ON APPROXIMATION OF MAXIMAL OPERATORS

HUGO AIMAR, MARILINA CARENA, AND BIBIANA IAFFEI

Abstract. We prove that the weak type $(1,1)$ for the maximal of a sequence of integral operators on a metric measure space X, follows from the uniform weak type on Dirac deltas of the restriction of the operators to a sequence of approximations of X.

1. Introduction

Let us start by a classical example, the Hardy-Littlewood maximal operator. The standard proofs of the weak type $(1,1)$ for this operator are based on covering lemmas. Besicovich type covering lemmas do not hold for general metrics. Wiener type, instead, are valid for general quasi-distances in finite metric (or Assouad) dimension spaces. Since the covering balls in Wiener's lemma are dilations of the selected balls, the doubling condition of the measure is the most usual tool to overcome this difficulty. In finite settings the strategy of Wiener becomes specially simple.

Let $[1,L]$ be the set of all integers between 1 and a given integer L larger than one, i.e. $[1,L]=\{i\in\mathbb{N}:1\leq i\leq L\}$. Let ρ be a distance on $[1,L]$. As usual $B_\rho(i,r)$ denotes the ρ-ball in $[1,L]$ centered at i and with radius $r>0$, $B_\rho(i,r)=\{j\in[1,L]:\rho(i,j)<r\}$. Let ν be a positive function defined on $[1,L]$. Given a subset E of $[1,L]$ we shall write $\nu(E)$ to denote the sum $\sum_{i\in E}\nu(i)$. Notice that

$$\begin{aligned}\nu(B_\rho(i,2r)) &= \frac{\nu(B_\rho(i,2r))}{\nu(B_\rho(i,r))}\nu(B_\rho(i,r))\\ &\leq \frac{\nu([1,L])}{\min\{\nu(j):j\in[1,L]\}}\nu(B_\rho(i,r)).\end{aligned}$$

So that ν is a doubling measure on $[1,L]$. In other words, the set

$$\mathbb{A}=\{A:\nu(B_\rho(i,2r))\leq A\,\nu(B_\rho(i,r)) \text{ for every } r>0 \text{ and every } i\in[1,L]\}$$

is non empty.

Theorem 1 (Hardy-Littlewood in $([1,L],\rho,\nu)$)**.** *For each $A\in\mathbb{A}$, $\lambda>0$ and any subset E of $[1,L]$, we have that*

$$\nu\left(\bigcup_{r>0}\left\{i\in[1,L]:\frac{\nu(E\cap B_\rho(i,r))}{\nu(B_\rho(i,r))}>\lambda\right\}\right)\leq\frac{A}{\lambda}\nu(E). \tag{1.1}$$

The proof of the above theorem can be obtained from the following discrete version of the Wiener covering lemma. Notice that the finite context makes easier the selection process than in continuous settings such as the Euclidean space.

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Lemma 2 (Wiener's lemma in $([1,L],\rho)$)**.** *Let E be a subset of $[1,L]$ and let $r : E \to \mathbb{R}^+$ be a given positive real function defined on E. Then there exists a subset F of E such that*

(a) *$B_\rho(i, r(i)) \cap B_\rho(j, r(j)) = \emptyset$ for every $i, j \in F$ with $i \neq j$;*
(b) *$E \subseteq \bigcup_{i\in F} B_\rho(i, 2r(i))$.*

Proof. Set $E_1 = E$ and $r_1 = \max_{i\in E_1} r(i) = r(i_1)$ for some $i_1 \in E_1$. Set $E_2 = E_1 \backslash B(i_1, 2r(i_1))$. If $E_2 = \emptyset$, we take $F = \{i_1\}$ and we are done. If $E_2 \neq \emptyset$ take $r_2 = \max_{i\in E_2} r(i) = r(i_2)$ for some $i_2 \in E_2$. Notice that in this case we have $r_2 \leq r_1$. Assuming that $E_1, E_2, \dots, E_{k-1}$ and $i_1, i_2, \dots, i_{k-1}$ have been constructed, set $E_k = E_{k-1}\backslash B_\rho(i_{k-1}, 2r(i_{k-1}))$. If $E_k \neq \emptyset$ we pick $i_k \in E_k$ such that $r(i_k) = r_k = \max_{i\in E_k} r(i)$. Notice that $r_j \leq r_k$ for $k \leq j$. Otherwise, if $E_k = \emptyset$, taking $F = \{i_i, i_2, \dots, i_{k-1}\}$ we see that $E \subseteq \bigcup_{m=1}^{k-1} B_\rho(i_m, 2r_m)$. On the other hand, if i_ℓ and i_j are two different points in F we have that que $B_\rho(i_\ell, r_\ell) \cap B_\rho(i_j, r_j) = \emptyset$. If fact, if $\ell < j$ and $z \in B_\rho(i_\ell, r_\ell) \cap B_\rho(i_j, r_j)$, then

$$\rho(i_\ell, i_j) \leq \rho(i_\ell, z) + \rho(z, i_j) < r_\ell + r_j \leq 2r_\ell.$$

So that $i_j \in B_\rho(i_\ell, 2r_\ell)$, which is impossible. Since E itself is a finite set, the selection stops. □

Proof of Theorem 1. Let us take $A \in \mathbb{A}$. Set

$$E_\lambda = \bigcup_{r>0} \left\{ i \in [1,L] : \lambda^{-1}\nu(E \cap B_\rho(i,r)) > \nu(B_\rho(i,r)) \right\}.$$

Notice that for each $i \in E_\lambda$ we have a positive number $r_i = r_i(E)$ such that

$$\nu(B_\rho(i, r(i))) < \frac{1}{\lambda}\nu(E \cap B_\rho(i, r(i))).$$

Applying Lemma 2 to this positive real function $r : E_\lambda \to \mathbb{R}^+$ we obtain a finite subset $F = \{i_1, \dots, i_m\}$ of E_λ such that (a) and (b) hold. Hence

$$\begin{aligned} \nu(E_\lambda) &\leq \nu\left(\bigcup_{j\in F} B_\rho(j, 2r(j))\right) \\ &\leq A \sum_{j\in F} \nu(B_\rho(j, r(j))) \\ &< \frac{A}{\lambda} \sum_{j\in F} \nu(E \cap B_\rho(j, r(j))) \\ &\leq \frac{A}{\lambda}\nu(E). \end{aligned}$$

□

Notice that Theorem 1 is nothing but the restricted weak type inequality for the Hardy-Littlewood maximal operator on $([1,L], \rho, \nu)$. In fact, with $f = \mathcal{X}_E$, the set

$$\bigcup_{r>0} \left\{ i \in [1,L] : \frac{\nu(E \cap B_\rho(i,r))}{\nu(B_\rho(i,r))} > \lambda \right\}$$

is the same as $\{i \in [1, L] : Mf(i) > \lambda\}$, where as usual

$$Mf(i) = \sup_{r>0} \frac{1}{\nu(B_\rho(i,r))} \int_{B_\rho(i,r)} |f| \, d\nu.$$

The above situation seems to be very particular because the basic set in which M is defined is the integer interval $[1, L]$. Nevertheless the generality of the above elementary result comes from the generality of the distance ρ. To illustrate this and the type of problems considered here, let us start by the most classical probability space: $X = [0,1]^n$ equipped with the Euclidean distance d and Lebesgue measure m.

Let S_j be the regular dyadic net $2^{-j}\mathbb{Z}^n \cap [0,1)^n = \{x^j_{\boldsymbol{\ell}} : \boldsymbol{\ell} = (\ell_1, \ldots, \ell_n),\ 1 \leq \ell_i \leq 2^j\}$. Since S_j contains 2^{jn} points, let us consider any one to one correspondence between S_j and the integer interval $[1, 2^{jn}]$. In other words we label each point in S_j with an integer number in $[1, 2^{jn}]$, so that $S_j = \{x^j_k : 1 \leq k \leq 2^{jn}\}$. Let μ_j be the probabilistic Borel measure on X supported on S_j, given by $\mu_j(\{x^j_k\}) = 2^{-nj}$, for every $x^j_k \in S_j$. Given a non-negative integer j, let us define a distance ρ_j on the integer interval $[1, 2^{jn}]$ by $\rho_j(k,i) = |x^j_k - x^j_i|$. The measure μ_j also gives rise to a measure ν_j on $[1, 2^{jn}]$, taking $\nu_j(\{k\}) = \mu_j(\{x^j_k\}) = 2^{-nj}$. Of course each

$$\mathbb{A}_j = \left\{A : A \text{ is a doubling constant for } ([1, 2^{nj}], \rho_j, \nu_j)\right\}$$

is non empty for each j. But more than that, it is easy to see that $\bigcap_{j=0}^{\infty} \mathbb{A}_j \neq \emptyset$. Let $\mathcal{A} = \inf \bigcap_{j=0}^{\infty} \mathbb{A}_j$, then the sequence of spaces of homogeneous type $([1, 2^{nj}], \rho_j, \nu_j)$ has $\mathcal{A}$ as a uniform doubling constant. Hence Theorem 1, can be applied to each space $([1, 2^{nj}], \rho_j, \nu_j)$ and (1.1) with $2^{nj}\lambda$ instead of λ, gives

$$\nu_j\left(\left\{k \in [1, 2^{nj}] : 2^{-nj} M_j \mathcal{X}_E(k) > \lambda\right\}\right) \leq \frac{\mathcal{A}}{2^{nj}\lambda} 2^{n^j} H = \frac{\mathcal{A}}{\lambda} H \tag{1.2}$$

for every j and every $\lambda > 0$, where E is a subset of $[1, 2^{nj}]$, H is the number of elements of E and

$$\begin{aligned} M_j \mathcal{X}_E(k) &= \sup_{r>0} \frac{1}{\nu_j(B_{\rho_j}(k,r))} \int_{B_{\rho_j}(k,r)} \mathcal{X}_E(i) \, d\nu_j(i) \\ &= \sup_{r>0} \frac{1}{\nu_j(B_{\rho_j}(k,r))} \nu_j(E \cap B_{\rho_j}(k,r)) \\ &= \sup_{r>0} \frac{\operatorname{card}(E \cap B_{\rho_j}(k,r))}{\operatorname{card}(B_{\rho_j}(k,r))}. \end{aligned}$$

Inequality (1.2) can be restated in (S_j, d, μ_j) in the following way

$$\mu_j\left(\left\{x^j_k \in S_j : \mathcal{M}_j g(x^j_k) > \lambda\right\}\right) \leq \frac{\mathcal{A}}{\lambda} H \tag{1.3}$$

for every $\lambda > 0$, where

$$\mathcal{M}_j f(x^j_k) = \sup_{r>0} \frac{1}{\mu_j(B_d(x^j_k, r))} \int_{B_d(x^j_k, r)} |f(x^j_i)| \, d\mu_j(x^j_i),$$

and $g = \sum_{k \in E} \delta_{x^j_k}$, with $\delta_{x^j_k}$ the "unit mass" at x^k_j given by $2^{-nj}\mathcal{X}_{\{x^j_k\}}$, and $H = \operatorname{card}(E)$.

In other words, Theorem 1 allows to obtain a uniform weak type $(1,1)$ inequality for the approximate Hardy-Littlewood operator on finite sums of Dirac deltas on an increasing approximation of the whole space $X = [0,1]^n$.

A basic question, regarding the above considerations, is wether or not a weak type inequality for a maximal operator of a sequence of integral operators on a general metric measure space (X, d, μ), can be obtained from such uniform weak type inequalities on finite sums of Dirac deltas on finite settings like (S_j, d, μ_j).

The previous works by one of the authors contained in [3] provide the ingredient in order to prove a positive result in this direction. There an extension of the results of Carrillo and de Guzmán concerning the weak type of maximal operators on finite sums of Dirac deltas (see [4]) to spaces of homogeneous type, is given.

A caveat for this program is provided by the example given by Akcoglu-Baxter-Bellow-Jones in [2]: the maximal of a sequence of convolution operators in $\mathbb{Z}$ which is of restricted weak type $(1,1)$ but not of weak type $(1,1)$.

In this note we give sufficient conditions on the metric measure space (X, d, μ) and on the kernel sequence, in such a way that a uniform discrete family of inequalities like (1.3) imply the weak type $(1,1)$ of the maximal operator defined by the given sequence on (X, d, μ).

In Section 2 we state the two results of this note, which are proved in Section 3.

2. Basic notation and statement of the results

Let us start by stating the precise properties of the setting for the result in this paper. Let (X, d) be a complete metric space. Assume that ω is a Borel measure with the following general structure

$$d\omega = w\, d\mu,$$

where w is a locally integrable non-negative function defined on X and μ is a *doubling* regular Borel measure on X. this means that there exists a constant $A > 0$ such that $0 < \mu(B_d(x, 2r)) \leq A\mu(B_d(x, r))$ for every $x \in X$ and $r > 0$. In other words (X, d, μ) is a *space of homogeneous type* with A as a doubling constant.

Let $\{k_\ell : \ell \in \mathbb{N}\}$ be a sequence of continuous kernels with compact support on $X \times X$. Given $f \in L^1(X)$ we define

$$K_\ell f(x) = \int_X k_\ell(x, y) f(y)\, d\omega(y),$$

and

$$K^* f(x) = \sup_\ell |K_\ell f(x)|.$$

Notice that from Fubini-Tonelli's theorem, $K_\ell f(x)$ is well defined for μ-almost every $x \in X$, and then $K^* f$ is a measurable function defined on X.

Let $\{(X_j, \omega_j) : j \in \mathbb{N}\}$ be a sequence of measure spaces such that

(a) each X_j is a Borel subset of X;
(b) $X_j \subseteq X_{j+1}$;
(c) $\bigcup_{j\in\mathbb{N}} X_j$ is dense in X;
(d) $\operatorname{supp} \omega_j \subseteq X_j$;
(e) $\omega_j \to \omega$ in the weak star convergence.

We shall use superscripts to denote points in a particular subspace X_j of X. In other words we write x^j to denote a generic point in X_j. We reserve the subscripts to denote different points in the same space. In the sequel we shall write $x^j_1, \dots, x^j_H$ to denote H points in X_j.

We shall consider two different restrictions to $X_j \times X_j$ of each k_ℓ of the given kernel sequence. The first one is just the usual restriction to $X_j \times X_j$. In other words

$k_\ell^j(x^j, y^j) = k_\ell(x^j, y^j)$ for $x^j, y^j \in X_j$. The second one, avoids the diagonal and is defined by $\mathcal{k}_\ell^j = k_\ell \mathcal{X}_{\triangle_j^c}$, where $\triangle_j^c$ is the complement of the diagonal in $X_j \times X_j$. In other words, $\mathcal{k}_\ell^j(x^j, y^j) = k_\ell(x^j, y^j)$ if x^j and y^j are two different points in X_j, and $\mathcal{k}_\ell^j(x^j, x^j) = 0$. Associated to these kernels we have the corresponding sequences of integral operators and their maximal operators. Precisely

$$K_\ell^j f(x^j) = \int k_\ell^j(x^j, y) f(y)\, d\omega_j(y) = \int_{X_j} k_\ell(x^j, y^j) f(y^j)\, d\omega_j(y^j)\,,$$

and

$$(K^{j^*} f)(x^j) = \sup_\ell \left| K_\ell^j f(x^j) \right|, \tag{2.1}$$

for $f \in L^1(X_j, \omega_j)$. In a similar way, we define

$$\mathcal{K}_\ell^j f(x^j) = \int_X \mathcal{k}_\ell^j(x^j, y) f(y)\, d\omega_j(y) = \int_{X_j - \{x^j\}} k_\ell(x^j, y^j) f(y^j)\, d\omega_j(y^j),$$

and

$$(\mathcal{K}^{j^*} f)(x^j) = \sup_\ell \left| \mathcal{K}_\ell^j f(x^j) \right| . \tag{2.2}$$

The main results of this note are contained in the following statements. The first one proves that the uniform weak type $(1,1)$ of $\mathcal{K}^{j^*}$ over finite sums of Dirac deltas on different points of (X_j, ω_j), is sufficient for the weak type $(1,1)$ of K^* on (X, ω).

Theorem 3. *Assume that $\omega(\{x\}) = 0$ for each $x \in X$. Let (X_j, ω_j) be a sequence satisfying (a) to (e). If there exists a constant C such that for every $\lambda > 0$ and every finite set $x_1^j, x_2^j, \dots, x_H^j$ of different points in X_j, we have*

$$\omega_j\left(\left\{ x^j \in X_j : \sup_{\ell \in \mathbb{N}} \left| \sum_{\substack{i=1,\dots,H \\ x^j \neq x_i^j}} k_\ell(x^j, x_i^j) \right| > \lambda \right\}\right) \le C\, \frac{H}{\lambda} \tag{2.3}$$

for every j, then K^* is of weak type $(1,1)$ on (X, ω).

Corollary 4. *If each X_j is finite, then the uniform restricted weak type $(1,1)$ of the sequence $\mathcal{K}^{j^*}$ in (X_j, ω_j) implies the weak type $(1,1)$ of K^* on (X, ω).*

The second result proves that the uniform weak type $(1,1)$ of K^{j^*} for linear combinations of Dirac deltas with positive integer coefficients on (X_j, ω_j), implies the weak type $(1,1)$ of K^* on (X, ω).

Theorem 5. *Let (X_j, ω_j) be a sequence satisfying (a) to (e). If there exists a constant C such that for every $\lambda > 0$ and every finite set $x_1^j, x_2^j, \dots, x_H^j$ of not necessarily different points in X_j, we have*

$$\omega_j\left(\left\{ x^j \in X_j : \sup_{\ell \in \mathbb{N}} \left| \sum_{i=1}^{H} k_\ell(x^j, x_i^j) \right| > \lambda \right\}\right) \le C\, \frac{H}{\lambda} \tag{2.4}$$

for every j, then K^* is of weak type $(1,1)$ on (X, ω).

Let us point out that the existence of a sequence of finite spaces (X_j, ω_j) as in Corollary 4 is contained in [1, Thm. 4.1] for X compact. Also in Euclidean spaces or even in general settings it is not difficult to build sequences (X_j, ω_j) satisfying those properties.

Let us make some remarks regarding the scope of Theorems 3 and 5. First of all let us point out that since the kernels k_ℓ are integrable, the study of the weak type $(1,1)$ of the associated maximal operator can be reduced to the case of non-negative kernels. With this observation in mind it is clear that the operator $\mathcal{K}^{j^*}$ which is involved in (2.3) is generally smaller than the operator K^{j^*} involved in inequality (2.4).

Not only from this point of view we see that hypothesis (2.3) is weaker than (2.4), but also because the class of "test functions" in Theorem 5 is larger than the class of test functions in Theorem 3. In fact, the former coincides with the class of all linear combinations of Dirac deltas with positive integer coefficients, the later instead is just the class of all finite sums of Dirac deltas on different points. Nevertheless the geometric hypothesis in Theorem 3, $\omega(\{x\}) = 0$ for each $x \in X$, can not be relaxed as the above mentioned example in [2] shows.

On the other hand, for some very classical settings and kernels such as some usual approximate identities on Euclidean spaces, $\mathcal{K}^{j^*}$ behaves much better than K^{j^*}. Precisely, uniform estimates of type (2.3) are possible while uniform estimates of type (2.4) are not.

Notice also that in the atomic case $\mathcal{K}^{j^*}$ does not give a good control of K^*. In fact if $X = X_j = \mathbb{Z}$ with the counting measure, and k_ℓ are supported on the diagonal of $X \times X$, $\mathcal{K}^{j^*}$ vanishes but generally K^* does not.

Finally let us point out that the hypothesis of continuity of the sequence of kernels can be relaxed. For example if there exists a sequence $\{\widetilde{k}_i : i \in \mathbb{N}\}$ of continuous and non-negative kernels such that there exists a constant C satisfying that for every $\ell \in \mathbb{N}$, $|k_\ell| \leq C\widetilde{k}_i$ for some $i \in \mathbb{N}$, and for every $i \in \mathbb{N}$ there exists $\ell \in \mathbb{N}$ such that $\widetilde{k}_i \leq C|k_\ell|$. Then the weak type for the maximal operator associated with the kernels k_ℓ is equivalent to the weak type for the maximal operator associated with the kernels $\widetilde{k}_\ell$.

3. Proof of Theorems 3 and 5

In this section, X, d, μ, ω and k_ℓ are as in Section 2. The main tools in the proof of Theorems 3 and 5 are the following extensions of the above mentioned result of Carrillo and de Guzmán. Its proof is contained in [3].

Lemma 6. (A) *K^* is of weak type $(1,1)$ if and only if there exists a constant $C > 0$ such that for every $\lambda > 0$ the inequality*

$$\omega\left(\left\{x \in X : \sup_\ell \left|\sum_{i=1}^{H} k_\ell(x, x_i)\right| > \lambda\right\}\right) \leq C\,\frac{H}{\lambda} \tag{3.1}$$

holds for every finite sequence $(x_1, x_2, \dots, x_H)$ of points in X.

(B) *If $\omega(\{x\}) = 0$ for every $x \in X$, then K^* is of weak type $(1,1)$ if and only if there exists a constant $C > 0$ such that (3.1) holds for every $\lambda > 0$ and every finite set $\{x_1, x_2, \dots, x_H\}$ of points in X.*

Of course, as usual, the weak type $(1,1)$ of K^* in (A) and (B) of the above lemma means that there exists a constant $\widetilde{C}$ which depends only on C such that

$$\omega\left(\{x \in X : K^*f(x) > \lambda\}\right) \leq \frac{\widetilde{C}}{\lambda}\|f\|_1,$$

for every $f \in L^1$ and every $\lambda > 0$.

Notice that in case (A), when no additional properties are required to the space, since repetition of the x_i's is allowed then (3.1) is equivalent to the weak type $(1,1)$ on the family of all linear combinations of Dirac deltas with positive integer coefficients. In (B), instead, a smaller class of test function is involved. In fact, since in the set $\{x_1, \dots, x_H\}$ we are assuming that $x_i \neq x_j$ if $i \neq j$, the test functions are finite sums of Dirac deltas supported at different points.

The above mentioned example in [2] shows that (B) is not possible in general atomic settings.

Proof of Theorem 3. We shall apply the result contained in (B) of Lemma 3.1. Hence we only have to prove that for every $\lambda > 0$ and every finite set $x_1, x_2, \dots, x_H$ of different points in X, we have

$$\omega\left(\left\{x \in X : \sup_{\ell}\left|\sum_{i=1}^{H} k_\ell(x, x_i)\right| > \lambda\right\}\right) \leq C\,\frac{H}{\lambda}, \tag{3.2}$$

where C is the constant in (2.3). Let us notice first that (3.2) is an immediate consequence of a uniform sequence of inequalities of the type

$$\omega\left(\left\{x \in X : \max_{1\leq\ell\leq N}\left|\sum_{i=1}^{H} k_\ell(x, x_i)\right| > \lambda\right\}\right) \leq C\,\frac{H}{\lambda},$$

with C independent of N. Let us the fix $N \in \mathbb{N}$ and $x_1, x_2, \dots, x_H$, H different points in X. Since $\bigcup_{j=1}^{\infty} X_j$ is dense in X and $X_j \subseteq X_{j+1}$, we can get a j_0 and a set $\{y_1^{j_0}, \dots, y_H^{j_0}\}$ of distinct points in X_{j_0} such that $y_i^{j_0}$ is as close to x_i as we wish, for each $i = 1, \dots, H$. For each $1 \leq \ell \leq N$ we write

$$\sum_{i=1}^{H} k_\ell(x, x_i) = \sum_{i=1}^{H}[k_\ell(x, x_i) - k_\ell(x, y_i^{j_0})] + \sum_{i=1}^{H} k_\ell(x, y_i^{j_0}).$$

Then for each $0 < \alpha < \lambda$ we have

$$\begin{aligned}\left\{x : \max_{1\leq\ell\leq N}\left|\sum_{i=1}^{H} k_\ell(x, x_i)\right| > \lambda\right\} &\subseteq \left\{x : \max_{1\leq\ell\leq N}\left|\sum_{i=1}^{H}[k_\ell(x, x_i) - k_\ell(x, y_i^{j_0})]\right| > \alpha\right\} \\ &\quad \cup \left\{x : \max_{1\leq\ell\leq N}\left|\sum_{i=1}^{H} k_\ell(x, y_i^{j_0})\right| > \lambda - \alpha\right\}\end{aligned}$$

Let us notice that, since each k_ℓ is uniformly continuous on $X \times X$, we can choose $y_i^{j_0}$ in such a way that the first set on the right hand side of the above inclusion becomes empty. With this set $Y = \{y_1^{j_0}, \dots, y_H^{j_0}\}$ so chosen, we have to get an estimate for

$$\omega\left(\left\{x \in X : \max_{1\leq\ell\leq N}\left|\sum_{i=1}^{H} k_\ell(x, y_i^{j_0})\right| > \lambda - \alpha\right\}\right)$$

Set

$$E = \left\{ x \in X : \max_{1 \le \ell \le N} \left| \sum_{i=1}^{H} k_\ell(x, y_i^{j_0}) \right| > \lambda - \alpha \right\} \quad \text{and} \quad E_j = E \cap X_j.$$

We shall prove that $\omega(E) \le C \frac{H}{\lambda - \alpha}$. From (2.3) we have that

$$\omega_j(E_j \backslash Y) \le C \, \frac{H}{\lambda - \alpha}$$

for every $j \ge j_0$. Since E is a bounded open subset of X, from the weak convergence of ω_j to ω and from the regularity of μ, we have that for each $\varepsilon > 0$ the inequality

$$\omega(E) < \omega_j(E) + \varepsilon = \omega_j(E_j) + \varepsilon$$

holds for j large enough. On the other hand, since

$$\omega_j(E_j) = \omega_j(E_j \cap Y) + \omega_j(E_j \backslash Y) \le \omega_j(E_j \cap Y) + C \, \frac{H}{\lambda - \alpha}$$

for $j \ge j_0$, we only have to prove that $\omega_j(E_j \cap Y) \to 0$ when $j \to \infty$. Since $E_j \cap Y \subseteq Y$ which is compact, again from the convergence of ω_j to ω, for each positive ε we have for j large enough

$$0 \le \omega_j(E_j \cap Y) \le \omega_j(Y) < \varepsilon + \omega(Y) = \varepsilon + \sum_{i=1}^{H} \omega(y_i^{j_0}) = \varepsilon.$$

Hence $\omega(E) \le C \frac{H}{\lambda - \alpha}$ for each α positive and less than λ. This proves the desired inequality. □

Proof of Theorem 5. We shall use part (A) of Lemma 6. Assume that $(x_1, \dots, x_H)$ is a sequence of not necessarily different points in X. We shall prove inequality (3.1) with the same constant C as in (2.4), which is our hypothesis. Let us fix a positive λ. Following the lines of the proof of Theorem 3, for each natural N and each $0 < \alpha < \lambda$ there exist $j_0 \in \mathbb{Z}$ and a subset $Y = \{y_1^{j_0}, \dots, y_H^{j_0}\}$ of X_{j_0} in such a way that

$$\max_{1 \le \ell \le N} \left| \sum_{i=1}^{H} [k_\ell(x, x_i) - k_\ell(x, y_i^{j_0})] \right| \le \alpha$$

for each $x \in X$. Hence

$$\omega \left(\left\{ x \in X : \max_{1 \le \ell \le N} \left| \sum_{i=1}^{H} k_\ell(x, x_i) \right| > \lambda \right\} \right) \le \omega(E),$$

with

$$E = \left\{ x \in X : \max_{1 \le \ell \le N} \left| \sum_{i=1}^{H} k_\ell(x, y_i^{j_0}) \right| > \lambda - \alpha \right\}.$$

Let us prove that $\omega(E) \le C \frac{H}{\lambda - \alpha}$. In fact, if $E_j = E \cap X_j$ from (2.4) we have that

$$\omega_j \left(E_j \right) \le C \, \frac{H}{\lambda - \alpha},$$

for $j \ge j_0$. Since E is a bounded open set, given $\varepsilon > 0$ there exists $j_1 = j_1(\varepsilon)$ such that

$$\omega(E) < \omega_j(E) + \varepsilon = \omega_j(E_j) + \varepsilon,$$

for $j \geq j_1$. So that for j large enough

$$\omega(E) < C\,\frac{H}{\lambda-\alpha} + 2\varepsilon,$$

which proves the theorem by letting first $\varepsilon \to 0$ and then $\alpha \to 0$. □

We shall conclude this section by briefly showing how (1.3) implies (2.3) with $X = [0,1]^n$, d the Euclidean distance, $\omega = m$ is Lebesgue measure on X, $X_j = S_j$ as in Section 1, $\omega_j = \mu_j$ and $k_\ell(x,y) = \frac{1}{m(B_d(x,2^{-\ell}))}\mathcal{X}_{B_d(x,2^{-\ell})}(y)$ for each $\ell \in \mathbb{N}$. Let us point out that even when the kernels $k_\ell(x,y)$ are not continuous, we can apply the remark at the end of Section 2 with

$$\widetilde{k}_\ell(x,y) = \frac{\varphi\left(2^\ell|x-y|\right)}{\int \varphi\left(2^\ell|x-z|\right)\,d(z)}$$

where φ is the continuous function defined on the non-negative real numbers by $\varphi(t) = 1$ for every t in the interval $[0,1]$, $\varphi(t) = 0$ if $t \geq 2$, and linear on $[1,2]$. It is not difficult to show that each $\widetilde{k}_\ell$ is continuous and that $2^{-n}k_\ell(x,y) \leq \widetilde{k}_\ell(x,y) \leq 2^n k_{\ell-1}(x,y)$.

In order to show that (1.3) implies (2.3), notice that (1.3) takes the following form

$$2^{-nj}\mathrm{card}\left(\left\{x_k^j \in S_j : \sup_{\ell\in\mathbb{N}} 2^{nj}\frac{\mathrm{card}\left(E\cap B_d(x_k^j,2^{-\ell})\right)}{\mathrm{card}\left(S_j\cap B_d(x_k^j,2^{-\ell})\right)} > \lambda\right\}\right) \leq \mathcal{A}\frac{H}{\lambda},$$

for every subset $E = \{x_{k_1}^j,\dots,x_{k_H}^j\}$ of S_j, every $j \in \mathbb{N}$ and every $\lambda > 0$. On the other hand (2.3) for this particular situation reads

$$2^{-nj}\mathrm{card}\left(\left\{x_k^j \in S_j : \sup_{\ell\in\mathbb{N}} \frac{1}{m(B_d(x_k^j,2^{-\ell}))}\sum_{\substack{i=1,\dots,H\\ x_{k_i}^j\neq x_k^j}} \mathcal{X}_{B_d(x_k^j,2^{-\ell})}(x_{k_i}^j) > \lambda\right\}\right) \leq C\frac{H}{\lambda}.$$

Hence the desired result shall be a consequence of the following inequalities:

$$\frac{1}{m(B_d(x_k^j,2^{-\ell}))}\sum_{\substack{i=1,\dots,H\\ x_{k_i}^j\neq x_k^j}} \mathcal{X}_{B_d(x_k^j,2^{-\ell})}(x_{k_i}^j) \leq 2^{n(j+1)}\frac{\mathrm{card}\left(E\cap B_d(x_k^j,2^{-\ell})\right)}{\mathrm{card}\left(S_j\cap B_d(x_k^j,2^{-\ell})\right)}, \tag{3.3}$$

for every $j,\ell \in \mathbb{N}$, every choice of $E = \{x_{k_1}^j,\dots,x_{k_H}^j\} \subseteq S_j$ and every generic point x_k^j in S_j.

In order to prove (3.3) ley us divide the analysis in two cases according to the relative sizes of j and ℓ. If $\ell > j$, there is nothing to prove since the left hand side in (3.3) vanishes because, in this case, the only point of S_j in $B_d(x_k^j,2^{-\ell})$ is x_k^j itself. Assume now that $\ell \leq j$. In this case we have that

$$\begin{aligned} 2^{-nj}\mathrm{card}\left(S_j\cap B_d(x_k^j,2^{-\ell})\right) &= \sum_{x_i^j\in S_j\cap B_d(x_k^j,2^{-\ell})} m(B_{d_\infty}(x_i^j,2^{-j})) \\ &\leq m(B_d(x_k^j,2^{-\ell+1})) \\ &\leq 2^n m(B_d(x_k^j,2^{-\ell})). \end{aligned}$$

Since clearly

$$\sum_{\substack{i=1,\dots,H\\ x^j_{k_i}\neq x^j_k}} \mathcal{X}_{B_d(x^j_k,2^{-\ell})}(x^j_{k_i}) \leq \operatorname{card}\left(\widetilde{E}\cap B_d(x^j_k,2^{-\ell})\right),$$

we have (3.3).

Applying Theorem 3 we obtain the weak type $(1,1)$ on $([0,1]^n,|\cdot|,m)$ for the maximal operator K^* associated with the sequence $\{k_\ell\}$, with constant for the weak type inequality which only depends on n. By standard homogeneity arguments for the Hardy-Littlewood maximal operator, this result extends to any cube of the form $[-2^i,2^i]^n$ with the same constant for every $i\in\mathbb{N}$. This implies the weak type $(1,1)$ of K^* on $(\mathbb{R}^n,|\cdot|,m)$ and hence the weak type $(1,1)$ of the classical Hardy-Littlewood maximal operator.

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Instituto de Matemática Aplicada del Litoral (CONICET-UNL), Departamento de Matemática (FIQ-UNL), Santa Fe, Argentina.
E-mail address: haimar@santafe-conicet.gov.ar

Instituto de Matemática Aplicada del Litoral (CONICET-UNL), Departamento de Matemática (FIQ-UNL), Santa Fe, Argentina.
E-mail address: mcarena@santafe-conicet.gov.ar

Instituto de Matemática Aplicada del Litoral (CONICET-UNL), Departamento de Matemática (FHUC-UNL), Santa Fe, Argentina.
E-mail address: biaffei@santafe-conicet.gov.ar