REGULARITY OF THE SCHRÖDINGER EQUATION FOR THE HARMONIC OSCILLATOR

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ABSTRACT. We consider the Schrödinger equation for the harmonic oscillator $i\partial_t u = Hu$, where $H = -\Delta + |x|^2$, with initial data in the Hermite–Sobolev space $H^{-s/2}L^2(\mathbb{R}^n)$. We obtain smoothing and maximal estimates and apply these to perturbations of the equation and almost everywhere convergence problems.

1. INTRODUCTION

We consider the regularity of the Schrödinger equation

$$i\partial_t u = Hu \tag{1}$$

with initial data $u(\,\cdot\,,0) = f$, where H is the Hermite operator defined by

$$H = -\Delta + |x|^2, \quad x \in \mathbb{R}^n. \tag{2}$$

This is an important model in quantum mechanics (see for example [7]).

The trigonometric polynomials are the eigenfunctions of Δ, and this is what makes the Fourier transform such an effective tool to attack the free equation, $i\partial_t u = -\Delta u$. Similarly, this enables us to measure the smoothness of the initial data with the fractional Sobolev spaces $W^{s,2}(\mathbb{R}^n) = (I - \Delta)^{-s/2}L^2(\mathbb{R}^n)$ defined via the Fourier transform.

The Schrödinger equation (1) has been considered with respect to these spaces (see for example [29]), however, the eigenfunctions of H are the Hermite functions which are also dense in $L^2(\mathbb{R}^n)$, and so it is often more efficient to decompose the initial data with these. Similarly, it seems in some sense more natural to measure the 'smoothness' of the initial data in the Hermite–Sobolev space $\mathcal{H}^s(\mathbb{R}^n) = H^{-s/2}L^2(\mathbb{R}^n)$.

Although the spectrum of H is discrete, recalling the free equation with periodic data (see for example [13]), our results will generally bear more resemblance to those for the nonperiodic free equation. In particular, by applying the projection estimates of Karadzov [8] and Koch–Tataru [12], in Section 3 we obtain 'smoothing' estimates which are unavailable in the periodic case.

In Section 4, we combine these estimates with the Strichartz estimates [10] and Wainger's Sobolev embedding theorem [27] to obtain the following result.

Theorem 1. *Let* $p \in [\frac{2(n+2)}{n}, \infty]$, $q \in [2, \infty)$, $\frac{n}{p} + \frac{2}{q} \leqslant \frac{n}{2}$ *and*

$$s(p,q) = n\Big(\frac{1}{2} - \frac{1}{p}\Big) - \frac{2}{q}.$$

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Then

$$\|e^{-itH}f\|_{L^p_x(\mathbb{R}^n,\,L^q_t[0,2\pi])} \leqslant C_s\|f\|_{\mathcal{H}^s(\mathbb{R}^n)}, \quad s \geqslant s(p,q), \tag{3}$$

and this is false when $s < s(p,q)$.

In Section 4 we also deduce the following almost everywhere convergence properties. In one spatial dimension, this is a consequence of Theorem 1, and in higher dimensions it follows from a second smoothing estimate that is proved in Section 3.

Corollary 1. *Let* $f \in \mathcal{H}^s(\mathbb{R}^n)$ *with* $s > 1/3$ *if* $n = 1$*, or* $s > 1/2$ *if* $n \geqslant 2$*. Then*

$$\lim_{t\to 0} e^{-itH}f(x) = f(x) \qquad a.e. \quad x \in \mathbb{R}^n.$$

Cowling [6] proved this convergence for data in $\mathcal{H}^s(\mathbb{R}^n)$ with $s > 1$. In one spatial dimension, this was improved by Torrea and the first author [2] (see [3] for a Laguerre version) to include data in $\mathcal{H}^s(\mathbb{R})$ with $s > 1/2$.

By a theorem of Thangavelu [21], $f \in W^{s,2}(\mathbb{R}^n)$ with compact support also belongs to $\mathcal{H}^s(\mathbb{R}^n)$, thus we recover a weaker version of the almost everywhere convergence result of Yajima [29] for data in $W^{s,2}(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ with $s > 1/2$.

Corollary 1 has subsequently been improved by Sjögren and Torrea [17] in one spatial dimension. They have proven that the convergence holds for data in either $W^{\frac14,2}(\mathbb{R})$ or $\mathcal{H}^{\frac14}(\mathbb{R})$, and this is sharp in the sense that for lower regularities the convergence is not guaranteed in either space.

Finally, in Section 5, we consider a perturbation of the equation (1) of the form

$$\begin{cases} i\frac{du}{dt} = \Big(-\Delta + |x|^2 + V(x)\Big)\, u \\ u(\,\cdot\,,0) = u_0 \in L^2(\mathbb{R}^n). \end{cases}$$

We prove that global existence of a solution is guaranteed when $n \geqslant 2$ and $\|V\|_{L^{n/2}}$ is sufficiently small. For $n \geqslant 3$ this can also be obtained via Theorem 1 combined with the arguments of Yajima [28].

Throughout, c and C will denote positive constants that may depend on the dimension n. Their values may change from line to line.

2. Preliminaries

In one dimension, the Hermite polynomials H_k are defined by

$$H_k(x) = (-1)^k\, e^{x^2}\frac{d^k}{dx^k}(e^{-x^2}), \quad x \in \mathbb{R},$$

and by normalization we obtain the Hermite functions,

$$h_k(x) = \frac{e^{-x^2/2}H_k(x)}{(\pi^{1/2}2^k k!)^{1/2}}, \quad x \in \mathbb{R}.$$

In higher dimensions, for each multi-index $\mathbf{k} = (k_j)_{j=1}^n \in \mathbb{N}_0^n$, the Hermite functions $h_{\mathbf{k}}$ are defined by

$$h_{\mathbf{k}}(x) = \prod_{j=1}^n h_{k_j}(x_j), \quad x = (x_1,\ldots,x_n) \in \mathbb{R}^n.$$

These are the eigenvectors of the Hermite operator defined in (2). In fact

$$Hh_{\mathbf{k}} = (2|\mathbf{k}| + n)\, h_{\mathbf{k}},$$

where $|\mathbf{k}| = \sum_{j=1}^n k_j$.

We consider the space of finite linear combinations of Hermite functions $\mathfrak{F}(\mathbb{R}^n)$,

$$f = \sum_{\mathbf{k}\in\mathbb{N}_0^n:\,|\mathbf{k}|\leqslant N} a_{\mathbf{k}} h_{\mathbf{k}},$$

where $a_{\mathbf{k}}$ are the Fourier–Hermite coefficients

$$a_{\mathbf{k}} = \int_{\mathbb{R}^n} f(x) h_{\mathbf{k}}(x) dx.$$

These are dense in $L^2(\mathbb{R}^n)$, and so, by the orthonormality of the Hermite functions,

$$\|f\|_{L^2(\mathbb{R}^n)} = \left(\sum_{\mathbf{k}\in\mathbb{N}_0^n} |a_{\mathbf{k}}|^2\right)^{1/2}, \tag{4}$$

and the Hermite–Sobolev norm is defined accordingly,

$$\|f\|_{\mathcal{H}^s(\mathbb{R}^n)} = \left(\sum_{\mathbf{k}\in\mathbb{N}_0^n} (2|\mathbf{k}|+n)^s |a_{\mathbf{k}}|^2\right)^{1/2}.$$

For initial data $f \in \mathfrak{F}(\mathbb{R}^n)$, the solution to the Schrödinger equation (1) can be written

$$e^{-itH} f = \sum_{\mathbf{k}\in\mathbb{N}_0^n} e^{-it(2|\mathbf{k}|+n)} a_{\mathbf{k}} h_{\mathbf{k}}. \tag{5}$$

Note that the solution is periodic in time. Comparing (4) with (5) we see that

$$\|e^{-itH} f\|_{L^2(\mathbb{R}^n)} = \|f\|_{L^2(\mathbb{R}^n)}, \quad t \in \mathbb{R}. \tag{6}$$

Finally, by the Mehler formula we also have the integral representation

$$e^{-itH} f(x) = \int_{\mathbb{R}^n} K_t(x,y)\, f(y)\, dy, \quad t \neq \frac{j\pi}{2},\ j \in \mathbb{Z}, \tag{7}$$

where

$$K_t(x,y) = \frac{1}{[2\pi i \sin(2t)]^{n/2}} \exp\left(\frac{i}{2}|x-y|^2 \cot(2t) - ix\cdot y \tan(t)\right).$$

3. Smoothing estimates

For the free equation, Kenig, Ponce and Vega [11] proved the sharp estimate

$$\sup_{x\in\mathbb{R}} \|e^{it\Delta} f(x)\|_{L^2_t(\mathbb{R})} \leqslant C\|f\|_{\dot{W}^{-\frac{1}{2},2}(\mathbb{R})},$$

where $\dot{W}^{s,2}(\mathbb{R}^n)$ denotes the homogeneous Sobolev space $(-\Delta)^{-s/2} L^2(\mathbb{R}^n)$. The estimate is false when the homogeneous space is replaced by the inhomogeneous one. For the harmonic oscillator, we prove something similar. Note that the spectrum of H is bounded away from the origin, so there is no distinction between the homogeneous and inhomogeneous Hermite–Sobolev spaces.

In order to get a global bound in space with no decay, we lose some regularity with respect to the free equation. The relationship between the decay and the regularity is sharp however. To see this, consider $f = h_k$, so that the inequality in the following proof can be reversed.

Theorem 2. *Let $1/6 \leqslant s \leqslant 1/2$. Then*

$$\sup_{x\in\mathbb{R}} (1+|x|)^{1/6-s} \|e^{-itH} f(x)\|_{L^2_t[0,2\pi]} \leqslant C_s \|f\|_{\mathcal{H}^{-s}(\mathbb{R})}.$$

Proof. As $\mathfrak{F}(\mathbb{R}) = H^{s/2}\mathfrak{F}(\mathbb{R})$ is dense in $\mathcal{H}^{-s}(\mathbb{R})$, it will suffice to consider $f \in \mathfrak{F}(\mathbb{R}^n)$ and we write $f = \sum_{k\in\mathbb{N}_0} a_k h_k$. Observe that by the orthogonality of the trigonometric polynomials,

$$\begin{aligned}\|e^{-itH} f(x)\|^2_{L^2_t[0,2\pi]} &= \int_0^{2\pi} \left(\sum_{j\in\mathbb{N}_0} e^{-it(2j+1)} a_j h_j(x)\right)\left(\sum_{k\in\mathbb{N}_0} e^{it(2k+1)} \overline{a}_k h_k(x)\right) dt \\ &= 2\pi \sum_{k\in\mathbb{N}_0} |a_k|^2 h_k^2(x).\end{aligned}$$

We use the following property of the Hermite functions which can be found in [20, pp. 26 , Lemma 1.5.1]: There exists a constant c such that

$$h_k(x) \leqslant c k^{-1/4}, \quad x \in [-R,R], \quad k \geqslant R^2. \tag{8}$$

Combining this with a second property which can be found in [19, pp. 242, Theorem 8.91.3] we obtain: Let $0 \leqslant \alpha \leqslant 1/3$. Then there exist constants c_0 and k_0 such that

$$c_0^{-1} k^{-\alpha/2-1/12} \leqslant \sup_{x\in\mathbb{R}} (1+|x|)^{-\alpha} h_k(x) \leqslant c_0 k^{-\alpha/2-1/12}, \quad k \geqslant k_0. \tag{9}$$

Thus, interchanging the sum and the supremum,

$$\sup_{x\in\mathbb{R}} (1+|x|)^{-2\alpha} \|e^{-itH} f(x)\|^2_{L^2_t[0,2\pi]} \leqslant 2\pi c_0^2 \sum_{k\in\mathbb{N}_0} \frac{(2k_0+1)^{\alpha+1/6}}{(2k+1)^{\alpha+1/6}} |a_k|^2.$$

Finally, by writing $s = \alpha + 1/6$, and taking the square root,

$$\sup_{x\in\mathbb{R}} (1+|x|)^{1/6-s} \|e^{-itH} f(x)\|_{L^2_t[0,2\pi]} \leqslant C_s \left(\sum_{k\in\mathbb{N}_0} (2k+1)^{-s} |a_k|^2\right)^{1/2},$$

as desired. □

For the free equation, Vega [16,24] (see also [9], [14], [30]) proved that for $n \geqslant 2$ and $p \geqslant \frac{2(n+1)}{n-1}$,

$$\|e^{it\Delta} f\|_{L^p_x(\mathbb{R}^n,\, L^2_t(\mathbb{R}))} \leqslant C_s \|f\|_{\dot{W}^{s,2}(\mathbb{R}^n)}, \quad s = n\Big(\frac{1}{2} - \frac{1}{p}\Big) - 1.$$

Note that s is negative in the range $p \in [\frac{2(n+1)}{n-1}, \frac{2n}{n-2})$.

In the following theorem we again lose some regularity with respect to the free equation, however we will see that it is sharp.

Theorem 3. *Let $n \geqslant 2$, $p \geqslant 2$ and*

$$s(p) = \begin{cases} \frac{1}{p} - \frac{1}{2}, & 2 \leqslant p \leqslant \frac{2(n+3)}{n+1} \\ \frac{n}{3}\Big(\frac{1}{2} - \frac{1}{p}\Big) - \frac{1}{3}, & \frac{2(n+3)}{n+1} \leqslant p \leqslant \frac{2n}{n-2} \\ n\Big(\frac{1}{2} - \frac{1}{p}\Big) - 1, & \frac{2n}{n-2} \leqslant p \leqslant \infty. \end{cases}$$

Then

$$\|e^{-itH} f\|_{L^p_x(\mathbb{R}^n,\, L^2_t[0,2\pi])} \leqslant C_s \|f\|_{\mathcal{H}^s(\mathbb{R}^n)}, \quad s \geqslant s(p),$$

and this is false when $s < s(p)$.

Proof. By density, it will suffice to consider $f \in \mathfrak{F}(\mathbb{R}^n)$ and we write $f = \sum_{\mathbf{k}\in\mathbb{N}_0^n} a_\mathbf{k} h_\mathbf{k}$. As before,

$$\begin{aligned}\|e^{-itH}f\|^2_{L^2_t[0,2\pi]} &= \int_0^{2\pi} \left(\sum_{\mathbf{j}\in\mathbb{N}_0^n} e^{-it(2|\mathbf{j}|+n)} a_\mathbf{j} h_\mathbf{j}\right)\left(\sum_{\mathbf{k}\in\mathbb{N}_0^n} e^{it(2|\mathbf{k}|+n)} \overline{a}_\mathbf{k} h_\mathbf{k}\right) dt \\ &= 2\pi \left(\sum_{\mathbf{j},\mathbf{k}\,:\; 2|\mathbf{k}|+n=2|\mathbf{j}|+n} a_\mathbf{j}\, \overline{a}_\mathbf{k} h_\mathbf{j}\, h_\mathbf{k}\right) \\ &= 2\pi \left(\sum_{\lambda\in\mathbb{N}} \sum_{\mathbf{j}\,:\,2|\mathbf{j}|+n=\lambda} \sum_{\mathbf{k}\,:\,2|\mathbf{k}|+n=\lambda} a_\mathbf{j}\, \overline{a}_\mathbf{k} h_\mathbf{j}\, h_\mathbf{k}\right).\end{aligned}$$

We recall the spectral projection operators P_λ defined by

$$P_\lambda f(x) = \sum_{2|\mathbf{k}|+n=\lambda} a_\mathbf{k} h_\mathbf{k}(x).$$

We see that

$$\|e^{-itH}f\|_{L^2_t[0,2\pi]} = (2\pi)^{1/2}\left(\sum_{\lambda\in\mathbb{N}} P_\lambda f \overline{P_\lambda f}\right)^{1/2} = (2\pi)^{1/2}\left(\sum_{\lambda\in\mathbb{N}} |P_\lambda f|^2\right)^{1/2},$$

and by Minkowski's inequality in $L^{p/2}_x$,

$$\|e^{-itH}f\|_{L^p_x(\mathbb{R}^n,\, L^2_t[0,2\pi])} \leqslant (2\pi)^{1/2}\left(\sum_{\lambda\in\mathbb{N}} \|P_\lambda f\|^2_{L^p(\mathbb{R}^n)}\right)^{1/2}. \tag{10}$$

Now by combining the results of Karadzhov [8, pp. 108, Theorem 3] and Koch–Tataru [12, pp. 376, Corollary 3.2], we have the sharp projection estimates

$$\|P_\lambda f\|^2_{L^p(\mathbb{R}^n)} \leqslant C\lambda^{s(p)}\|P_\lambda f\|^2_{L^2(\mathbb{R}^n)}, \tag{11}$$

where $s(p)$ is as in the statement of the theorem. By orthogonality,

$$\|P_\lambda f\|^2_{L^2(\mathbb{R}^n)} = \sum_{\mathbf{k}\,:\,2|\mathbf{k}|+n=\lambda} |a_\mathbf{k}|^2,$$

so that using (11) we see that

$$\sum_{\lambda\in\mathbb{N}} \|P_\lambda f\|^2_{L^p(\mathbb{R}^n)} \leqslant \sum_{\lambda\in\mathbb{N}} \lambda^s \|P_\lambda f\|^2_{L^2(\mathbb{R}^n)} = \|f\|^2_{\mathcal{H}^s(\mathbb{R}^n)}. \tag{12}$$

The argument is completed by substituting (12) into (10).

To see that these estimates are sharp we observe that $|e^{-itH}P_\lambda f| = |P_\lambda f|$ so that $\|e^{-itH}P_\lambda f\|_{L^2_t[0,2\pi]} = (2\pi)^{1/2}|P_\lambda f|$. Thus, an improvement of the previous estimate would yield improved estimates for the spectral projection operator, which is not possible (see [12]). □

For the free equation, Vega [24,25] (see also [11]) proved that for all $\alpha > 1$,

$$\left(\int_{-\infty}^{\infty}\int_{\mathbb{R}^n} |e^{it\Delta}f(x)|^2 \frac{dxdt}{(1+|x|)^\alpha}\right)^{1/2} \leqslant C_\alpha \|f\|_{\dot{W}^{-\frac{1}{2},2}(\mathbb{R}^n)}.$$

On the other hand, Sjölin [18] and Constantin–Saut [5] independently proved a similar estimate for data in the inhomogeneous Sobolev space. This was subsequently refined by Ben-Artzi–Klainerman [1] and Kato–Yajima [9] for $n \geqslant 2$, so that for all $\alpha > 2$,

$$\left(\int_{-\infty}^{\infty}\int_{\mathbb{R}^n} |e^{it\Delta}f(x)|^2 \frac{dxdt}{(1+|x|)^\alpha}\right)^{1/2} \leqslant C_\alpha \|f\|_{W^{-\frac{1}{2},2}(\mathbb{R}^n)}, \tag{13}$$

and this is false when $\alpha < 2$ (see [27]). In an involved argument, Yajima [29] proved that if one integrates over a compact integral of time, then (13) holds for $\alpha > 1$ with Δ replaced by a class of operators that includes both Δ and H. Considering $\mathcal{H}^{-\frac{1}{2}}(\mathbb{R}^n)$ instead of $W^{-\frac{1}{2},2}(\mathbb{R}^n)$ enables the following simple proof more in the spirit of [11].

Theorem 4. *For all $\alpha > 1$,*

$$\left(\int_0^{2\pi}\int_{\mathbb{R}^n} |e^{-itH}f(x)|^2 \frac{dxdt}{(1+|x|)^\alpha}\right)^{1/2} \leqslant C_\alpha \|f\|_{\mathcal{H}^{-s}(\mathbb{R}^n)}, \quad s \geqslant 1/2,$$

and this is false if $s < 1/2$.

Proof. By density, it will suffice to consider $f \in \mathfrak{F}(\mathbb{R}^n)$ and we write $f = \sum_{\mathbf{k}\in\mathbb{N}_0^n} a_{\mathbf{k}} h_{\mathbf{k}}$. Observe that by the orthogonality of the trigonometric polynomials,

$$\begin{aligned}\|e^{-itH}f\|^2_{L^2_t[0,2\pi]} &= \int_0^{2\pi}\left(\sum_{\mathbf{j}\in\mathbb{N}_0^n} e^{-it(2|\mathbf{j}|+n)} a_{\mathbf{j}} h_{\mathbf{j}}\right)\left(\sum_{\mathbf{k}\in\mathbb{N}_0^n} e^{it(2|\mathbf{k}|+n)} \overline{a}_{\mathbf{k}} h_{\mathbf{k}}\right) dt\\ &= 2\pi\left(\sum_{\mathbf{j},\mathbf{k}\,:\,j_1=k_1+|\overline{\mathbf{k}}|-|\overline{\mathbf{j}}|} a_{\mathbf{j}}\, \overline{a}_{\mathbf{k}} h_{j_1} h_{k_1} h_{\overline{\mathbf{j}}}\, h_{\overline{\mathbf{k}}}\right),\end{aligned}$$

where $\overline{\mathbf{j}} = (j_2,\ldots,j_n)$ and $\overline{\mathbf{k}} = (k_2,\ldots,k_n)$. By Fubini's theorem,

$$\begin{aligned}&\int_0^{2\pi}\int_{[-R,R]\times\mathbb{R}^{n-1}} |e^{-itH}f(x)|^2 dxdt\\ =&2\pi\left(\sum_{\mathbf{j},\mathbf{k}\,:\,j_1=k_1+|\overline{\mathbf{k}}|-|\overline{\mathbf{j}}|} a_{\mathbf{j}}\, \overline{a}_{\mathbf{k}} \int_{-R}^{R} h_{j_1}(x_1)h_{k_1}(x_1)dx_1 \int_{\mathbb{R}^{n-1}} h_{\overline{\mathbf{j}}}(\overline{x})h_{\overline{\mathbf{k}}}(\overline{x})d\overline{x}\right),\end{aligned}$$

so that, by the orthonormality of the Hermite functions in $n-1$ variables,

$$\int_0^{2\pi}\int_{[-R,R]\times\mathbb{R}^{n-1}} |e^{-itH}f(x)|^2 dxdt = 2\pi\sum_{\mathbf{k}} |a_{\mathbf{k}}|^2 \int_{-R}^{R} h^2_{k_1}(x_1)dx_1.$$

Of course, we can repeat the argument for each variable, and so for $i = 1,\ldots,n$,

$$\int_0^{2\pi}\int_{[-R,R]^n} |e^{-itH}f(x)|^2 dxdt \leqslant 2\pi\sum_{\mathbf{k}} |a_{\mathbf{k}}|^2 \int_{-R}^{R} h^2_{k_i}(x_i)dx_i. \tag{14}$$

Now by property (8),

$$\int_{-R}^{R} h^2_{k_i}(x_i)\,dx_i \leqslant C\frac{R}{k_i^{1/2}}.$$

Note that the inequality follows from the orthonormality of the Hermite functions when $k_i^{1/2} \leqslant R$. Substituting into (14), we see that

$$\int_0^{2\pi}\int_{[-R,R]^n} |e^{-itH}f(x)|^2 dxdt \leqslant CR\sum_{\mathbf{k}}(2k_i+1)^{-1/2}|a_{\mathbf{k}}|^2. \tag{15}$$

Now we can decompose our function $f=\sum_{i=1}^n f_i$, where $f_i=\sum_{\mathbf{k}} a_{\mathbf{k}}^i h_{\mathbf{k}}$ and

$$a_{\mathbf{k}}^i = \begin{cases} a_{\mathbf{k}}, & k_i \geqslant k_j \text{ for all } j\neq i, \text{ and } k_i\neq k_j \text{ for all } j<i \\ 0, & \text{otherwise.}\end{cases}$$

By (15), we see that for $i=1,\dots,n$,

$$\begin{aligned}\int_0^{2\pi}\int_{[-R,R]^n} |e^{-itH}f_i(x)|^2 dxdt &\leqslant CR\sum_{\mathbf{k}}(2k_i+1)^{-1/2}|a_{\mathbf{k}}^i|^2\\ &\leqslant Cn^{1/2}R\sum_{\mathbf{k}}(2|\mathbf{k}|+n)^{-1/2}|a_{\mathbf{k}}^i|^2,\end{aligned}$$

where we have used the fact that $nk_i \geqslant |\mathbf{k}|$ when $a_{\mathbf{k}}^i\neq 0$. By Minkowski's inequality followed by Cauchy–Schwarz,

$$\left(\int_0^{2\pi}\int_{[-R,R]^n} |e^{-itH}f(x)|^2 dxdt\right)^{1/2} \leqslant Cn^{3/4}R^{1/2}\left(\sum_{\mathbf{k}}(2|\mathbf{k}|+n)^{-1/2}|a_{\mathbf{k}}|^2\right)^{1/2},$$

and the result follows by summing dyadic pieces.

To see that the estimate is sharp with respect to the regularity, we consider g_N defined by

$$g_N(x)=h_{4N}(x_1)h_0(x_2)\dots h_0(x_n).$$

Note that

$$\begin{aligned}\|e^{-itH}g_N\|^2_{L^2([0,2\pi]\times[0,1]^n)} &= 2\pi\left(\int_0^1 h_{4N}^2(x_1)dx_1\int_0^1 e^{-x_2^2}dx_2\dots\int_0^1 e^{-x_n^2}dx_n\right)\\ &= C\int_0^1 h_{4N}^2(x_1)dx_1.\end{aligned}$$

Now by the following Lemma 1, h_{4k} take values $\approx k^{-1/4}$ when x belongs to one of $\approx k^{1/2}$ subintervals of $[0,1]$ of length $k^{-1/2}$. Thus

$$\int_0^1 h_{4k}^2(x)dx \geqslant ck^{-1/2},$$

so that

$$\|e^{-itH}g_N\|_{L^2([0,1]^n\times[0,2\pi])} \geqslant CN^{-1/4}.$$

Now as $\|g_N\|_{\mathcal{H}^s(\mathbb{R}^n)}=(8N+n)^{s/2}$, letting N tend to infinity, we see that $s\geqslant -1/2$ is a necessary condition. $\square$

It would be interesting to know if the previous theorem is sharp with respect to α, however we do not know. To see that it was sharp with respect to the regularity, we used the following lemma which we now prove.

Lemma 1. *Let $I_m \subset [0,1]$ denote intervals of length $\frac{1}{\sqrt{k}}$, centered at $x_m = \frac{\sqrt{2}\pi m}{\sqrt{k}}$. Then there exist positive constants c_0, k_0 and μ such that*

$$c_0^{-1}k^{-1/4} \leqslant h_{4k}(x) \leqslant c_0 k^{-1/4}$$

for all $k \geqslant k_0$ when $x \in I_m$ and $m = \lfloor \sqrt{k}/\mu \rfloor, \dots, \lfloor \sqrt{2k}/\mu \rfloor$.

Proof. For k an even integer, there is an explicit formula for the Hermite functions given by

$$h_k(x) = \frac{2}{\pi^{3/4}}(-1)^{k/2}\frac{2^{k/2}}{\sqrt{k!}}e^{\frac{x^2}{2}}\int_0^\infty e^{-s^2}s^k\cos(2xs)ds \tag{16}$$

(see [19, pp. 107]). Note that by the formula for the Gamma function and a change of variables,

$$\int_0^\infty e^{-s^2}s^k\cos(2xs)ds \leqslant \int_0^\infty e^{-s^2}s^k ds = \frac{1}{2}\Gamma\left(\frac{k+1}{2}\right).$$

We will see later that this bound will suffice to provide the upper bound, so we concentrate on the lower bound.

Consider an interval I_m of length $\frac{1}{\sqrt{k}}$ with center $x_m = \frac{\sqrt{2}\pi m}{\sqrt{k}}$, where

$$m = \lfloor \sqrt{k}/\mu \rfloor, \dots, \lfloor \sqrt{2k}/\mu \rfloor,$$

with μ to be chosen later. We split the integral

$$\begin{aligned}\int_0^\infty e^{-s^2}s^k\cos(2xs)ds &= \int_0^{\sqrt{\frac{k}{2}}(1-\frac{1}{8m})} + \int_{\sqrt{\frac{k}{2}}(1-\frac{1}{8m})}^{\sqrt{\frac{k}{2}}} + \int_{\sqrt{\frac{k}{2}}}^{\sqrt{\frac{k}{2}}(1+\frac{1}{8m})} + \int_{\sqrt{\frac{k}{2}}(1+\frac{1}{8m})}^\infty \\ &=: I_1 + I_2 + I_3 + I_4.\end{aligned}$$

The function $e^{-s^2}s^k$ attains its unique local maximum when $s = \sqrt{k/2}$, and so is positive and increasing in $(0, \sqrt{k/2}(1-\frac{1}{8m}))$. By the second mean value theorem for integrals, there exists a point c such that

$$\begin{aligned}|I_1| &\leqslant e^{-s^2}s^k\Big|_{s=\sqrt{\frac{k}{2}}(1-\frac{1}{8m})}\int_c^{\sqrt{\frac{k}{2}}(1-\frac{1}{8m})}\cos(2xs)ds \\ &\leqslant e^{-\frac{k}{2}(1-\frac{1}{8m})^2}\left(\frac{k}{2}\right)^{\frac{k}{2}}\left(1-\frac{1}{8m}\right)^k\frac{1}{x}.\end{aligned}$$

Squaring out and using the fact that $m \leqslant \lfloor \sqrt{2k}/\mu \rfloor$ and $1/x < \mu$, for sufficiently large k,

$$|I_1| \leqslant \mu e^{-\frac{\mu^2}{256}}e^{\frac{k}{8m}}\left(1-\frac{1}{8m}\right)^k e^{-\frac{k}{2}}\left(\frac{k}{2}\right)^{\frac{k}{2}}.$$

On the other hand, $\cos(2xs)$ is positive on the interval $(\sqrt{k/2}(1-\frac{1}{8m}), \sqrt{k/2})$ for $x \in I_m$, and strictly greater than $\cos(3/2)$ on $(\sqrt{k/2}(1-\frac{1}{16m}), \sqrt{k/2})$, so that

$$I_2 \geqslant c\int_{\sqrt{\frac{k}{2}}(1-\frac{1}{16m})}^{\sqrt{\frac{k}{2}}} e^{-s^2}s^k ds.$$

Now, we are integrating over an interval of length $\geqslant c\mu$, so considering the smallest value of the integrand,

$$I_2 \geqslant c\mu e^{-\frac{k}{2}(1-\frac{1}{16m})^2} \left(\frac{k}{2}\right)^{\frac{k}{2}} \left(1 - \frac{1}{16m}\right)^k.$$

Squaring out as before and using the fact that $m \geqslant \lfloor \sqrt{k}/\mu \rfloor$, we have

$$I_2 \geqslant c\mu e^{-\frac{\mu^2}{512}} e^{\frac{k}{16m}} \Big(1 - \frac{1}{16m}\Big)^k e^{-\frac{k}{2}} \left(\frac{k}{2}\right)^{\frac{k}{2}}.$$

Now, $e^{kx}(1-x)^k$ is a decreasing function on $[0,1]$, so we can also write

$$I_2 \geqslant c\mu e^{-\frac{\mu^2}{512}} e^{\frac{k}{8m}} \Big(1 - \frac{1}{8m}\Big)^k e^{-\frac{k}{2}} \left(\frac{k}{2}\right)^{\frac{k}{2}}.$$

Comparing with the upper bound for $|I_1|$, and choosing μ sufficiently large, this yields

$$I_1 + I_2 \geqslant c e^{\frac{k}{8m}} \Big(1 - \frac{1}{8m}\Big)^k e^{-\frac{k}{2}} \left(\frac{k}{2}\right)^{\frac{k}{2}},$$

and by a completely analogous argument we also have

$$I_3 + I_4 \geqslant c e^{-\frac{k}{8m}} \Big(1 + \frac{1}{8m}\Big)^k e^{-\frac{k}{2}} \left(\frac{k}{2}\right)^{\frac{k}{2}}.$$

Now as

$$e^{\frac{k}{8m}} \geqslant (1 + \frac{1}{8m})^k \quad \text{ and } \quad e^{-\frac{k}{8m}} \geqslant (1 - \frac{1}{8m})^k,$$

we see that

$$c\Big(1 - \frac{1}{64m^2}\Big)^k e^{-\frac{k}{2}} \left(\frac{k}{2}\right)^{\frac{k}{2}} \leqslant \int_0^\infty e^{-s^2} s^k \cos(2xs) ds \leqslant \frac{1}{2}\Gamma\left(\frac{k+1}{2}\right).$$

Finally, as $m^2 \approx k$ and

$$\Gamma\left(\frac{k+1}{2}\right) = 2\sqrt{\pi}\frac{k!}{2^k \left(\frac{k}{2}\right)!}, \tag{17}$$

(see [19, pp. 14]), by (16) we have

$$c_0 \frac{2^{k/2}}{\sqrt{k!}} e^{-\frac{k}{2}} \left(\frac{k}{2}\right)^{\frac{k}{2}} \leqslant h_k(x) \leqslant c_1 \frac{2^{k/2}}{\sqrt{k!}} \frac{k!}{2^k \left(\frac{k}{2}\right)!}$$

for $k/2$ even, and the proof is completed by Stirling's formula. □

4. Pointwise convergence

We are able to obtain the convergence result of Corollary 1, for the case $n \geqslant 2$, as a consequence of Theorem 4.

Proof of Corollary 1, case $n \geqslant 2$. By the Cauchy–Schwarz inequality, functions $F : [0, 2\pi] \to \mathbb{C}$ that satisfy

$$\Big\| \sum_{\substack{\lambda \in \mathbb{Z} \\ \lambda \neq 0}} |\lambda|^\alpha \widehat{F}(\lambda)\, e^{-it\lambda} \Big\|_{L^2[0,2\pi]} < \infty, \quad \alpha > 1/2,$$

are in fact continuous, where $\widehat{F}$ denotes the Fourier transform of F. Writing

$$e^{-itH}f(x) = \sum_{\lambda\in\mathbb{N}}\left(\sum_{\mathbf{k}\,:\,2|\mathbf{k}|+n=\lambda} a_{\mathbf{k}}h_{\mathbf{k}}(x)\right)e^{-it\lambda} = \sum_{\lambda\in\mathbb{N}} P_\lambda f(x)\,e^{-it\lambda}, \tag{18}$$

by Theorem 4, we have

$$\begin{aligned}\Big\|\,\big\|\sum_{\lambda\in\mathbb{N}}|\lambda|^\alpha P_\lambda f\,e^{-it\lambda}\big\|_{L^2_t[0,2\pi]}\Big\|_{L^2_x(B_R)} &= \big\|e^{-itH}H^\alpha f\big\|_{L^2([0,2\pi]\times B_R)}\\ &\leqslant C_R\|f\|_{\mathcal{H}^{2\alpha-1/2}(\mathbb{R}^n)}.\end{aligned}$$

Thus, when $f\in\mathcal{H}^s(\mathbb{R}^n)$ with $s>1/2$, we see that $e^{-itH}f(x)$ is a continuous function of t for almost every $x\in B_R$. Writing

$$\mathbb{R}^n = \bigcup_{j\in\mathbb{Z}} B_{2^j}\backslash B_{2^{j-1}},$$

we see that the set of divergence is null, which proves Corollary 1 for $n\geqslant 2$. $\square$

For the one dimensional improvement, we appeal to the Strichartz estimates that will also enable us to complete the proof of Theorem 1. The integral representation (7) can be combined with the machinery of Keel and Tao [10] so that

$$\|e^{-itH}f\|_{L^q_t([0,2\pi],L^p_x(\mathbb{R}^n))}\leqslant C_p\,\|f\|_{L^2(\mathbb{R}^n)} \tag{19}$$

when $q\geqslant 2$ and $\frac{n}{p}+\frac{2}{q}=\frac{n}{2}$, excluding the case $(p,q,n)\neq(\infty,2,2)$. Koch and Tataru [12] proved (19) for a more general class of operators that includes H, and also noted that there can be no such estimates for p outside of $[2,\frac{2n}{n-2}]$. Applying Hölder in the temporal integral yields (19) in the range $p\in[2,\frac{2n}{n-2}]$ when $\frac{n}{p}+\frac{2}{q}\geqslant\frac{n}{2}$, excluding the case $(p,q,n)\neq(\infty,2,2)$. We will see later that, modulo the exceptional case, the estimate is completely sharp in the sense that (19) cannot hold when $\frac{n}{p}+\frac{2}{q}<\frac{n}{2}$.

Theorem 3 and (19) are the key ingredients in the proof of Theorem 1. For the best known results in this direction for the free equation see [9, 11, 14–16].

Proof of Theorem 1. For $1<r<q<\infty$, we recall the following fractional Sobolev inequality due to Wainger [26]:

$$\Big\|\sum_{\substack{\lambda\in\mathbb{Z}\\ \lambda\neq 0}}|\lambda|^{-\alpha}\widehat{F}(\lambda)\,e^{-it\lambda}\Big\|_{L^q[0,2\pi]}\leqslant C\,\|F\|_{L^r[0,2\pi]},\qquad \alpha=\frac{1}{r}-\frac{1}{q}.$$

In particular, by (18) we see that

$$\begin{aligned}\|e^{-itH}f(x)\|_{L^q_t[0,2\pi]} &\leqslant C\left\|\sum_{\lambda\in\mathbb{N}}|\lambda|^\alpha P_\lambda f(x)\,e^{-it\lambda}\right\|_{L^r_t[0,2\pi]}\\ &= C\left\|\sum_{\mathbf{k}\in\mathbb{N}_0^n}(2|\mathbf{k}|+n)^\alpha a_{\mathbf{k}}h_{\mathbf{k}}(x)\,e^{-it(2|\mathbf{k}|+n)}\right\|_{L^r_t[0,2\pi]},\end{aligned} \tag{20}$$

where $f=\sum_{\mathbf{k}\in\mathbb{N}_0^n}a_{\mathbf{k}}h_{\mathbf{k}}$ is initially a member of $\mathfrak{F}(\mathbb{R}^n)$.

In the range $p \in [\frac{2n}{n-2}, \infty]$, with $n \geqslant 3$, by taking $r = 2$ in (20) and applying Theorem 3, we see that

$$\begin{aligned} \|e^{-itH}f\|_{L^p_x(\mathbb{R}^n,\, L^q_t[0,2\pi])} &\leqslant C \left(\sum_{k\in\mathbb{N}_0^n} (2|\mathbf{k}|+n)^s (2|\mathbf{k}|+n)^{1-\frac{2}{q}} |a_{\mathbf{k}}|^2 \right)^{1/2} \\ &\leqslant C\,\|f\|_{\mathcal{H}^{s+1-\frac{2}{q}}(\mathbb{R}^n)}, \end{aligned}$$

where $s = n(\frac{1}{2} - \frac{1}{p}) - 1$. This yields the desired inequality.

For the range $p \in [\frac{2(n+2)}{n}, \frac{2n}{n-2})$ ($p \in [6, \infty]$ when $n = 1$), we apply Minkowski's integral inequality to (19), so that

$$\|e^{-itH}f\|_{L^p_x(\mathbb{R}^n, L^{q_0}_t[0,2\pi])} \leqslant C_p \|f\|_{L^2(\mathbb{R}^n)}, \tag{21}$$

where $\frac{n}{p} + \frac{2}{q_0} = \frac{n}{2}$. Now combining (21) with (20), with $r = q_0$, we see that

$$\begin{aligned} \|e^{-itH}f\|_{L^p_x(\mathbb{R}^n,\, L^q_t[0,2\pi])} &\leqslant C \left(\sum_{k\in\mathbb{N}_0^n} (2|\mathbf{k}|+n)^{\frac{2}{q_0}-\frac{2}{q}} |a_{\mathbf{k}}|^2 \right)^{1/2} \\ &\leqslant C\,\|f\|_{\mathcal{H}^{\frac{2}{q_0}-\frac{2}{q}}(\mathbb{R}^n)}, \end{aligned}$$

and as $\frac{2}{q_0} - \frac{2}{q} = n(\frac{1}{2} - \frac{1}{p}) - \frac{2}{q}$, we are done.

To see that this is sharp with respect to the regularity we consider g_N defined by

$$g_N = \sum_{\mathbf{k}\,:\, N\leqslant k_j<2N} h_{4\mathbf{k}}.$$

When $|t| \leqslant \frac{1}{100nN}$ and $|\mathbf{k}| \leqslant 2nN$, we have

$$\left|\Re(e^{-it(8|\mathbf{k}|+n)} - 1)\right| = |\cos t(8|\mathbf{k}|+n) - 1| < 1/2,$$

so that

$$\begin{aligned} |e^{-itH}g_N| &\geqslant \left| \sum_{\mathbf{k}\,:\, N\leqslant k_j<2N} h_{4\mathbf{k}} \right| - \left| \sum_{\mathbf{k}\,:\, N\leqslant k_j<2N} [\cos(t(8|\mathbf{k}|+n)) - 1]\, h_{4\mathbf{k}} \right| \\ &\geqslant \left| \sum_{\mathbf{k}\,:\, N\leqslant k_j<2N} h_{4\mathbf{k}} \right| - \frac{1}{2} \sum_{\mathbf{k}\,:\, N\leqslant k_j<2N} |h_{4\mathbf{k}}|. \end{aligned}$$

Thus, by the following Lemma 2, if $|x_j| < \frac{c_1}{2N^{1/2}}$ for all $j = 1, \ldots, n$, then

$$|e^{-itH}g_N(x)| \geqslant \frac{1}{2} \sum_{\mathbf{k}\,:\, N\leqslant k_j<2N} h_{4\mathbf{k}}(x) \geqslant cN^{n-\frac{n}{4}}.$$

Calculating, we see that

$$\|e^{-itH}g_N\|_{L^p_x(\mathbb{R}^n,\, L^q_t[0,2\pi])} \geqslant cN^{\frac{3n}{4}-\frac{n}{2p}-\frac{1}{q}}.$$

On the other hand, $\|g_N\|_{\mathcal{H}^s(\mathbb{R}^n)} \leqslant CN^{\frac{s}{2}+\frac{n}{2}}$, so that letting N tend to infinity, for (3) to hold, it is necessary that

$$s \geqslant n\Big(\frac{1}{2} - \frac{1}{p}\Big) - \frac{2}{q}.$$

□

We note that by the same calculation,

$$\|e^{-itH}g_N\|_{L^q_t([0,2\pi],\,L^p_x(\mathbb{R}^n))} \geqslant cN^{\frac{3n}{4}-\frac{n}{2p}-\frac{1}{q}},$$

so that, taking $s=0$, we see that the Strichartz estimates (19) are also sharp.

Now we complete the proof of Corollary 1.

Proof of Corollary 1, case $n=1$. We again appeal to [26]. There it was proven that functions

$$F(t)=\sum_{\lambda\in\mathbb{N}}\widehat{F}(\lambda)\,e^{-it\lambda}$$

which satisfy

$$\Big\|\sum_{\lambda\in\mathbb{N}}|\lambda|^\alpha\widehat{F}(\lambda)\,e^{-it\lambda}\Big\|_{L^q[0,2\pi]}<\infty,\quad \alpha>1/q,$$

are also continuous. By Theorem 1 we see that for certain $q<\infty$,

$$\Big\|\sum_{\lambda\in\mathbb{N}}|\lambda|^\alpha P_\lambda f(x)\,e^{-it\lambda}\Big\|_{L^p_x(\mathbb{R},\,L^q_t[0,2\pi])}\leqslant C\|f\|_{\mathcal{H}^s(\mathbb{R})},\quad \alpha=\frac{1}{2}\Big(s-\big(\frac{1}{2}-\frac{1}{p}\big)+\frac{2}{q}\Big).$$

In particular, taking $p=6$ and $s>\frac{1}{3}$, we see that $\alpha>1/q$ so that $t\to e^{-itH}f(x)$ is continuous for almost every $x\in\mathbb{R}$. $\square$

Almost everywhere convergence results can also be obtained from maximal inequalities. By an appropriate dyadic decomposition, Theorem 1 implies that

$$\big\|\sup_{t\in\mathbb{R}}|e^{-itH}f|\,\big\|_{L^p_x(\mathbb{R}^n)}\leqslant C_s\,\|f\|_{\mathcal{H}^s(\mathbb{R}^n)},\quad s>n\big(\frac{1}{2}-\frac{1}{p}\big),\quad p\geqslant\frac{2(n+2)}{n}.$$

Curiously, and unlike the free case, this is not trivial even when $p=\infty$. Indeed, for a dyadic piece $f_N=\sum_{N\leqslant|\mathbf{k}|\leqslant 2N}a_{\mathbf{k}}h_{\mathbf{k}}$, we can write

$$\sup_{x\in\mathbb{R}^n,\,t\in[0,2\pi]}|e^{-itH}f_N(x)|\leqslant\sup_{x\in\mathbb{R}^n}\Bigg(\sum_{N\leqslant|\mathbf{k}|\leqslant 2N}|h_{\mathbf{k}}(x)|^2\Bigg)^{1/2}\Bigg(\sum_{N\leqslant|\mathbf{k}|\leqslant 2N}|a_{\mathbf{k}}|^2\Bigg)^{1/2},$$

however, the property (9) only provides the estimate

$$\sup_{x\in\mathbb{R}^n}\Bigg(\sum_{N\leqslant|k|\leqslant 2N}|h_{\mathbf{k}}(x)|^2\Bigg)^{1/2}\leqslant CN^{\frac{1}{2}\frac{5n}{6}}.$$

On the other hand, using the local property (8),

$$\sup_{x\in B_R}\Bigg(\sum_{N\leqslant|k|\leqslant 2N}|h_{\mathbf{k}}(x)|^2\Bigg)^{1/2}\leqslant CN^{\frac{1}{2}\frac{n}{2}},$$

and so we recover a local version of our estimate. Theorem 1 tells us that global estimates are indeed possible even though this is not immediately apparent. Thangavelu [22] noted a similar phenomenon for the Bochner–Reisz problem for Hermite expansions.

As we saw in the previous section, necessary conditions for the harmonic oscillator are harder to see than for the free equation. To see that Theorem 1 was sharp with respect to the regularity, we used the following lemma, which we now prove.

Lemma 2. *There exist positive constants c_0 and c_1 such that*

$$h_{4k}(x) \geqslant c_0 k^{-1/4}$$

for all $k \in \mathbb{N}$ when $|x| < c_1 k^{-1/2}$.

Proof. For k an even integer, $|x| < \frac{1}{4\sqrt{k}}$ and $0 < s < \sqrt{k}$, we have $\cos(2xs) > 1/2$, so that

(22)
$$\begin{aligned}\left|\int_0^\infty e^{-s^2} s^k \cos(2xs)ds\right| &\geqslant \int_0^{\sqrt{k}} e^{-s^2} s^k \cos(2xs)ds - \left|\int_{\sqrt{k}}^\infty e^{-s^2} s^k \cos(2xs)ds\right| \\ &\geqslant \frac{1}{2}\int_0^{\sqrt{k}} e^{-s^2} s^k ds - \int_{\sqrt{k}}^\infty e^{-s^2} s^k ds \\ &= \frac{1}{2}\int_0^\infty e^{-s^2} s^k ds - \frac{3}{2}\int_{\sqrt{k}}^\infty e^{-s^2} s^k ds.\end{aligned}$$

Now, by the formula for the Gamma function and a change of variables,

$$\frac{1}{2}\int_0^\infty e^{-s^2} s^k ds = \frac{1}{4}\Gamma\left(\frac{k+1}{2}\right). \tag{23}$$

On the other hand, making the change of variables $r = \frac{s}{\sqrt{2}}$,

$$\begin{aligned}\int_{\sqrt{k}}^\infty e^{-s^2} s^k ds &\leqslant e^{-\frac{k}{2}} \int_{\sqrt{k}}^\infty e^{-\frac{s^2}{2}} s^k ds \leqslant e^{-\frac{k}{2}} \int_0^\infty e^{-\frac{s^2}{2}} s^k ds \\ &\leqslant \sqrt{2}\left(\frac{2}{e}\right)^{\frac{k}{2}} \int_0^\infty e^{-r^2} r^k dr = \frac{\sqrt{2}}{2}\left(\frac{2}{e}\right)^{\frac{k}{2}} \Gamma\left(\frac{k+1}{2}\right) \\ &\leqslant \frac{1}{16}\Gamma\left(\frac{k+1}{2}\right)\end{aligned} \tag{24}$$

for all $k \geqslant k_0 = 2\frac{\log(16)}{\log\frac{e}{2}}$ (since $h_k(0) > 0$ when $k/2$ is even, it is sufficient to prove the assertion for $k \geqslant k_0$).

Substituting (23) and (24) into (22), we obtain

$$\left|\int_0^\infty e^{-s^2} s^k \cos(2xs)ds\right| \geqslant \frac{1}{8}\Gamma\left(\frac{k+1}{2}\right),$$

so that from (16), we see that

$$h_k(x) \geqslant \frac{1}{8\pi^{3/4}} \frac{2^{k/2}}{\sqrt{k!}} \Gamma\left(\frac{k+1}{2}\right)$$

for all $k \geqslant k_0$ when $|x| < \frac{1}{4\sqrt{k}}$ and $k/2$ is even.

Now, from (17), we have

$$h_k(x) \geqslant \frac{1}{4\pi^{1/4}} \frac{\sqrt{k!}}{2^{k/2}\left(\frac{k}{2}\right)!},$$

and the result follows by Stirling's formula as before. ∎

5. The forced harmonic oscillator

We consider the Cauchy problem for the Schrödinger equation of the form

$$\text{(FHO)}\qquad \begin{cases} i\frac{du}{dt}+\Delta u=|x|^2u+V(x,t)\,u\\ u(\,\cdot\,,0)=u_0\in L^2(\mathbb{R}^n). \end{cases}$$

We note in passing that the cubic equation, $i\frac{du}{dt}+\Delta u=|x|^2u+|u|^2\,u$, has been extensively considered in connection with Bose–Einstein condensation (see for example [4] or [23].)

In the following theorem, when $n\geqslant 3$ the hypothesis $\|V\|_{L^q_x(\mathbb{R}^n,\,L^\infty_t[0,\infty))}$ sufficiently small can be changed to $\|V\|_{L^\infty_t([0,\infty),L^q_x(\mathbb{R}^n))}$ sufficiently small, by using estimate (19) instead of Theorem 3.

Theorem 5. *Let $n\geqslant 2$ and $\frac{2}{p}+\frac{1}{q}=1$, and suppose that $\|V\|_{L^q_x(\mathbb{R}^n,\,L^\infty_t[0,\infty))}$ is sufficiently small, where $q\in[\frac{n}{2},\infty]$. Then there exists a unique global solution of (FHO) belonging to $C([0,\infty),L^2_x(\mathbb{R}^n))\cap L^p_x(\mathbb{R}^n,\,L^2_{loc}[0,\infty))$.*

Proof. We use the standard contraction mapping argument. By Duhamel's formula

$$u(x,t)=e^{-itH}u_0-i\int_0^t e^{-i(t-\tau)H}V(\cdot,\tau)u(\cdot,\tau)(x)\,d\tau.$$

For $2\leqslant p\leqslant\frac{2n}{n-2}$, by Theorem 3, there exists a constant $C_0>1$ such that

$$\|e^{-itH}f\|_{L^p_x(\mathbb{R}^n,\,L^2_t[0,2\pi])}\leqslant C_0\|f\|_{L^2(\mathbb{R}^n)},\tag{25}$$

and, by duality, this yields

$$\Big\|\int_0^t e^{i\tau H}G(\,\cdot\,,\tau)\,d\tau\Big\|_{L^2_x(\mathbb{R}^n)}\leqslant C_0\,\|G\|_{L^{p'}_x(\mathbb{R}^n,L^2_t[0,2\pi])},\qquad t\in[0,2\pi].\tag{26}$$

Now, by various applications of Fubini's theorem,

$$\begin{aligned}
&\int_{\mathbb{R}^n}\int_0^{2\pi}\int_0^t e^{-i(t-\tau)H}VF(\cdot,\tau)(x)\,d\tau\,G(x,t)\,dxdt\\
&=\int_0^{2\pi}\int_0^t\int_{\mathbb{R}^n} e^{-i(t-\tau)H}VF(\cdot,\tau)(x)\,G(x,t)\,dx\,d\tau\,dt\\
&=\int_0^{2\pi}\int_0^t\int_{\mathbb{R}^n} e^{i\tau H}VF(\cdot,\tau)(x)\,e^{-itH}G(x,t)\,dx\,d\tau\,dt\\
&=\int_{\mathbb{R}^n}\int_0^t e^{i\tau H}VF(\cdot,\tau)(x)\,d\tau\int_0^{2\pi}e^{-itH}G(x,t)\,dt\,dx,
\end{aligned}$$

where the second equality follows using the orthogonality of the Hermite functions. Thus, by the Cauchy–Schwarz inequality followed by two applications of (26) and duality,

$$\Big\|\int_0^t e^{-i(t-\tau)H}V(\cdot,\tau)F(\cdot,\tau)(x)\,d\tau\Big\|_{L^p_x(\mathbb{R}^n,L^2_t[0,2\pi])}\leqslant C_0^2\|VF\|_{L^{p'}_x(\mathbb{R}^n,L^2_t[0,2\pi])}.\tag{27}$$

We define the Banach space $X=C([0,2\pi],L^2_x(\mathbb{R}^n))\cap L^p_x(\mathbb{R}^n,L^2_t[0,2\pi])$ via the norm

$$\|u\|_X=\sup_{t\in[0,2\pi]}\|u(\,\cdot\,,t)\|_{L^2_x(\mathbb{R}^n)}+\|u\|_{L^p_x(\mathbb{R}^n,\,L^2_t[0,2\pi])},$$

and the nonlinear map $\mathcal{L}: X \to X$ by

$$\mathcal{L}F = e^{-itH}u_0 - i\int_0^t e^{-i(t-\tau)H}V(\cdot,\tau)F(\cdot,\tau)(x)\,d\tau.$$

By (5) and the conservation of the L^2 norm (6) we see that

$$\|e^{-itH}u_0\|_X \leqslant (C_0+1)\|u_0\|_{L^2(\mathbb{R}^n)},$$

and combining (26) and (27), we also have

$$\Big\|i\int_0^t e^{-i(t-\tau)H}V(\cdot,\tau)F(\cdot,\tau)(x)\,d\tau\Big\|_X \leqslant (C_0+C_0^2)\|V\|_{L^q_x(\mathbb{R}^n,L^\infty_t[0,\infty))}\|F\|_X;$$

here we have used the fact that

$$\|VF\|_{L^{p'}_x L^2_t} \leqslant \|V\|_{L^q_x L^\infty_t}\|F\|_{L^p_x L^2_t}, \quad \frac{2}{p}+\frac{1}{q}=1.$$

Thus we see that $\mathcal{L}$ maps $\{F : \|F\|_X \leqslant 2(C_0+1)\|u_0\|_{L^2(\mathbb{R}^n)}\}$ into itself provided $(C_0+C_0^2)\|V\|_{L^q_x(\mathbb{R}^n,L^\infty_t[0,\infty))} \leqslant \frac{1}{2}$. This also guarantees that

$$\|\mathcal{L}(F-G)\|_X \leqslant \frac{1}{2}\|F-G\|_X, \tag{28}$$

so that by the contraction mapping principle, there exists a solution. Now although the L^2–norm may have increased in size, we know that it is at least finite, so by iterating the process, replacing u_0 with $u(\cdot,2k\pi)$, $k\in\mathbb{N}$, we obtain a global solution.

To see that the solution is unique in $L^p_x(\mathbb{R}^n,\,L^2_{loc}[0,\infty))$, suppose that u_1 and u_2 are solutions. Then by (27) as before, we see that

$$\|u_1-u_2\|_{L^p_x(\mathbb{R}^n,\,L^2_t[2k\pi,2(k+1)\pi])} \leqslant \frac{1}{2}\|u_1-u_2\|_{L^p_x(\mathbb{R}^n,\,L^2_t[2k\pi,2(k+1)\pi])}$$

for all $k\geqslant 0$, so they are in fact the same. □

We remark that the iteration in the previous argument could be avoided by considering the estimate

$$\|e^{-itH}f\|_{L^p_x(\mathbb{R}^n,\,L^2_t[0,T])} \leqslant C_0\sqrt{T}\,\|f\|_{L^2(\mathbb{R}^n)},$$

however, in doing so, our hypothesis would worsten. We would then require that $T\|V\|_{L^q_x(\mathbb{R}^n,\,L^\infty_t[0,T))}$ be sufficiently small.

6. Final remarks

We combine the Strichartz estimates with the orthogonality of the trigonometric polynomials to obtain some mysterious inequalities for the Hermite functions. Observe that for $f=\sum_{\mathbf{k}\in E}a_{\mathbf{k}}h_{\mathbf{k}}$, where $E\subset\mathbb{N}_0^n$,

$$\begin{aligned}\|e^{-itH}f\|^4_{L^4_t[0,2\pi]} &= \int_0^{2\pi}\left|\left(\sum_{\mathbf{j}\in E}e^{-it(2|\mathbf{j}|+2)}a_{\mathbf{j}}h_{\mathbf{j}}\right)\left(\sum_{\mathbf{k}\in E}e^{it(2|\mathbf{k}|+2)}\overline{a}_{\mathbf{k}}h_{\mathbf{k}}\right)\right|^2 dt\\ &= 2\pi\left(\sum_{\substack{\mathbf{i},\mathbf{j},\mathbf{k},\mathbf{l}\in E\\ |\mathbf{i}|+|\mathbf{k}|=|\mathbf{j}|+|\mathbf{l}|}}a_{\mathbf{i}}\overline{a}_{\mathbf{j}}a_{\mathbf{k}}\overline{a}_{\mathbf{l}}h_{\mathbf{i}}h_{\mathbf{j}}h_{\mathbf{k}}h_{\mathbf{l}}\right).\end{aligned}$$

In two spatial dimensions, by (19),

$$\Big\|\|e^{-itH}f\|_{L^4_t[0,2\pi]}\Big\|_{L^4_x(\mathbb{R}^2)} \leqslant C\|f\|_2,$$

so that setting $a_{\mathbf{k}}=1$, we have

$$\sum_{\substack{\mathbf{i},\mathbf{j},\mathbf{k},\mathbf{l}\in E\\ |\mathbf{i}|+|\mathbf{k}|=|\mathbf{j}|+|\mathbf{l}|}} \int_{\mathbb{R}^2} h_{\mathbf{i}}h_{\mathbf{j}}h_{\mathbf{k}}h_{\mathbf{l}} \leqslant CN^2, \quad \#E=N.$$

In one spatial dimension, by the same procedure we obtain

$$\sum_{i,j,k,l,m\in E} \int_{\mathbb{R}} h_ih_jh_kh_lh_mh_{i+k+m-j-l} \leqslant CN^3, \quad \#E=N.$$

We see that there is cancelation. A better understanding of this cancelation would presumably yield improved results.

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