \\ &= \sum_{q\in\mathcal{Q}} \nu_n(\{x_n^q\}) \\ &\leq \nu_n(B^4), \end{aligned}$$

as desired. □

The following result states, as well as the doubling property for the limit space, that the approximating sequence $(Y_n, \mu_n) := T^n(Y_0, \mu_0)$ is uniformly increasing doubling for adequate inicial spaces $(Y_0, \mu_0) \in \mathcal{E}$.

Theorem 3.3. *Let (X,d) be a compact metric space with finite metric dimension. Let $\Phi = \{\phi_i : i = 1,\dots,M\}$ be a family of contractive similitudes on X with the OSC, such that*

$$d\left(\phi_i(x), \phi_i(y)\right) = \frac{1}{a} d(x,y)$$

for every $x, y \in X$ and $1 \le i \le M$, where $a > 1$. Let T be the contractive mapping on $\mathscr{X}$ induced by Φ. If $(Y_0, \mu_0) \in \mathcal{E}$ satisfies $Y_0 \subset U$, where U is a set for the OSC for Φ, then there exists a constant $A' \ge 1$ such that $(Y_n, \mu_n) \in \mathcal{D}^{5a^{-n}}(A')$ for every n, and then the limit space (Y_∞, μ_∞) is a space of homogeneous type.

Proof. Fix $(Y_0, \mu_0) \in \mathcal{E}$ with $Y_0 \subset U$. First notice that

$$\mu_n(\boldsymbol{\phi_i}(Y_0)) = M^{-n},$$

for every n and every $\boldsymbol{i} \in \{1, 2\dots, M\}^n$. In fact, for a fixed $\boldsymbol{i} \in \{1, 2\dots, M\}^n$ we have that

$$\begin{aligned}
\mu_n(\boldsymbol{\phi_i}(Y_0)) &= M^{-n} \sum_{\boldsymbol{j} \in \{1,2\dots,M\}^n} \mu_0\left(\boldsymbol{\phi_j}^{-1}(\boldsymbol{\phi_i}(Y_0))\right) \\
&= M^{-n}\mu_0(Y_0) + M^{-n} \sum_{\substack{\boldsymbol{j} \in \{1,2\dots,M\}^n \\ \boldsymbol{j} \neq \boldsymbol{i}}} \mu_0\left(\boldsymbol{\phi_j}^{-1}(\boldsymbol{\phi_i}(Y_0)\right).
\end{aligned}$$

Since $\mu_0(Y_0) = 1$ and $\boldsymbol{\phi_j}^{-1}(\boldsymbol{\phi_i}(Y_0)) = \emptyset$ for every choice of $\boldsymbol{j} \neq \boldsymbol{i}$ (see Lemma 1.1 (b)), we have the claim.

On the other hand, fix $x_0 \in Y_0$. Since $d(Y_0, \partial U) > 0$ we can apply Theorem 2.2 to obtain a constant $A \ge 1$ such that $T^n(\{x_0\}, \delta_{x_0}) \in \mathcal{D}(A)$ for every $n \in \mathbb{N}$. Then, applying Theorem 3.2 with

$$M_n = M^n, \quad Y_n = \bigcup_{\boldsymbol{i} \in \{1,\dots,M\}^n} \boldsymbol{\phi_i}(Y_0), \quad d_n = a^{-n}, \quad (X_n, \nu_n) = T^n(\{x_0\}, \delta_{x_0}),$$

we have that $(Y_n, \mu_n) \in \mathcal{D}^{5a^{-n}}(A^2)$. Finally, from Proposition 3.1 we can conclude that $(Y_\infty, \mu_\infty) \in \mathcal{D}(A^8)$. □

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