WEIGHTED INEQUALITIES FOR NEGATIVE POWERS OF SCHRÖDINGER OPERATORS

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Abstract. In this article we obtain boundedness of the operator $(-\Delta + V)^{-\alpha/2}$ from $L^{p,\infty}(w)$ into weighted bounded mean oscillation type spaces $BMO^{\beta}_{\mathcal{L}}(w)$ under appropriate conditions on the weight w. We also show that these weighted spaces also have a point-wise description for $0 < \beta < 1$. Finally, we study the behaviour of the operator$(-\Delta + V)^{-\alpha/2}$ when acting on $BMO^{\beta}_{\mathcal{L}}(w)$.

1. Introduction

Let us consider the Schrödinger operator on $\mathbb{R}^d$ with $d \geq 3$,

$$\mathcal{L} = -\Delta + V$$

where $V \geq 0$ is a function satisfying, for some $q > \frac{d}{2}$, the reverse Hölder inequality

$$\left(\frac{1}{|B|} \int_B V(y)^q \, dy \right)^{1/q} \leq \frac{C}{|B|} \int_B V(y) \, dy$$

for every ball $B \subset \mathbb{R}^d$. The set of functions with the last property is usually denoted by RH_q.

It is well known that negative powers of the Schrödinger operator can be expressed in terms of the heat diffusion semigroup generated by $\mathcal{L}$ as

$$\mathcal{I}_\alpha f(x) = \mathcal{L}^{-\alpha/2} f(x) = \int_0^\infty e^{-t\mathcal{L}} f(x) \, t^{\alpha/2} \, \frac{dt}{t}, \quad \alpha > 0.$$

For each $t > 0$ the operator $e^{-t\mathcal{L}}$ is an integral operator with kernel $k_t(x, y)$ having a better behaviour far away form the diagonal than the classical heat kernel. Some useful properties of k_t where obtained in [6], [3] and [4]. As a consequence $\mathcal{I}_\alpha f$ turns out to be finite a.e. even if f belongs to L^p with p greater than the critical index d/α. Particularly, in [1] the authors proved that $\mathcal{I}_\alpha$ maps $L^{d/\alpha}$ into an appropriate substitute of L^∞ denoted by $BMO_{\mathcal{L}}$ which in fact is smaller than the classical BMO space of John-Nirenberg.

In this work we extend and improve their result by analysing the behaviour of $\mathcal{I}_\alpha$ on weighted weak L^p spaces with $p \geq d/\alpha$ for a suitable class of weights. In order to do that we introduce a family of spaces $BMO^{\beta}_{\mathcal{L}}(w)$ that includes, as a particular case, the space $BMO_{\mathcal{L}}$. We point out that in the case of $w \equiv 1$ and $p = d/\alpha$, we obtain a better result than that in [1] since L^p is strictly contained in weak L^p.

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It is worth mentioning that for $w \equiv 1$ the spaces $BMO^{\beta}_{\mathcal{L}}$ are the duals of the H^p-spaces introduced in [2] and [4], as it can be easily checked from the atomic decomposition given there. For $\beta = 0$ such representation was already pointed out in [1].

We also study the behaviour of $\mathcal{I}_\alpha$ on $BMO^{\beta}_{\mathcal{L}}(w)$ proving that, under appropriate conditions on the weight, they are transformed into $BMO^{\beta+\alpha}_{\mathcal{L}}(w)$. In proving such result we give a point-wise characterization of our spaces $BMO^{\beta}_{\mathcal{L}}(w)$ when $0 < \beta < 1$, which we believe to be of independent interest.

Finally, we remark that when the potential V belongs to $RH_{d/2}$, as it is the case of the Hermite differential operator, the classes of weights for which we prove our boundedness results coincide with those obtained in [5] for $V = 0$.

This article is organized as follows. In Section 2 we introduce the family of spaces $BMO^{\beta}_{\mathcal{L}}(w)$ and we prove some basic properties. In particular the aforementioned point-wise description is given in Proposition 4. The remaining two sections contain the main results: Section 3 is devoted to the analysis of $\mathcal{I}_\alpha$ acting on $L^{p,\infty}(w)$ while Section 4 deals with the boundedness on $BMO^{\beta}_{\mathcal{L}}(w)$.

2. $BMO^{\beta}_{\mathcal{L}}(w)$ spaces

For a given potential $V \in RH_q$, with $q > \frac{d}{2}$, we introduce the function

$$\rho(x) = \sup \left\{ r > 0 : \; \frac{1}{r^{d-2}} \int_{B(x,r)} V \le 1 \right\}, \quad x \in \mathbb{R}^d.$$

Due to the above assumptions $\rho(x)$ is finite for all $x \in \mathbb{R}^d$. This auxiliary function plays an important role in the estimates of the operators and in the description of the spaces associated to $\mathcal{L}$ (see [1], [3], [4], [7]).

The following propositions contain some properties of ρ that will be useful in the sequel.

Proposition 1 (Lemma 1.4 in [7]). *There exist C and $k_0 \ge 1$ such that,*

$$C^{-1}\rho(x)\left(1+\frac{|x-y|}{\rho(x)}\right)^{-k_0} \;\le\; \rho(y) \;\le\; C\rho(x)\left(1+\frac{|x-y|}{\rho(x)}\right)^{\frac{k_0}{k_0+1}} \tag{1}$$

for all $x, y \in \mathbb{R}^d$.

Throughout this work, we denote $w(E) = \int_E w$ for every measurable subset $E \subset \mathbb{R}^d$, and $CB = B(x, Cr)$, for $x \in \mathbb{R}^d$, $r > 0$ and $C > 0$.

Proposition 2 ([2]). *There exists a sequence of points $\{x_k\}_{k=1}^{\infty}$ in $\mathbb{R}^d$, so that the family $B_k = B(x_k, \rho(x_k))$, $k \ge 1$, satisfies*

(1) $\cup_k B_k = \mathbb{R}^d$.

(2) There exists N such that, for every $k \in \mathbb{N}$, $card\{j : \; 4B_j \cap 4B_k \neq \emptyset\} \le N$.

We denote by L^1_{loc} the set of locally integrable functions of $\mathbb{R}^d$. For $\eta \ge 1$ and w a weight, i.e. $w \ge 0$ and $w \in L^1_{\text{loc}}$, we say that $w \in D_\eta$ if there exists a constant C such that

$$w(tB) \le C\, t^{d\eta}\, w(B),$$

for every ball $B \subset \mathbb{R}^d$ and $t \ge 1$.

It is easy to see that a weight w belongs to $D = \cup_{\eta \geq 1} D_\eta$ if and only if it satisfies the doubling condition

$$w(2B) \leq C w(B).$$

For $\beta \geq 0$ we define the space $BMO_{\mathcal{L}}^{\beta}(w)$ as the set of functions f in L^1_{loc} satisfying for every ball $B = B(x, R)$, with $x \in \mathbb{R}^d$ and $R > 0$,

$$\int_B |f - f_B| \leq C\, w(B)\, |B|^{\beta/d}, \quad \text{with } f_B = \frac{1}{|B|}\int_B f, \tag{2}$$

and

$$\int_B |f| \leq C\, w(B)\, |B|^{\beta/d}, \quad \text{if } R \geq \rho(x). \tag{3}$$

Let us note that if (3) is true for some ball B then (2) holds for the same ball, so we might ask to (2) only for balls with radii lower than $\rho(x)$.

The constants in (2) and (3) are independent of the choice of B but may depend on f. A norm in the space $BMO_{\mathcal{L}}^{\beta}(w)$ can be given by the maximum of the two infima of the constants that satisfy (2) and (3) respectively.

The case $\beta = 0$ and $w \equiv 1$ was introduced in [1] as a natural substitute of L^∞ in the context of the semigroup generated by the operator $\mathcal{L}$. As in that case we can replace condition (3) by the following weaker condition (4) that only takes into account critical balls.

Proposition 3. *Let $\beta \geq 0$ and $w \in D_\eta$. If $\{x_k\}_{k=1}^\infty$ is a sequence as in Proposition 2, then a function f belongs to $BMO_{\mathcal{L}}^{\beta}(w)$ if, and only if, f satisfies* (2) *for every ball, and*

$$\int_{B(x_k,\rho(x_k))} |f| \leq C w(B(x_k, \rho(x_k)))\, |\rho(x_k)|^\beta, \quad \textit{for all } k \geq 1. \tag{4}$$

Proof. Let f satisfy (4), and let $B = B(x, R)$ be a ball with radius $R > \rho(x)$. From Proposition 2 the set

$$F = \{k : \ B \cap B_k \neq \emptyset\}$$

is finite and

$$\sum_{k\in F} \int_{B_k} w \leq (N + 1) \int_{\cup_{k\in F} B_k} w\ , \tag{5}$$

where N is the constant controlling the overlapping (see Proposition 2).

It is easy to see that for some constant C, $B_k \subset CB$ for every $k \in F$. In fact, if $z \in B_k \cap B$, from (1),

$$\begin{aligned}
\rho(x_k) &\leq C\rho(z)\left(1 + \frac{|x_k - z|}{\rho(x_k)}\right)^{k_0} \leq C2^{k_0}\rho(z) \\
&\leq C2^{k_0}\rho(x)\left(1 + \frac{|x - z|}{\rho(x)}\right)^{\frac{k_0}{k_0+1}} \leq C2^{k_0}\rho(x)\left(1 + \frac{R}{\rho(x)}\right) \\
&\leq C2^{k_0+1}R,
\end{aligned}$$

then

$$\begin{aligned}\int_B |f| &\le \sum_{k\in F}\int_{B\cap B_k}|f| \le \sum_{k\in F}\int_{B_k}|f|\\ &\le C\sum_{k\in F} w(B_k)\,|B_k|^{\beta/d} \le C\,|B|^{\beta/d}\sum_{k\in F}\int_{B_k} w\\ &\le C\,|B|^{\beta/d}\int_{\cup_{k\in F}B_k} w \le C|B|^{\beta/d}\int_{CB} w.\end{aligned}$$

Since we assumed that w is doubling the last expression is bounded up to a constant by $w(B)\,|B|^{\beta/d}$. □

Corollary 1. *A function f belongs to $BMO^{\beta}_{\mathcal{L}}(w)$ if, and only if, condition* (2) *is satisfied for every ball $B = B(x,R)$ with $x\in\mathbb{R}^d$ and $R<\rho(x)$, and*

$$\int_{B(x,\rho(x))}|f| \le Cw(B(x,\rho(x)))\,|\rho(x)|^{\beta}, \quad \textit{for all } x\in\mathbb{R}^d. \tag{6}$$

For $\beta>0$ and $w\in L^1_{\text{loc}}$, we define

$$W_\beta(x,r) = \int_{B(x,r)}\frac{w(z)}{|z-x|^{d-\beta}}\,dz.$$

for all $x\in\mathbb{R}^d$ and $r>0$.

We introduce a kind of Lipschitz space $\Lambda^{\beta}_{\mathcal{L}}(w)$ as the set of functions f such that

$$|f(x)-f(y)| \;\le\; C\;[W_\beta(x,|x-y|) + \; W_\beta(y,|x-y|)] \tag{7}$$

and

$$|f(x)| \;\le\; C\,W_\beta(x,\rho(x)) \tag{8}$$

for almost all x and y in $\mathbb{R}^d$.

It is possible to define a norm in these spaces by taking the maximum of the two infima of the constants that satisfy equations (7) and (8) respectively.

Remark 1. For almost every $x\in\mathbb{R}^d$, $W_\beta(x,r)$ is finite for all $r>0$, it is always increasing as a function of r. Also, if w satisfies the doubling condition, then we have

$$W_\beta(x,2r)\le C\,W_\beta(x,r), \tag{9}$$

for almost every $x\in\mathbb{R}^d$ and $r>0$, where the constant C does not depend on r or x.

Proposition 4. *If $0<\beta<1$ and w satisfies the doubling condition, then*

$$\Lambda^{\beta}_{\mathcal{L}}(w) = BMO^{\beta}_{\mathcal{L}}(w),$$

and the norms are equivalent.

Proof. Let f be in $BMO^{\beta}_{\mathcal{L}}(w)$ with $\|f\|_{BMO^{\beta}_{\mathcal{L}}(w)} = 1$, x and y in $\mathbb{R}^d$. Since f satisfies (2), from [5] (Proposition 1.3) we obtain

$$|f(x)-f(y)|\le C\;\;[W_\beta(x,2|x-y|) + \; W_\beta(y,2|x-y|)]$$

for all x and y Lebesgue points of f. Hence, Remark 1 implies that f satisfies (7).

To verify (8), if $x \in \mathbb{R}^d$ is a Lebesgue point of f, and $B = B(x, \rho(x))$, from (7) and condition (3), we have

$$\begin{aligned} |f(x)| &\leq \frac{1}{|B|}\int_B |f(x)-f(y)|\,dy \;+\; \frac{1}{|B|}\int_B |f(y)|\,dy \\ &\leq \frac{C}{|B|}\left(\int_B W_\beta(x,|x-y|)dy + \int_B W_\beta(y,|x-y|)dy + \; w(B)\,|B|^{\frac{\beta}{d}}\right). \end{aligned} \tag{10}$$

In the last sum, by Remark 1, the first term is

$$\int_B W_\beta(x,|x-y|)dy \leq |B|\,W_\beta(x,\rho(x)).$$

For the second term of (10), if $y \in B$, we have $B(y,|x-y|) \subset B(x, 2\rho(x))$, then

$$\begin{aligned} \int_B W_\beta(y,|x-y|)\,dy \;&\leq\; \int_{B(x,2\rho(x))} w(z)\left(\int_B \frac{1}{|z-y|^{d-\beta}}\,dy\right)dz \\ &\leq\; C\,|B|^{\beta/d} w(B) \;\leq\; C\,|B| W_\beta(x,\rho(x)). \end{aligned}$$

Finally, the last term of (10) is bounded by

$$|B|^{\frac{\beta}{d}-1} w(B) \;\leq\; W_\beta(x,\rho(x)),$$

and we have shown that (8) is satisfied.

In order to prove the other inclusion, consider $\|f\|_{\Lambda^\beta_{\mathcal{L}}(w)} = 1$. From [5] (Proposition 1.3) we have (7) implie (2). To see condition (3), let $x \in \mathbb{R}^d$ and $R \geq \rho(x)$. If $y \in B(x,R)$, from Proposition 1,

$$\rho(y) \;\leq\; C\rho(x)\left(1+\frac{|x-y|}{\rho(x)}\right)^{\frac{k_0}{k_0+1}} \;\leq\; C\rho(x)\left(\frac{R}{\rho(x)}\right)^{\frac{k_0}{k_0+1}} \;\leq\; CR$$

and thus by (8)

$$\begin{aligned} \int_{B(x,R)} |f(y)|dy &\leq \int_{B(x,R)}\int_{B(y,\rho(y))} \frac{w(z)}{|z-y|^{d-\beta}}dz\,dy \\ &\leq \int_{B(x,CR)} w(z)\int_{B(x,R)} \frac{1}{|z-y|^{d-\beta}}dy\,dz \\ &\leq C\,R^\beta w(B(x,CR)) \\ &\leq C\,|B(x,R)|^{\beta/d} w(B(x,R)), \end{aligned}$$

where in the last inequality we have used the fact that w is doubling. □

Remark 2. Observe that in the last proof (see inequality (10)) we have shown that (6) and (7) with $|x-y| < \rho(x)$ implies (8), and thus conditions (6) and (7) imply that f belongs to $\Lambda^\beta_{\mathcal{L}}(w)$.

3. $\mathcal{I}_\alpha$ ON $L^{p,\infty}(w)$ SPACES

We begin by stating a series of lemmas that will be useful in proving the main results. We omit the proofs though we provide references where they can be found.

For $L^{p,\infty}(w)$, $p > 1$, we mean the space of measurable functions f such that

$$[f]_{p,w} \;=\; \left(\sup_{t>0} t^p \left|\{x:\; \frac{|f(x)|}{w(x)} > t\}\right|\right)^{1/p} \tag{11}$$

is finite. The quantity (11) is not a norm (triangular inequality fails) but it turns to be equivalent to a norm. Clearly, the Lebesgue spaces $L^p(w) = \{f : \int_{\mathbb{R}^d} |f/w|^p < \infty\}$ are continuously embedded in $L^{p,\infty}(w)$.

As usual p' denotes the Hölder conjugate exponent of p.

Lemma 1 ([5]). *Let $p > 1$ and w a weight in $RH_{p'}$. If f is a locally integrable function and B is a ball in $\mathbb{R}^d$ then, there exists a constant C such that*

$$\int_B |f| \leq C\, w(B)|B|^{-\frac{1}{p}}\, [f]_{p,w}.$$

For $t > 0$ let k_t be the kernel of $e^{-t\mathcal{L}}$. Then, the kernel of $\mathcal{I}_\alpha$ is given by the formula

$$K_\alpha(x,y) = \int_0^\infty k_t(x,y)\, t^{\alpha/2}\, \frac{dt}{t}. \tag{12}$$

Some estimates of k_t are presented below.

Lemma 2 ([6]). *Given $N > 0$, there exists a constant C such that for all x and y in $\mathbb{R}^d$,*

$$k_t(x,y) \leq C\, t^{-d/2}\, e^{-\frac{|x-y|^2}{C\,t}} \left(1 + \frac{\sqrt{t}}{\rho(x)} + \frac{\sqrt{t}}{\rho(y)}\right)^{-N}.$$

As a consequence of the previous lemma we have

$$K_\alpha(x,y) \ \leq\ \frac{C}{|x-y|^{d-\alpha}} \tag{13}$$

for all x and y in $\mathbb{R}^d$.

Lemma 3 ([4] Proposition 4.11). *Given $N > 0$ and $0 < \delta < \min(1, 2 - \frac{d}{q})$, there exists a constant C such that*

$$|k_t(x,y) - k_t(x_0,y)| \leq C \left(\frac{|x-x_0|}{\sqrt{t}}\right)^{\delta} t^{-d/2}\, e^{-\frac{|x-y|^2}{C\,t}} \left(1 + \frac{\sqrt{t}}{\rho(x)} + \frac{\sqrt{t}}{\rho(y)}\right)^{-N},$$

for all x, y and x_0 in $\mathbb{R}^d$ with $|x - x_0| < \sqrt{t}$.

A function ψ is rapidly decaying (see [3]) if for every $N > 0$ there exists a constant C_N such that

$$|\psi(x)| \leq C_N (1 + |x|)^{-N}.$$

If ψ is a real function on $\mathbb{R}^d$ and $t > 0$, we define

$$\psi_t(x) = \frac{1}{t^{d/2}} \psi\left(\frac{x}{\sqrt{t}}\right).$$

We will also need some estimates for the kernel

$$q_t(x,y) = k_t(x,y) - \tilde{k}_t(x,y),$$

for all $x, y \in \mathbb{R}^d$ and $t > 0$, where $\tilde{k}_t$ is the kernel of the classical heat operator $e^{-t\Delta}$.

Lemma 4 ([3]). *There exists a rapidly decaying function $\psi \geq 0$ such that*

$$q_t(x,y) \leq C \left(\frac{\sqrt{t}}{\rho(x)}\right)^{2-\frac{d}{q}} \psi_t(x-y),$$

for all x, y in $\mathbb{R}^d$ and $t > 0$.

Lemma 5 ([3]). *For every $0 < \delta < \min(1, 2 - \frac{d}{q})$ and C, there exists a rapidly decaying function ψ such that*

$$|q_t(x, y + h) - q_t(x, y)| \leq C' \left(\frac{|h|}{\rho(x)} \right)^{\delta} \psi_t(x - y)$$

for all x, y and in $\mathbb{R}^d$ and $t > 0$, with $|h| < C\rho(y)$ and $|h| < \frac{|x-y|}{4}$.

In [1] the authors obtain boundedness of $\mathcal{I}_\alpha$ from $L^{d/\alpha}$ into $BMO_{\mathcal{L}} = BMO^0_{\mathcal{L}}$. The next theorem presents an extension of this result to L^p spaces with p greater than d/α. Moreover, $L^{p,\infty}$ spaces are considered instead of L^p.

Theorem 1. *Let us assume that the potential $V \in RH_q$ with $q \geq d/2$ and set $\delta_0 = \min\{1, 2 - \frac{d}{q}\}$. Let $0 < \alpha < d$, $\frac{d}{\alpha} \leq p < \frac{d}{(\alpha - \delta_0)^+}$ and $w \in RH_{p'} \cap D_\eta$, where $1 \leq \eta < 1 - \frac{\alpha}{d} + \frac{\delta_0}{d} + \frac{1}{p}$, then the operator $\mathcal{I}_\alpha$ is bounded from $L^{p,\infty}(w)$ into $BMO^{\alpha - d/p}_{\mathcal{L}}(w)$.*

Proof. We need the following claim: if f is a locally integrable function and B a ball in $\mathbb{R}^d$, then

$$\frac{1}{w(B)} \int_B \mathcal{I}_\alpha(|f \chi_{2B}|) \leq C\, |B|^{\frac{\alpha}{d} - \frac{1}{p}}\, [f]_{p,w}. \tag{14}$$

To get this estimate, from (13), we have

$$\frac{1}{w(B)} \int_B \mathcal{I}_\alpha(|f \chi_{2B}|) \leq C\, \frac{1}{w(B)} \int_B \int_{2B} \frac{|f(y)|}{|x - y|^{d - \alpha}}\, dy\, dx.$$

Let x_0 be the center of B and r its radius. Applying Tonelli's Theorem, the last integral is

$$\begin{aligned} \int_{2B} |f(y)| \int_B \frac{dx}{|x - y|^{d - \alpha}}\, dy &\leq C\, r^\alpha \int_{2B} |f(y)|\, dy \\ &\leq C\, w(B) |B|^{\frac{\alpha}{d} - \frac{1}{p}} [f]_{p,w}, \end{aligned}$$

where the last inequality is due to Lemma 1, finishing the proof of (14).

In order to see that $\mathcal{I}_\alpha f$ is in $BMO^{\alpha - d/p}_{\mathcal{L}}(w)$, in view of Corollary 1, it is enough to check that there exists a constant C such that the two following conditions hold:

(i) For any $x_0 \in \mathbb{R}^d$

$$\frac{1}{w(B(x_0, \rho(x_0)))} \int_{B(x_0, \rho(x_0))} |\mathcal{I}_\alpha f| \leq C\, |B(x_0, \rho(x_0))|^{\frac{\alpha}{d} - \frac{1}{p}} [f]_{p,w}.$$

(ii) For every ball $B = B(x_0, r)$ with $r < \rho(x_0)$ and some constant c_B

$$\frac{1}{w(B)} \int_B |\mathcal{I}_\alpha f(x) - c_B|\, dx \leq C\, |B|^{\frac{\alpha}{d} - \frac{1}{p}}\, [f]_{p,w}.$$

We first prove (i). Let $B = B(x_0, R)$ with $R = \rho(x_0)$. Splitting $f = f_1 + f_2$, with $f_1 = f\, \chi_{2B}$, by the claim (14), we have

$$\frac{1}{w(B)} \int_B |\mathcal{I}_\alpha f_1| \leq C\, |B|^{\frac{\alpha}{d} - \frac{1}{p}}\, [f]_{p,w}.$$

To deal with f_2, we split the integral representation of $\mathcal{I}_\alpha$ as follows. Let $x \in B$,

$$\mathcal{I}_\alpha f_2(x) = \int_0^{R^2} e^{-t\mathcal{L}} f_2(x)\, t^{\alpha/2 - 1}\, dt + \int_{R^2}^\infty e^{-t\mathcal{L}} f_2(x)\, t^{\frac{\alpha}{2} - 1}\, dt. \tag{15}$$

For $x \in B$ and $y \in \mathbb{R}^d \setminus 2B$, we have $|x_0 - y| \le C\,|x-y|$, then for the first term of (15), we have

$$\begin{aligned}\left|\int_0^{R^2} e^{-t\mathcal{L}} f_2(x)\, t^{\alpha/2-1}\, dt\right| &= \left|\int_0^{R^2} \int_{\mathbb{R}^d\setminus 2B} k_t(x,y)\, f(y)\, dy\, t^{\frac{\alpha}{2}-1}\, dt\right| \\ &\le C \int_0^{R^2} \int_{\mathbb{R}^d\setminus 2B} \frac{1}{t^{d/2}}\, e^{-\frac{|x-y|^2}{t}}\, |f(y)|\, dy\, t^{\frac{\alpha}{2}-1}\, dt \\ &\le C \int_0^{R^2} t^{\frac{-d+\alpha}{2}-1} \int_{\mathbb{R}^d\setminus 2B} \left(\frac{t}{|x-y|^2}\right)^{M/2} |f(y)|\, dy\, dt \\ &\le C \int_0^{R^2} t^{\frac{M-d+\alpha}{2}-1}\, dt \int_{\mathbb{R}^d\setminus 2B} \frac{|f(y)|}{|x_0-y|^M}\, dy\ ,\end{aligned}$$

where M is a constant to be determined later and C depends on M.

Splitting the domain of the second integral into dyadic annuli $2^{k+1}B \setminus 2^k B$, and applying Lemma 1 we get

$$\begin{aligned}\int_{(2B)^c} \frac{|f(y)|}{|x_0-y|^M}\, dy &= \sum_{k=1}^{\infty} \int_{2^{k+1}B\setminus 2^k B} \frac{|f(y)|}{|x_0-y|^M}\, dy \\ &\le \frac{1}{R^M} \sum_{k=1}^{\infty} \frac{1}{2^{kM}} \int_{2^{k+1}B} |f(y)|\, dy \\ &\le C\, R^{-\frac{d}{p}-M}\, [f]_{p,w} \sum_{k=1}^{\infty} w(2^{k+1}B)\, 2^{-k(\frac{d}{p}+M)} \\ &\le C w(B)\, R^{-\frac{d}{p}-M}\, [f]_{p,w} \sum_{k=1}^{\infty} 2^{-k(\frac{d}{p}+M-d\eta)},\end{aligned} \tag{16}$$

where the last inequality follows from the fact that $w \in D_\eta$.

The last series converges if $M > d\eta - \frac{d}{p}$. Therefore, for such M,

$$\begin{aligned}\left|\int_0^{R^2} e^{-t\mathcal{L}} f_2(x)\, t^{\alpha/2-1}\, dt\right| &\le C\, w(B)\, R^{-\frac{d}{p}-M}\, [f]_{p,w} \int_0^{R^2} t^{(M-d+\alpha)/2-1}\, dt \\ &= C\, w(B)\, |B|^{\frac{\alpha}{d}-\frac{1}{p}-1}\, [f]_{p,w}\ .\end{aligned}$$

For the second term of (15), we use the extra decay of the kernel $k_t(x,y)$ given by Lemma 2. Thus, we can choose M as above and $N \ge M$ so that,

$$\begin{aligned}\left|\int_{R^2}^{\infty} e^{-t\mathcal{L}} f_2(x)\, t^{\alpha/2-1}\, dt\right| &= \int_{R^2}^{\infty} \int_{\mathbb{R}^d\setminus 2B} k_t(x,y)\, |f(y)|\, dy\, t^{\alpha/2-1}\ dt \\ &\le C \int_{R^2}^{\infty} \int_{\mathbb{R}^d\setminus 2B} t^{(\alpha-d-N)/2-1}\, \rho(x)^N\, e^{-\frac{|x-y|^2}{t}}\, |f(y)|\, dy\, dt\end{aligned}$$

and the last expression is bounded by

$$C\, \rho(x)^N \int_{R^2}^{\infty} t^{(\alpha-d-N)/2-1} \int_{\mathbb{R}^d\setminus 2B} \left(\frac{t}{|x-y|^2}\right)^{M/2} |f(y)|\, dy\, dt.$$

As $x \in B$, we have $\rho(x) \sim \rho(x_0) = R$. Then, the last expression is bounded by a constant times

$$R^N \int_{R^2}^{\infty} t^{(M+\alpha-d-N)/2-1}\, dt \int_{\mathbb{R}^d \setminus 2B} \frac{|f(y)|}{|x_0 - y|^M}\, dy\ .$$

Since $M + \alpha - d - N < 0$, the integral in t converges. Then, splitting the second integral in the same way as before, the last term is bounded by

$$C\, w(B)\, R^{\alpha - \frac{d}{p} - d}\, [f]_{p,w} = C\, w(B)\, |B|^{\frac{\alpha}{d} - \frac{1}{p} - 1}\, [f]_{p,w}$$

and we have proved (i).

Now we will see (ii). Let $B = \{x \in \mathbb{R}^d \ :\ |x - x_0| < r\}$, with $r < \rho(x_0)$. We set $f = f_1 + f_2$ with $f_1 = f\, \chi_{2B}$ and

$$c_B = \int_{r^2}^{\infty} e^{-t\mathcal{L}} f_2(x_0)\, t^{\alpha/2-1}\, dt\,.$$

By the claim (14) we have

$$\begin{aligned}\frac{1}{w(B)} \int_B |\mathcal{I}_\alpha(f) - c_B| &\le \frac{1}{w(B)} \int_B \mathcal{I}_\alpha(|f_1|) + \frac{1}{w(B)} \int_B |\mathcal{I}_\alpha(f_2) - c_B| \\ &\le C\, |B|^{\alpha/d - 1/p}\, [f]_{p,w} + \frac{1}{w(B)} \int_B |\mathcal{I}_\alpha(f_2) - c_B|\,.\end{aligned}$$

For the second term, we will show that

$$|\mathcal{I}_\alpha f_2(x) - c_B| \le C\, w(B)\, |B|^{\frac{\alpha}{d} - \frac{1}{p} - 1}\, [f]_{p,w}\ . \tag{17}$$

Let x be in B and split $\mathcal{I}_\alpha f_2(x)$ as in (15). For the first term we can proceed as before to obtain that

$$\left| \int_0^{r^2} e^{-t\mathcal{L}} f_2(x)\, t^{\alpha/2-1}\, dt \right| \le C\, w(B)\, |B|^{\frac{\alpha}{d} - \frac{1}{p} - 1}\, [f]_{p,w}\ .$$

The remaining part, by the definition of c_B, is bounded by

$$\left| \int_{r^2}^{\infty} e^{-t\mathcal{L}} f_2(x)\, t^{\alpha/2-1}\, dt - c_B \right| \le \int_{r^2}^{\infty} \int_{\mathbb{R}^d \setminus 2B} |k_t(x,y) - k_t(x_0,y)|\, |f(y)|\, dy\, t^{\alpha/2-1}\, dt$$

and by Lemma 3, for any $0 < \delta < \delta_0$ the last integral is majorised

$$C_\delta \int_{r^2}^{\infty} \int_{\mathbb{R}^d \setminus 2B} \left(\frac{|x - x_0|}{\sqrt{t}} \right)^{\delta} t^{-d/2}\, e^{-\frac{|x-y|^2}{C\,t}}\, |f(y)|\, dy\, t^{\alpha/2-1}\, dt\ .$$

Since $|x_0 - x| < r$, applying Fubini's Theorem the last integral is bounded by

$$r^\delta \int_{\mathbb{R}^d \setminus 2B} |f(y)| \int_{r^2}^{\infty} t^{-(d-\alpha+\delta)/2}\, e^{-\frac{|x-y|^2}{C\,t}}\, \frac{dt}{t}\, dy\ .$$

Now, changing variables $s = \frac{|x-y|^2}{t}$ we obtain the bound

$$r^\delta \int_{\mathbb{R}^d \setminus 2B} \frac{|f(y)|}{|x - y|^{d-\alpha+\delta}}\, dy \int_0^{\infty} s^{(d-\alpha+\delta)/2}\, e^{-s/C}\, \frac{ds}{s}.$$

Since the integral in s is finite, we only need to estimate the integral in y. We perform the same calculation as in (16) with $M = d - \alpha + \delta$. But now, to make the series convergent we need $\eta < 1 - \alpha/d + \delta/d + 1/p$ which holds true by our assumption on η, and taking δ close enough to δ_0. Notice this is the only place where we have used the condition on the size of η. In this way the above expression can be controlled by $w(B)r^{\alpha-\frac{d}{p}-d}\,[f]_{p,w}$ and so (17) is proved. $\square$

4. $\mathcal{I}_\alpha$ ON $BMO^\beta_\mathcal{L}(w)$ SPACES

The definition of $BMO^\beta_\mathcal{L}(w)$ only establishes a control for the averages over balls with radii greater than ρ at their centres (see (3)). However, for lower radii some kind of estimate can be proved.

Lemma 6. *Let $w \in D_\eta$ with $\eta \geq 1$ and $f \in BMO^\beta_\mathcal{L}(w)$. Then, for every ball $B = B(x,r)$, we have*

$$\int_B |f| \;\leq\; C\,\|f\|_{BMO^\beta_L(w)}\; w(B)|B|^{\beta/d}\max\left\{1, \left(\frac{\rho(x)}{r}\right)^{d\eta-d+\beta}\right\},$$

if $\eta > 1$ or $\beta > 0$, and

$$\int_B |f| \;\leq\; C\,\|f\|_{BMO_\mathcal{L}(w)}\; w(B)\;\max\left\{1, 1+\log\left(\frac{\rho(x)}{r}\right)\right\},$$

if $\eta = 1$ and $\beta = 0$.

Proof. Let $f \in BMO^\beta_\mathcal{L}(w)$. If $r \geq \rho(x)$ the conclusion follows from condition (3). If $r < \rho(x)$, let $j_0 = \lfloor \log_2(\frac{\rho(x)}{r})\rfloor + 1$, where $\lfloor\cdot\rfloor$ denotes the greatest integer function. Then

$$\begin{aligned}\frac{1}{|B|}\int_B |f| \;&\leq\; 2^d\sum_{j=0}^{j_0-1}\frac{1}{|2^jB|}\int_{2^jB}|f(z)-f_{2^jB}|dz + \frac{1}{|2^{j_0}B|}\int_{2^{j_0}B}|f| \\ &\leq\; C\,\|f\|_{BMO^\beta_\mathcal{L}(w)}\sum_{j=0}^{j_0} w(2^jB)|2^jB|^{\frac{\beta}{d}-1},\end{aligned}$$

since $r2^{j_0} \geq \rho(x)$. Using now that $w \in D_\eta$, we get

$$\begin{aligned}\int_B |f| \leq C\|f\|_{BMO^\beta_\mathcal{L}(w)} w(B)|B|^{\beta/d}\sum_{j=0}^{j_0} 2^{j(d\eta-d+\beta)} \\ \leq\; C\,\|f\|_{BMO^\beta_\mathcal{L}(w)}\; w(B)|B|^{\beta/d}\left(\frac{\rho(x)}{r}\right)^{d\eta-d+\beta},\end{aligned}$$

in the case $\eta > 1$ or $\beta > 0$. If $\eta = 1$ and $\beta = 0$, we have

$$\sum_{j=1}^{j_0} 2^{j(d\eta-d+\beta)} \;=\; j_0 \;\leq\; 1+\log_2\left(\frac{\rho(x)}{r}\right),$$

and the proof is finished. $\square$

Theorem 2. *Let us assume that the potential $V \in RH_q$ with $q \geq d/2$ and set $\delta_0 = \min\{1, 2-\frac{d}{q}\}$. Let $0 < \alpha < 1$, $\beta \geq 0$, $\alpha+\beta < \delta_0$ and, $w \in D_\eta$ with $1 \leq \eta < 1+\frac{\delta_0-\alpha-\beta}{d}$, then the operator $\mathcal{I}_\alpha$ is bounded from $BMO^\beta_\mathcal{L}(w)$ into $BMO^{\beta+\alpha}_\mathcal{L}(w)$.*

Proof. Since $\alpha > 0$, $BMO^{\beta+\alpha}_{\mathcal{L}}(w) = \Lambda^{\beta+\alpha}_{\mathcal{L}}(w)$ with equivalent norms, due to Proposition 4. Hence we can prove boundedness from $BMO^{\beta}_{\mathcal{L}}(w)$ into $\Lambda^{\beta+\alpha}_{\mathcal{L}}(w)$. Let $f \in BMO^{\beta}_{\mathcal{L}}(w)$. We will see that for x and y in $\mathbb{R}^d$, we have

$$|\mathcal{I}_\alpha f(x) - \mathcal{I}_\alpha f(y)| \leq C\|f\|_{BMO^{\beta}_{\mathcal{L}}(w)} \left[W_{\beta+\alpha}(x, |x-y|) + W_{\beta+\alpha}(y, |x-y|)\right] \tag{18}$$

provided $|x - y| < \rho(x)$, and

$$\int_{B(x,\rho(x))} |\mathcal{I}_\alpha f(u)|\, du \;\leq\; \|f\|_{BMO^{\beta}_{\mathcal{L}}(w)}\, \rho(x)^{\beta+\alpha}\, w(B(x, \rho(x))). \tag{19}$$

The above inequalities (18) and (19) would imply that $\mathcal{I}_\alpha f$ belongs to $\Lambda^{\beta+\alpha}_{\mathcal{L}}(w)$ (see Remark 2).

Suppose $\|f\|_{BMO^{\beta}_{\mathcal{L}}(w)} = 1$ and let us start with (19). We split the inner integral, as usual, in local and global parts. If we call $B = B(x, \rho(x))$, then

$$\int_B |\mathcal{I}_\alpha f(u)|\, du \leq \int_B \left(\int_{2B} + \int_{(2B)^c} \right) K_\alpha(u,z)\, |f(z)|\, dz\, du.$$

By estimate (13), the first term is bounded by

$$\begin{aligned} \int_B \int_{2B} \frac{|f(z)|}{|u-z|^{d-\alpha}}\, dz\, du &\leq \int_{2B} |f(z)|\, dz \int_B \frac{1}{|u-x|^{d-\alpha}}\, du \\ &\leq C\|f\|_{BMO^{\beta}_{\mathcal{L}}(w)}\, \rho(x)^{\alpha+\beta}\, w(B). \end{aligned}$$

For the second term, using Lemma 2 and the change of variables $s = \frac{|u-z|^2}{Ct}$, we have

$$\begin{aligned} &\int_B \int_0^\infty \int_{(2B)^c} k_t(u,z)|f(z)|\, dz\, t^{\alpha/2}\, \frac{dt}{t}\, du \\ &\quad \leq C \int_B \int_0^\infty \int_{(2B)^c} t^{-(d-\alpha+N)/2}\, e^{-\frac{|u-z|^2}{Ct}}\, \rho(u)^N |f(z)|\, dz\, \frac{dt}{t}\, du \\ &\quad \leq C \int_0^\infty s^{(d-\alpha+N)/2}\, e^{-s} \frac{ds}{s} \int_B \int_{(2B)^c} \rho(u)^N \frac{|f(z)|}{|u-z|^{d-\alpha+N}}\, dz\, du. \end{aligned}$$

If $u \in B(x, \rho(x))$ then $\rho(u) \leq C\rho(x)$ (Proposition 1), and also $|u - z| > |x - z|/2$ for all $z \in B(x, 2\rho(x))^c$. Hence, the last expression is bounded by

$$C\, \rho(x)^{N+d} \int_{(2B)^c} \frac{|f(z)|}{|x-z|^{d-\alpha+N}}\, dz. \tag{20}$$

If we call $B_j = 2^j B$, we may split the last integral into annuli, use that $f \in BMO^\beta_{\mathcal{L}}(w)$ and $w \in D_\eta$ to obtain

$$\begin{aligned}\int_{(2B)^c} \frac{|f(z)|}{|x-z|^{d-\alpha+N}}\,dz &\le \sum_{k=1}^{\infty} \int_{B_{k+1}\setminus B_k} \frac{|f(z)|}{|x-z|^{d-\alpha+N}}\,dz \\ &\le \rho(x)^{-d+\alpha-N} \sum_{k=1}^{\infty} 2^{-k(d-\alpha+N)} \int_{B_{k+1}} |f(z)|\,dz \\ &\le C\ \rho(x)^{-d+\alpha+\beta-N} \sum_{k=1}^{\infty} 2^{-k(d-\alpha-\beta+N)} w(B_{k+1}) \\ &\le C\ \rho(x)^{-d+\alpha+\beta-N}\, w(B) \sum_{k=1}^{\infty} 2^{-k(d-\alpha-\beta+N-d\eta)}.\end{aligned}$$

If we choose N large enough, the last sum is finite, thus (20) is bounded by a constant times

$$\rho(x)^{\alpha+\beta}\, w(B(x,\rho(x))),$$

and we have shown that (19) is satisfied.

To see (18), let $|x-y| < \rho(x)$,

$$\begin{aligned}|\mathcal{I}_\alpha f(x) - \mathcal{I}_\alpha f(y)| \ &\le \ \left| \int_0^{\rho(x)^2} \int_{\mathbb{R}^d} [k_t(x,z) - k_t(y,z)]\, f(z)\, dz\, t^{\alpha/2} \frac{dt}{t} \right| \\ &\quad + \left| \int_{\rho(x)^2}^{\infty} \int_{\mathbb{R}^d} [k_t(x,z) - k_t(y,z)]\, f(z)\, dz\, t^{\alpha/2} \frac{dt}{t} \right|. \end{aligned} \tag{21}$$

For the first term, if $t > \rho(x)^2$, since $|x-y| < \rho(x)$, we have $|x-y| < \sqrt{t}$, hence Lemma 3 allows us to get

$$\begin{aligned}&\int_{\rho(x)^2}^{\infty} \int_{\mathbb{R}^d} |k_t(x,z) - k_t(y,z)|\, |f(z)|\, dz\, t^{\frac{\alpha}{2}-1}\, dt \\ &\quad\le\ C_\delta\, |x-y|^\delta \int_{\rho(x)^2}^{\infty} \int_{\mathbb{R}^d} e^{-\frac{|x-z|^2}{C\,t}}\, |f(z)|\, dz\, t^{(-d+\alpha-\delta)/2} \frac{dt}{t},\end{aligned} \tag{22}$$

for each $0 < \delta < \delta_0$. If $t > \rho(x)^2$, calling $B = B(x,\sqrt{t})$ we estimate the inner integral as

$$\int_{\mathbb{R}^d} e^{-\frac{|x-z|^2}{C\,t}}\, |f(z)|\, dz \ \le\ C \int_B |f| \ +\ t^{M/2} \sum_{k=0}^{\infty} \int_{2^{k+1}B\setminus 2^k B} \frac{|f(z)|}{|x-z|^M}\, dz,$$

for some $M > 1$ to be chosen. Since $f \in BMO^\beta_{\mathcal{L}}(w)$ and $t > \rho(x)^2$, the first integral is bounded by $w(B) t^{\beta/2}$. To deal with the sum in k, we use again $f \in BMO^\beta_{\mathcal{L}}(w)$, and then $w \in D_\eta$, to obtain

$$\begin{aligned} t^{M/2}\sum_{k=0}^{\infty}\int_{2^{k+1}B\setminus 2^kB}\frac{|f(z)|}{|x-z|^M}\,dz \;&\le\; 2\sum_{k=0}^{\infty}2^{-kM}\int_{2^{k+1}B}|f| \\ &\le C\,t^{\beta/2}\sum_{k=0}^{\infty}2^{-k(M-\beta)}w(2^{k+1}B) \\ &\le C\,t^{\beta/2}w(B)\sum_{k=0}^{\infty}2^{-k(M-\beta-d\eta)}, \end{aligned}$$

and the sum is finite for M large enough. Therefore, since $|x-y|<\rho(x)<\sqrt{t}$ and $-d+\alpha+\beta-\delta+d\eta<0$ choosing δ close to δ_0, (22) is bounded by

$$\begin{aligned} &|x-y|^\delta\int_{\rho(x)^2}^{\infty}w(B(x,\sqrt{t}))t^{(-d+\alpha+\beta-\delta)/2}\frac{dt}{t} \\ &\quad\le\; C\,|x-y|^{\delta-d\eta}w(B(x,|x-y|))\int_{|x-y|^2}^{\infty}t^{(-d+\alpha+\beta-\delta+d\eta)/2}\frac{dt}{t} \\ &\quad\le\; C\,w(B(x,|x-y|))\;|x-y|^{-d+\alpha+\beta} \\ &\quad\le\; C\,W_{\alpha+\beta}(x,|x-y|). \end{aligned}$$

To deal with the second term of (21), we set

$$q_t(x,y)=k_t(x,y)-\tilde{k}_t(x,y),$$

for all $x,y\in\mathbb{R}^d$ and $t>0$, where $\tilde{k}_t$ is the classical heat kernel as before. Then we have,

$$\left|\int_0^{\rho(x)^2}\int_{\mathbb{R}^d}[k_t(x,z)-k_t(y,z)]\;f(z)\,dz\,t^{\alpha/2}\,\frac{dt}{t}\right|\;\le\;I+II,$$

where

$$I\;=\;\left|\int_0^{\rho(x)^2}\int_{\mathbb{R}^d}[q_t(x,z)-q_t(y,z)]\,f(z)\,dz\,t^{\alpha/2}\,\frac{dt}{t}\right|$$

and

$$II\;=\;\left|\int_0^{\rho(x)^2}\int_{\mathbb{R}^d}[\tilde{k}_t(x,z)-\tilde{k}_t(y,z)]\,f(z)\,dz\,t^{\alpha/2}\,\frac{dt}{t}\right|.$$

To estimate I, calling $B=B(x,4|x-y|)$, we split $\mathbb{R}^d$ into two regions and write

$$I\;\le\;I_1+I_2+I_3,$$

with

$$I_1=\int_0^{\rho(x)^2}\int_{B^c}|q_t(x,z)-q_t(y,z)|\,|f(z)|\,dz\,t^{\alpha/2}\,\frac{dt}{t},$$

$$I_2=\int_0^{\rho(x)^2}\int_{B}|q_t(x,z)|\,|f(z)|\,dz\,t^{\alpha/2}\,\frac{dt}{t}$$

and

$$I_3=\int_0^{\rho(x)^2}\int_{B}|q_t(y,z)|\,|f(z)|\,dz\,t^{\alpha/2}\,\frac{dt}{t}.$$

If $z \in B^c$, we are in the hypothesis of Lemma 5 and therefore, given $0<\delta<\delta_0$, there exists a rapidly decaying function ψ such that

$$\begin{aligned} I_1 &\le C|x-y|^\delta \int_0^{\rho(x)^2}\int_{B^c}\frac{\psi_t(z-x)}{\rho(z)^\delta}\,|f(z)|\,dz\,t^{\alpha/2}\,\frac{dt}{t}\\ &\le C\left(\frac{|x-y|}{\rho(x)}\right)^\delta\int_0^{\rho(x)^2}\int_{B^c}\left(1+\frac{|x-z|}{\rho(x)}\right)^{\delta k_0}\psi_t(z-x)\,|f(z)|\,dz\,t^{\alpha/2}\,\frac{dt}{t}, \end{aligned}$$

where in the last inequality we have used Proposition 1.

The inner integral is

$$\begin{aligned} &\int_{B^c}\left(1+\frac{|x-z|}{\rho(x)}\right)^{\delta k_0}\psi_t(z-x)\,|f(z)|\,dz\\ &\qquad=\sum_{j=0}^{\infty}\int_{B_j\setminus B_{j-1}}\left(1+\frac{|x-z|}{\rho(x)}\right)^{\delta k_0}\psi_t(z-x)\,|f(z)|\,dz, \end{aligned}$$

where $B_j = B(x, 2^{j+3}|x-y|)$. Thus $I_1 \le I_{11}+I_{12}$, where

$$I_{11} = C\left(\frac{|x-y|}{\rho(x)}\right)^\delta\int_0^{\rho(x)^2}\sum_{j=0}^{j_0}\int_{B_j\setminus B_{j-1}}\left(1+\frac{|x-z|}{\rho(x)}\right)^{\delta k_0}\psi_t(z-x)\,|f(z)|\,dz\,t^{\alpha/2}\,\frac{dt}{t},$$

with $j_0 = \left\lfloor \log_2\left(\frac{\rho(x)}{|x-y|}\right)\right\rfloor$, and I_{12} the same but summing up from j_0+1. If $j \le j_0$ and $z \in B_j\setminus B_{j-1}$, then $\left(1+\frac{|x-z|}{\rho(x)}\right)^{\delta k_0} \le C$, and since $\psi_t(z-x) \le C\,t^{\epsilon/2}/|x-z|^{d+\epsilon}$, for some $\epsilon>0$ fixed, we obtain

$$\begin{aligned} I_{11} &\le C\left(\frac{|x-y|}{\rho(x)}\right)^\delta\int_0^{\rho(x)^2} t^{(\alpha+\epsilon)/2}\,\frac{dt}{t}\sum_{j=0}^{j_0}\int_{B_j\setminus B_{j-1}}\frac{|f(z)|}{|x-z|^{d+\epsilon}}\,dz\\ &\le C\frac{|x-y|^{\delta-d-\epsilon}}{\rho(x)^{\delta-\alpha-\epsilon}}\sum_{j=0}^{j_0}2^{-j(d+\epsilon)}\int_{B_j}|f(z)|\,dz. \end{aligned}$$

From Lemma 6 and the fact that $w \in D_\eta$, in the case $\eta>1$ or $\beta>0$,

$$\begin{aligned} \sum_{j=0}^{j_0}2^{-j(d+\epsilon)}\int_{B_j}|f(z)|\,dz &\le C\sum_{j=0}^{j_0}2^{-j(d+\epsilon)}w(B_j)|B_j|^{\beta/d}\left(\frac{\rho(x)}{2^{j+3}|x-y|}\right)^{d\eta-d+\beta}\\ &\le C\,\frac{\rho(x)^{d\eta-d+\beta}}{|x-y|^{d\eta-d}}w(B)\sum_{j=0}^{j_0}2^{-j\epsilon}\\ &\le C\,\frac{\rho(x)^{d\eta-d+\beta}}{|x-y|^{d\eta-d}}w(B). \end{aligned}$$

Therefore, we have

$$\begin{aligned} I_{11} &\le C\left(\frac{|x-y|}{\rho(x)}\right)^{\delta-\alpha-\beta-d\eta+d-\epsilon}\frac{w(B)}{|x-y|^{d-\alpha-\beta}}\\ &\le C\,\frac{w(B)}{|x-y|^{d-\alpha-\beta}}, \end{aligned}\tag{23}$$

since by hypothesis $1 \le \eta < \frac{\delta_0-\alpha-\beta}{d}+1$ and $|x-y|<\rho(x)$, and thus $\delta-\alpha-\beta-d\eta+d-\epsilon>0$, choosing ϵ small enough and δ close to δ_0.

As for the case $\beta = 0$ and $\eta = 1$, using Lemma 6 and the inequality

$$1 + \log(t) \leq C t^{\epsilon/2}, \tag{24}$$

for $t > 1/8$, we arrive to the same estimate of I_{11} proceeding as before.

Next we estimate I_{12}. For $M > \delta k_0 + d\eta + \beta$, we have $\psi_t(z-x) \leq C \dfrac{t^{(M-d)/2}}{|z-x|^M}$. Also if $z \in B_j \setminus B_{j-1}$ for $j > j_0$, then $|x-z| > \rho(x)$. Therefore

$$\begin{aligned} I_{12} &\leq C \left(\frac{|x-y|}{\rho(x)}\right)^{\delta+\delta k_0} \int_0^{\rho(x)^2} t^{(M-d+\alpha)/2} \frac{dt}{t} \sum_{j=j_0+1}^{\infty} 2^{j\delta k_0} \int_{B_j \setminus B_{j-1}} \frac{|f(z)|}{|z-x|^M}\, dz \\ &\leq C \frac{|x-y|^{\delta+\delta k_0 - M}}{\rho(x)^{\delta+\delta k_0 - M + d - \alpha}} \sum_{j=j_0+1}^{\infty} 2^{-j(M-\delta k_0)} \int_{B_j} |f(z)|\, dz. \end{aligned}$$

Since for $j > j_0$, the radius of B_j is $2^{j+3}|x-y| > \rho(x)$, then

$$\int_{B_j} |f(z)|\, dz \;\leq\; C\, w(B_j)|B_j|^{\beta/d} \;\leq\; C\, 2^{j(d\eta+\beta)}|x-y|^{\beta} w(B),$$

and thus

$$\begin{aligned} I_{12} &\leq C \left(\frac{|x-y|}{\rho(x)}\right)^{-M+\delta k_0+\delta-\alpha+d} \frac{w(B)}{|x-y|^{d-\alpha-\beta}} \sum_{j=j_0+1}^{\infty} 2^{-j(M-\delta k_0 - d\eta-\beta)} \\ &\leq C \left(\frac{|x-y|}{\rho(x)}\right)^{d-d\eta+\delta-\alpha-\beta} \frac{w(B)}{|x-y|^{d-\alpha-\beta}} \\ &\leq C \frac{w(B)}{|x-y|^{d-\alpha-\beta}}, \end{aligned} \tag{25}$$

with an appropriate choice of δ.

To deal with I_2, let $M > d$. From Lemma 4, being $t < \rho(x)^2$,

$$|q_t(x,z)| \leq C \left(\frac{\sqrt{t}}{\rho(x)}\right)^{\delta_0} \frac{1}{t^{d/2}} \left(1 + \frac{|x-z|}{\sqrt{t}}\right)^{-M}. \tag{26}$$

Then we may write

$$I_2 = I_{21} + I_{22},$$

where

$$I_{21} \;=\; C \int_0^{|x-y|^2} \int_B |q_t(x,z)|\,|f(z)|\, dz\, t^{\alpha/2} \frac{dt}{t}$$

and

$$I_{22} \;=\; \int_{|x-y|^2}^{\rho(x)^2} \int_B |q_t(x,z)|\,|f(z)|\, dz\, t^{\alpha/2} \frac{dt}{t}.$$

To take care of I_{21} let $B_t = B(x,\sqrt{t})$ and $N = \left\lfloor \log_2\left(\frac{4|x-y|}{\sqrt{t}}\right)\right\rfloor$. Using estimate (26), we have

$$\begin{aligned}
\int_B |q_t(x,z)|\,|f(z)|\,dz \;&\le \frac{t^{\frac{\delta_0-d}{2}}}{\rho(x)^{\delta_0}}\left(\int_{B_t}|f| \;+\; t^{M/2}\int_{B\setminus B_t}\frac{|f(z)|}{|x-z|^M}\,dz\right)\\
&\le \frac{t^{\frac{\delta_0-d}{2}}}{\rho(x)^{\delta_0}}\left(\int_{B_t}|f| \;+\; t^{M/2}\sum_{j=0}^{N}\int_{2^{j+1}B_t\setminus 2^jB_t}\frac{|f(z)|}{|x-z|^M}\,dz\right)\\
&\le C\,\frac{t^{\frac{\delta_0-d}{2}}}{\rho(x)^{\delta_0}}\left(\sum_{j=0}^{N+1}2^{-jM}\int_{2^jB_t}|f|\right),
\end{aligned}$$

and since every ball in the last sum has its radius less than $8\rho(x)$, we can apply Lemma 6 and that $w\in D_\eta$, to obtain

$$\begin{aligned}
\int_B |q_t(x,z)|\,|f(z)|\,dz \;&\le\; C\,\frac{t^{(\delta_0-d\eta)/2}}{\rho(x)^{\delta_0-d\eta+d-\beta}}\left(\sum_{j=0}^{N}2^{-j(M-d+d\eta)}w(2^jB_t)\right)\\
&\le\; C\,\frac{t^{(\delta_0-d\eta)/2}}{\rho(x)^{\delta_0-d\eta+d-\beta}}w(B_t)\left(\sum_{j=0}^{\infty}2^{-j(M-d)}\right)\\
&\le\; C\,\frac{t^{(\delta_0-d\eta)/2}}{\rho(x)^{\delta_0-d\eta+d-\beta}}w(B_t),
\end{aligned}$$

where the last sum is finite since $M>d$.

Hence,

$$\begin{aligned}
I_{21} \;&\le \frac{C}{\rho(x)^{\delta_0-d\eta+d-\beta}}\int_0^{|x-y|^2} t^{(\delta_0+\alpha-d\eta)/2}w(B_t)\,\frac{dt}{t}\\
&\le \frac{C}{\rho(x)^{\delta_0-d\eta+d-\beta}}\int_0^{|x-y|^2} t^{(\delta_0-\beta-d\eta+d)/2}\,W_{\alpha+\beta}(x,\sqrt{t})\,\frac{dt}{t}\\
&\le \frac{C}{\rho(x)^{\delta_0-d\eta+d-\beta}}\int_0^{|x-y|^2} t^{(\delta_0-\beta-d\eta+d)/2}\,\frac{dt}{t}\;W_{\alpha+\beta}(x,|x-y|)\\
&\le C\left(\frac{|x-y|}{\rho(x)}\right)^{\delta_0-\beta-d\eta+d} W_{\alpha+\beta}(x,|x-y|).
\end{aligned}\tag{27}$$

Since $\delta_0-\beta-d\eta+d>\alpha>0$, and $|x-y|<\rho(x)$, we have $I_{21}\le W_{\alpha+\beta}(x,|x-y|)$.

To deal with I_{22} we use again (26) and Lemma 6, to get

$$\begin{aligned}
I_{22} \;&=\; \int_{|x-y|^2}^{\infty}\int_B |q_t(x,z)|\,|f(z)|\,dz\,t^{\alpha/2}\,\frac{dt}{t}\\
&\le\; C\,\rho(x)^{-\delta_0}\int_{|x-y|^2}^{\infty} t^{(\alpha+\delta_0-d)/2}\,\frac{dt}{t}\int_B|f|\\
&\le\; C\,\frac{w(B)}{|x-y|^{d-\alpha-\beta}}\left(\frac{|x-y|}{\rho(x)}\right)^{d-d\eta+\delta_0-\beta}\\
&\le\; C\,\frac{w(B)}{|x-y|^{d-\alpha-\beta}} \le C\,W_{\alpha+\beta}(x,|x-y|),
\end{aligned}\tag{28}$$

since $d - d\eta + \delta_0 - \beta > \alpha > 0$, and $|x-y| < \rho(x)$.

The case $\beta = 0$ and $\eta = 1$ is performed using Lemma 6 and inequality (24) with $\epsilon < \delta_0$, following the same steps as in (27) and (28) respectively.

We can also obtain that

$$I_3 \leq C\, W_{\alpha+\beta}(x, |x-y|) \tag{29}$$

following the same lines as in I_2 but exchanging x by y and integrating over $B(y, 8|x-y|)$.

From (23), (25), (27), (28) and (29) we obtain

$$I \ \leq\ C\, W_{\alpha+\beta}(x, |x-y|).$$

To see $II \leq C\, W_{\alpha+\beta}(x, |x-y|)$ we refer to the reader to [5], p. 238. In fact, since $\tilde{k}_t$ is a convolution kernel,

$$\int_{\mathbb{R}^d} [\tilde{k}_t(x,z) - \tilde{k}_t(y,z)]\, dz = 0.$$

So we have

$$II \ = \ \left| \int_0^{\rho(x)^2} \int_{\mathbb{R}^d} [\tilde{k}_t(x,z) - \tilde{k}_t(y,z)]\, [f(z) - f_B]\, dz\, t^{\alpha/2} \frac{dt}{t} \right| \leq II_1 + II_2,$$

where

$$II_1 \ = \ \int_0^{\rho(x)^2} \int_B |\tilde{k}_t(x,z) - \tilde{k}_t(y,z)|\, |f(z) - f_B|\, dz\, t^{\alpha/2} \frac{dt}{t}$$

and

$$II_2 \ = \ \int_0^{\rho(x)^2} \int_{B^c} |\tilde{k}_t(x,z) - \tilde{k}_t(y,z)|\, |f(z) - f_B|\, dz\, t^{\alpha/2} \frac{dt}{t}$$

with $B = B(x, |x-y|)$.

Applying

$$|\tilde{k}_t(x,z) - \tilde{k}_t(y,z)| \leq C\, \frac{e^{-\frac{|x-y|}{Ct}}}{t^{d/2-1}} |x-y|\, |x-z|$$

and changing variables $s = \frac{t}{|x-y|}$, we have

$$\begin{aligned} II_2 \ &\leq\ |x-y| \int_{B^c} |f(z) - f_B|\, |x-z| \int_0^{\rho(x)^2} \frac{e^{-\frac{|x-y|}{Ct}}}{t^{d/2-1}} t^{\alpha/2} \frac{dt}{t}\, dz \\ &\leq\ |x-y| \int_{B^c} \frac{|f(z) - f_B|}{|x-z|^{d-\alpha+1}}\, dz. \end{aligned}$$

Since $w \in D_\eta$, from Lemma 4.7 in [5], the last expression is bounded by

$$|x-y| \int_{B^c} \frac{w(z)}{|x-z|^{d-\alpha-\beta+1}}\, dz \ \leq\ Cw(B)|x-y|^{d-\alpha-\beta},$$

where the last inequality is due to Lemma 3.9 in [5].

To deal with II_1,

$$\begin{aligned} \int_0^{\rho(x)^2} \int_B |\tilde{k}_t(x,z)|\, |f(z) - f_B|\, dz\, t^{\alpha/2} \frac{dt}{t} \ &\leq\ C \int_0^{\infty} \int_B \frac{e^{-\frac{|x-y|}{Ct}}}{t^{d/2}}\, |f(z) - f_B|\, dz\, t^{\alpha/2} \frac{dt}{t} \\ &=\ C \int_B \frac{|f(z) - f_B|}{|x-z|^{d-\alpha}}\, dz \end{aligned}$$

and denoting $B_j = 2^{-j}B$, we obtain

$$\begin{aligned}
\int_B \frac{|f(z)-f_B|}{|x-z|^{d-\alpha}}\,dz &= \sum_{k=0}^{\infty}\int_{B_j\setminus B_{j+1}} \frac{|f(z)-f_B|}{|x-z|^{d-\alpha}} \\
&\le C\sum_{k=0}^{\infty}\left(\frac{2^j}{|x-y|}\right)^{d-\alpha}\int_{B_j}|f(z)-f_B| \\
&\le C\sum_{k=0}^{\infty}\left(\frac{2^j}{|x-y|}\right)^{d-\alpha-\beta} w(B_j) \\
&\le C\sum_{k=0}^{\infty}\left(\frac{2^j}{|x-y|}\right)^{d-\alpha-\beta} w(B_j\setminus B_{j+1}) \\
&\le C\sum_{k=0}^{\infty}\int_{B_j\setminus B_{j+1}}\frac{w(z)}{|x-z|^{d-\alpha-\beta}}\,dz \\
&= C\int_B \frac{w(z)}{|x-z|^{d-\alpha-\beta}}\,dz,
\end{aligned}$$

finishing the proof of the theorem. □

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