ODD BMO($\mathbb{R}$) FUNCTIONS AND CARLESON MEASURES IN THE BESSEL SETTING

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ABSTRACT. In this paper we characterize the odd functions in BMO($\mathbb{R}$) by using Carleson measures associated with Poisson and heat semigroups for Bessel operators.

1. INTRODUCTION

As it is well-known, a measurable function f belongs to the classical space of functions of bounded mean oscillation, BMO($\mathbb{R}$), when

$$||f||_{BMO(\mathbb{R})} = \sup_{I \subset \mathbb{R}} \frac{1}{|I|} \int_I |f(x) - f_I| dx < \infty,$$

where the supremum is taken over all bounded intervals I in $\mathbb{R}$, $f_I = \frac{1}{|I|} \int_I f(x)dx$ and $|I|$ denotes the length of I.

In this paper we deal with the space $\mathrm{BMO_o}(\mathbb{R})$ of all the odd functions in BMO($\mathbb{R}$). This space was considered in [5]. A useful property is the following. Let $1 \leq p < \infty$. A measurable function f is in $\mathrm{BMO_o}(\mathbb{R})$ if and only if there exists $C > 0$ such that

$$\left\{ \frac{1}{|I|} \int_I |f(x) - f_I|^p dx \right\}^{1/p} \leq C, \tag{1.1}$$

for every interval $I = (a, b)$, $0 < a < b < \infty$, and

$$\left\{ \frac{1}{|I|} \int_I |f(x)|^p dx \right\}^{1/p} \leq C, \tag{1.2}$$

for each interval $I = (0, b)$, $0 < b < \infty$. Moreover, the quantity $\inf\{C > 0 : (1.1) \text{ and } (1.2) \text{ hold}\}$ is equivalent to $||f||_{BMO(\mathbb{R})}$.

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It is well-known that the classical space $\mathrm{BMO}(\mathbb{R}^n)$ can be characterized by means of Carleson measures. We say that a positive measure μ on $(0,\infty)\times(0,\infty)$ is Carleson, when there exists $C>0$ such that, for every interval $I\subset(0,\infty)$

$$\frac{\mu(I\times(0,|I|))}{|I|}\leq C.$$

We denote by $||\mu||_{\mathcal{C}}=\sup_{I\subset(0,\infty),\, I \text{ interval}}\frac{\mu(I\times(0,|I|))}{|I|}$, when μ is a Carleson measure on $(0,\infty)\times(0,\infty)$.

The aim of this paper is to characterize the space $\mathrm{BMO_o}(\mathbb{R})$ in terms of Carleson measures involving the heat and Poisson semigroups associated with the Bessel operator $\Delta_\lambda=-x^{-\lambda}Dx^{2\lambda}Dx^{-\lambda}$, $\lambda>0$.

Assume that $\lambda>0$. For each $y\in(0,\infty)$, the function $\varphi_y(x)=\sqrt{xy}J_{\lambda-1/2}(xy)$, $x\in(0,\infty)$, is an eigenfunction of Δ_λ. According to [14, §3.2 (5) and (6)] we have that

$$\Delta_\lambda(\sqrt{xy}J_{\lambda-\frac{1}{2}}(xy))=y^2\sqrt{xy}J_{\lambda-\frac{1}{2}}(xy),\quad x,y\in(0,\infty). \tag{1.3}$$

Here J_ν represents the Bessel function of the first kind and order ν.
The heat semigroup $\{W_t^\lambda\}_{t>0}$ associated with the Bessel operator Δ_λ is defined by

$$W_t^\lambda(f)(x)=\int_0^\infty W_t^\lambda(x,y)f(y)dy,\quad t,x\in(0,\infty),$$

where the heat kernel is given by

$$W_t^\lambda(x,y)=\int_0^\infty e^{-tz^2}\varphi_x(z)\varphi_y(z)dz,\quad t,x,y\in(0,\infty).$$

According to [14, §13.31 (1)] it has that

$$W_t^\lambda(x,y)=\frac{\sqrt{xy}}{2t}I_{\lambda-\frac{1}{2}}\left(\frac{xy}{2t}\right)e^{-\frac{x^2+y^2}{4t}},\quad t,x,y\in(0,\infty),$$

where I_ν denotes the modified Bessel function of the first kind and order ν.

The Poisson semigroup $\{P_t^\lambda\}_{t>0}$ for Δ_λ is defined by

$$P_t^\lambda(f)(x)=\int_0^\infty P_t^\lambda(x,y)f(y)dy,\quad t,x\in(0,\infty),$$

being

$$P_t^\lambda(x,y)=\int_0^\infty e^{-tz}\varphi_x(z)\varphi_y(z)dz,\quad t,x,y\in(0,\infty).$$

By taking into account [10, (16.4)] (see also [15]) we can write

$$(1.4) \qquad P_t^\lambda(x,y)=\frac{2\lambda(xy)^\lambda t}{\pi}\int_0^\pi \frac{(\sin\theta)^{2\lambda-1}}{((x-y)^2+t^2+2xy(1-\cos\theta))^{\lambda+1}}d\theta, \quad t,x,y\in(0,\infty).$$

The main result of the paper is showed in the following theorem. By BMO_+ we denote the space of all those $f\in L^1_{\mathrm{loc}}[0,\infty)$ such that the odd extension f_{o} of f to $\mathbb{R}$ is in $\mathrm{BMO}(\mathbb{R})$. We consider the natural norm on BMO_+.

Theorem 1.5. *Let $\lambda>0$. Assume that $f\in L^1_{\mathrm{loc}}[0,\infty)$. The following assertions are equivalent.*

(i) $f\in BMO_+$.

(ii) $(1+x^2)^{-1}f\in L^1(0,\infty)$ and the measure

$$d\mu_f(x,t)=\left|t^2\frac{\partial}{\partial s}W_s^\lambda(f)(x)_{\big|_{s=t^2}}\right|^2\frac{dxdt}{t}$$

is Carleson on $(0,\infty)\times(0,\infty)$.

(iii) $(1+x^2)^{-1}f\in L^1(0,\infty)$ and the measure

$$d\gamma_f(x,t)=\left|t\frac{\partial}{\partial t}P_t^\lambda(f)(x)\right|^2\frac{dxdt}{t}$$

is Carleson on $(0,\infty)\times(0,\infty)$.

Moreover, the quantities $||f||^2_{\mathrm{BMO}_+}$, $||\mu_f||_{\mathcal{C}}$ and $||\gamma_f||_{\mathcal{C}}$ are equivalent.

The next sections are devoted to give a proof of Theorem 1.5. On the way of the proof we obtain new characterizations of the subspace $H^1_{\mathrm{o}}(\mathbb{R})$ of $H^1(\mathbb{R})$ constituted by the odd functions which belong to the Hardy space $H^1(\mathbb{R})$. The space $H^1_{\mathrm{o}}(\mathbb{R})$ was considered by Fridli [6], who described it by using what he called Telyakovskii transform, a local Hilbert transform studied by Andersen and Muckenhoupt [1]. Fridli also obtained a description of the space $H^1_{\mathrm{o}}(\mathbb{R})$ in terms of (odd)-atoms (see [6, Theorem 2.1]). A measurable function a on $(0,\infty)$ is an (odd)-atom when it satisfies one of the following properties:

(a) $a=\frac{1}{\delta}\chi_{(0,\delta)}$, for some $\delta>0$, where $\chi_{(0,\delta)}$ denotes as usual the characteristic function on the interval $(0,\delta)$;

(b) there exists a bounded interval $I\subset(0,\infty)$ such that supp $a\subset I$, $\int_I a(x)dx=0$, and $||a||_\infty\le|I|^{-1}$.

In Proposition 4.1 we characterize the Hardy space $H^1_{\mathrm{o}}(\mathbb{R})$ by using nontangential Littlewood-Paley g-functions associated to the Poisson semigroup for the Bessel operators Δ_λ.

Assume that $f \in BMO_o(\mathbb{R})$. If a represents the odd extension to $\mathbb{R}$ of an (odd)-atom, we denote by

$$\Phi_f(a) = \int_{-\infty}^{+\infty} f(y)a(y)dy.$$

By arguing in a standard way (see, for instance [11, ps. 142 and 143]) we can prove that Φ_f admits a unique extension to $H^1_o(\mathbb{R})$ defining an element of the dual space $(H^1_o(\mathbb{R}))'$ of $H^1_o(\mathbb{R})$ and being $||\Phi_f||_{(H^1_o(\mathbb{R}))'} \leq C||f||_{BMO_o(\mathbb{R})}$. Moreover, according to [4, Corollary 1.6], [2, Proposition 32(b)] and [5, Proposition 3.1(ii)], the mapping $f \longrightarrow \Phi_f$ is an isomorphism between $BMO_o(\mathbb{R})$ and $(H^1_o(\mathbb{R}))'$ and $||f||_{BMO_o(\mathbb{R})}$ is equivalent to $||\Phi_f||_{(H^1_o(\mathbb{R}))'}$, for every $f \in BMO_o(\mathbb{R})$.

The Hankel transform defined by

$$h_\lambda(f)(x) = \int_0^\infty \sqrt{xy} J_{\lambda-\frac{1}{2}}(xy) f(y)dy, \quad x \in (0, \infty),$$

will be useful in the sequel. h_λ is an isometry in $L^2(0,\infty)$ ([13, p. 473 (1)]). By using the results presented in [7] and [8] a convolution operation for h_λ can be defined. The main properties of the Hankel transform h_λ was established in [16].

Throughout this paper we denote by C a suitable positive constant that is not necessarily the same in each ocurrence.

2. Proof of $(i) \Longrightarrow (ii)$ in Theorem 1.5

To simplify we write, for every $t > 0$,

$$Q_t^\lambda(f) = t^2 \frac{\partial}{\partial s} W_s^\lambda(f)\Big|_{s=t^2}.$$

We can see that

$$Q_t^\lambda(f)(x) = \int_0^\infty t^2 \frac{\partial}{\partial s} W_s^\lambda(x,y)\Big|_{s=t^2} f(y)dy, \quad t, x \in (0,\infty).$$

In effect, according to [9, (5.7.9)] we have, for each $x, y, s \in (0, \infty)$,

$$\begin{aligned} \frac{\partial}{\partial s} W_s^\lambda(x,y) &= \frac{\partial}{\partial s}\left[\frac{(xy)^\lambda}{(2s)^{\lambda+\frac{1}{2}}}\left(\frac{xy}{2s}\right)^{-\lambda+\frac{1}{2}} I_{\lambda-\frac{1}{2}}\left(\frac{xy}{2s}\right) e^{-\frac{x^2+y^2}{4s}}\right] \\ &= -\frac{(xy)^\lambda}{(2s)^{\lambda+\frac{1}{2}}}\left(\frac{\lambda+\frac{1}{2}}{s} - \frac{x^2+y^2}{4s^2}\right)\left(\frac{xy}{2s}\right)^{-\lambda+\frac{1}{2}} I_{\lambda-\frac{1}{2}}\left(\frac{xy}{2s}\right) e^{-\frac{x^2+y^2}{4s}} \\ &\quad - \frac{(xy)^\lambda}{(2s)^{\lambda+\frac{1}{2}}} \frac{(xy)^2}{4s^3}\left(\frac{xy}{2s}\right)^{-\lambda-\frac{1}{2}} I_{\lambda+\frac{1}{2}}\left(\frac{xy}{2s}\right) e^{-\frac{x^2+y^2}{4s}}. \end{aligned} \tag{2.1}$$

Hence, by [9, (5.11.10)] we get

$$\left|\frac{\partial}{\partial s}W_s^\lambda(x,y)\right| \le C\frac{e^{-\frac{(x-y)^2}{4s}}}{s^{\frac{3}{2}}}\left(1+\frac{x^2+y^2}{s}\right),\quad x,y,s\in(0,\infty).$$

Denote by S_Q^λ the square function

$$S_Q^\lambda(f)(x)=\left(\int_0^\infty |Q_t^\lambda(f)(x)|^2\frac{dt}{t}\right)^{\frac{1}{2}},\quad x\in(0,\infty).$$

The following result concerning to the boundedness of this square function follows easily by taking into account that h_λ is an isometry in $L^2(0,\infty)$ and that, according to [12, Lemma 1], for each $f\in L^2(0,\infty)$,

$$Q_t^\lambda(f)=-t^2h_\lambda(e^{-t^2y^2}y^2h_\lambda(f)),\quad t\in(0,\infty).$$

Lemma 2.2. *Let $\lambda>0$. It has that*

$$||S_Q^\lambda(f)||_{L^2(0,\infty)}=\frac{1}{2\sqrt{2}}||f||_{L^2(0,\infty)},\quad f\in L^2(0,\infty).$$

Assume now that $f\in \mathrm{BMO}_+$. Since $f\in L^1_{\mathrm{loc}}[0,\infty)$, [11, p. 141] leads to $(1+x^2)^{-1}f\in L^1(0,\infty)$.

To see that the measure

$$d\mu_f(x,t)=\left|t^2\frac{\partial}{\partial s}W_s^\lambda(f)(x)\Big|_{s=t^2}\right|^2\frac{dxdt}{t}=|Q_t^\lambda(f)(x)|^2\frac{dxdt}{t}$$

is Carleson on $(0,\infty)\times(0,\infty)$, and $||\mu_f||_{\mathcal{C}}\le C||f||^2_{\mathrm{BMO}_+}$, we will establish that, for a certain $C>0$,

$$\frac{1}{|I|}\int_0^{|I|}\int_I |Q_t^\lambda(f)(x)|^2\frac{dxdt}{t}\le C||f||^2_{\mathrm{BMO}_+},\tag{2.3}$$

for every $I=(a,b)$, $0\le a<b<\infty$.

Assume that $I=(a,b)$ with $0\le a<b<\infty$. As in the classical case ([11, p. 160]) we split f as follows

$$f=(f-f_{2I})\chi_{2I}+(f-f_{2I})\chi_{(0,\infty)\setminus 2I}+f_{2I}=f_1+f_2+f_3,$$

where $2I$ denotes the double of the interval I in $(0,\infty)$, that is, if $x_I=\frac{a+b}{2}$, then $2I=(x_I-|I|,x_I+|I|)\cap(0,\infty)$.

By Lemma 2.2 we get

$$
\begin{aligned}
\frac{1}{|I|}\int_0^{|I|}\int_I |Q_t^\lambda(f_1)(x)|^2\frac{dxdt}{t} &\leq \frac{1}{|I|}\int_I |S_Q^\lambda(f_1)(x)|^2dx \leq \frac{1}{|I|}||S_Q^\lambda(f_1)||^2_{L^2(0,\infty)}\\
&\leq \frac{C}{|I|}\int_{2I}|f(x)-f_{2I}|^2dx \leq C||f||^2_{\mathrm{BMO}_+}. \qquad (2.4)
\end{aligned}
$$

On the other hand, by using [12, Lemma 1] we can write

$$
W_s^\lambda(x,y)=\frac{1}{2^{2\lambda}\Gamma(\lambda+\frac{1}{2})}\frac{(xy)^\lambda}{s^{\lambda+\frac{1}{2}}}\int_0^\infty e^{-\frac{z^2}{4s}}D_\lambda(x,y,z)z^{2\lambda}dz, \quad s,x,y\in(0,\infty).
$$

Here

$$
D_\lambda(x,y,z)=\begin{cases} \dfrac{2^{2-2\lambda}\Gamma(\lambda+\frac{1}{2})}{\sqrt{\pi}\Gamma(\lambda)}\dfrac{[(z^2-(x-y)^2)((x+y)^2-z^2)]^{\lambda-1}}{(xyz)^{2\lambda-1}}, & |x-y|<z<x+y,\\ 0, & \text{otherwise,}\end{cases}
$$

and $\int_0^\infty D_\lambda(x,y,z)z^{2\lambda}dz=1$, $x,y\in(0,\infty)$ ([8]).

Hence, if $s,x,y\in(0,\infty)$,

$$
\frac{\partial}{\partial s}W_s^\lambda(x,y)=\frac{(xy)^\lambda}{2^{2\lambda}\Gamma(\lambda+\frac{1}{2})s^{\lambda+\frac{3}{2}}}\int_{|x-y|}^{x+y}\left(-\lambda-\frac{1}{2}+\frac{z^2}{4s}\right)e^{-\frac{z^2}{4s}}D_\lambda(x,y,z)z^{2\lambda}dz.
$$

We infer that,

$$
\begin{aligned}
\left|\frac{\partial}{\partial s}W_s^\lambda(x,y)\right| &\leq C\frac{(xy)^\lambda}{s^{\lambda+\frac{3}{2}}}\int_{|x-y|}^{x+y}e^{-\frac{z^2}{8s}}D_\lambda(x,y,z)z^{2\lambda}dz\\
&\leq C\frac{(xy)^\lambda}{s^{\lambda+\frac{3}{2}}}\int_{|x-y|}^{x+y}\frac{1}{\left(1+\frac{z^2}{8s}\right)^{\lambda+1}}D_\lambda(x,y,z)z^{2\lambda}dz\\
&\leq C\frac{(xy)^\lambda}{s}\int_{|x-y|}^{x+y}\frac{\sqrt{s}}{(s+z^2)^{\lambda+1}}D_\lambda(x,y,z)z^{2\lambda}dz \quad s,x,y\in(0,\infty).
\end{aligned}
$$

By [12, (4.1) and (6.7)] and [10, p. 86 (b)] one obtains,

$$
\left|\frac{\partial}{\partial s}W_s^\lambda(x,y)\right|\leq\frac{C}{\sqrt{s}(s+|x-y|^2)}, \quad s,x,y\in(0,\infty).
$$

Then,

$$
\left|t^2\frac{\partial}{\partial s}W_s^\lambda(x,y)\Big|_{s=t^2}\right|\leq C\frac{t}{t^2+|x-y|^2}\leq C\frac{t}{(t+|x-y|)^2}, \quad t,x,y\in(0,\infty).
$$

A standard procedure leads to

$$\begin{aligned}
|Q_t^\lambda(f_2)(x)| &\leq C\int_{(0,\infty)\setminus 2I}\frac{t}{(t+|x-y|)^2}|f(y)-f_{2I}|dy\\
&\leq C\int_{(0,\infty)\setminus 2I}\frac{t}{(t+|x_I-y|)^2}|f(y)-f_{2I}|dy\\
&\leq C\sum_{k=1}^{\infty}\int_{\{y\in(0,\infty):2^{k-1}|I|\leq|x_I-y|<2^k|I|\}}\frac{t}{(t+|x_I-y|)^2}|f(y)-f_{2I}|dy\\
&\leq C\sum_{k=1}^{\infty}\frac{t}{2^{2k}|I|^2}\left(\int_{\{y\in(0,\infty):|x_I-y|<2^k|I|\}}|f(y)-f_{2^{k+1}I}|dy+2^k|I||f_{2^{k+1}I}-f_{2I}|\right)\\
&\leq C\frac{t}{|I|}\sum_{k=1}^{\infty}\frac{1}{2^k}\left(\frac{1}{2^k|I|}\int_{2^{k+1}I}|f(y)-f_{2^{k+1}I}|dy+|f_{2^{k+1}I}-f_{2I}|\right)\\
&\leq C\frac{t}{|I|}||f||_{BMO_+},\quad x\in I, t>0.
\end{aligned}$$

We have taken into account that if $k\in\mathbb{N}\setminus\{0\}$ and $2^k|I|>x_I$ then $2^{k+1}I=(0,x_I+2^k|I|)\subset(0,2^{k+1}|I|)$ and

$$\begin{aligned}
\int_{2^{k+1}I}|f(y)-f_{2^{k+1}I}|dy &\leq \int_0^{2^{k+1}|I|}(|f(y)|+|f_{2^{k+1}I}|)dy\\
&\leq 2^{k+1}|I|\left(||f||_{BMO_+}+\frac{1}{2^k|I|}\int_0^{2^{k+1}|I|}|f(y)|dy\right)\leq 2^{k+3}|I|||f||_{BMO_+}.
\end{aligned}$$

Hence

$$\frac{1}{|I|}\int_0^{|I|}\int_I|Q_t^\lambda(f_2)(x)|^2\frac{dxdt}{t}\leq C\frac{||f||_{BMO_+}^2}{|I|^3}\int_0^{|I|}tdt\int_I dx\leq C||f||_{BMO_+}^2. \tag{2.5}$$

Finally we have to analyze

$$\frac{1}{|I|}\int_0^{|I|}\int_I|Q_t^\lambda(f_3)(x)|^2\frac{dxdt}{t}=\frac{|f_{2I}|^2}{|I|}\int_0^{|I|}\int_I|Q_t^\lambda(1)(x)|^2\frac{dxdt}{t}.$$

In the classical case this term does not appear ([11, p. 160]).

We first observe that, since $2I=(x_I-|I|,x_I+|I|)$,

$$|f_{2I}|\leq\frac{1}{|I|}\int_0^{x_I+|I|}|f(y)|dy\leq\frac{x_I+|I|}{|I|}||f||_{BMO_+}. \tag{2.6}$$

In order to estimate

$$\frac{1}{|I|}\int_0^{|I|}\int_I|Q_t^\lambda(1)(x)|^2\frac{dxdt}{t}$$

we write, for $t, x \in (0,\infty)$,

$$Q_t^\lambda(1)(x) = t^2 \left(\int_0^{\frac{t^2}{x}} + \int_{\min\{\frac{t^2}{x},\frac{x}{2}\}}^{\frac{x}{2}} + \int_{\max\{\frac{t^2}{x},\frac{x}{2}\}}^{\max\{\frac{t^2}{x},\frac{3x}{2}\}} + \int_{\max\{\frac{t^2}{x},\frac{3x}{2}\}}^{\infty} \right) \frac{\partial}{\partial s} W_s^\lambda(x,y)_{|s=t^2} dy.$$

According to (2.1) it has

$$\left| \frac{\partial}{\partial s} W_s^\lambda(x,y) \right| \leq C \frac{(xy)^\lambda}{s^{\lambda+\frac{3}{2}}} e^{-\frac{x^2+y^2}{8s}},$$

provided that $s, x, y \in (0,\infty)$ and $xy \leq s$. Then

$$\left| \int_0^{\frac{t^2}{x}} \frac{\partial}{\partial s} W_s^\lambda(x,y)_{|s=t^2} dy \right| \leq C \frac{x^\lambda e^{-\frac{x^2}{8t^2}}}{t^{2\lambda+3}} \int_0^\infty e^{-\frac{y^2}{8t^2}} y^\lambda dy \leq C \frac{x^\lambda e^{-\frac{x^2}{8t^2}}}{t^{\lambda+2}}, \quad t, x \in (0,\infty).$$

Hence

$$\frac{1}{|I|} \int_0^{|I|} \int_I \left| t^2 \int_0^{\frac{t^2}{x}} \frac{\partial}{\partial s} W_s^\lambda(x,y)_{|s=t^2} dy \right|^2 \frac{dxdt}{t} \leq \frac{C}{|I|} \int_I \int_0^\infty \frac{x^{2\lambda}}{t^{2\lambda+1}} e^{-\frac{x^2}{4t^2}} dtdx \leq C. \tag{2.7}$$

Also, if $x_I > |I|$ we can write

$$\begin{aligned} \frac{1}{|I|} \int_0^{|I|} \int_I \left| t^2 \int_0^{\frac{t^2}{x}} \frac{\partial}{\partial s} W_s^\lambda(x,y)_{|s=t^2} dy \right|^2 \frac{dxdt}{t} &\leq \frac{C}{|I|} \int_0^{|I|} \int_I \frac{x^{2\lambda}}{t^{2\lambda+1}} e^{-\frac{x^2}{4t^2}} dxdt \\ &\leq \frac{C}{|I|} \int_0^{|I|} \int_I \frac{x^{2\lambda}}{t^{2\lambda+1}} \left(\frac{t^2}{x^2} \right)^{\lambda+\frac{3}{2}} dxdt \\ &\leq \frac{C}{|I|} \int_0^{|I|} t^2 dt \int_I \frac{1}{x^3} dx \leq C|I|^3 \frac{x_I}{(4x_I^2 - |I|^2)^2}. \end{aligned} \tag{2.8}$$

Then according to (2.6), (2.7) and (2.8) we obtain

$$\frac{|f_{2I}|^2}{|I|} \int_0^{|I|} \int_I \left| t^2 \int_0^{\frac{t^2}{x}} \frac{\partial}{\partial s} W_s^\lambda(x,y)_{|s=t^2} dy \right|^2 \frac{dxdt}{t} \leq C \begin{cases} ||f||^2_{BMO_+}, & \text{when } x_I < |I|, \\ \frac{x_I |I|}{(2x_I - |I|)^2} ||f||^2_{BMO_+}, & \text{when } x_I > |I|. \end{cases}$$

Thus we conclude that

$$\frac{|f_{2I}|^2}{|I|} \int_0^{|I|} \int_I \left| t^2 \int_0^{\frac{t^2}{x}} \frac{\partial}{\partial s} W_s^\lambda(x,y)_{|s=t^2} dy \right|^2 \frac{dxdt}{t} \leq C ||f||^2_{BMO_+}. \tag{2.9}$$

On the other hand, by [9, (5.7.9)] one has

$$\begin{aligned}\frac{\partial}{\partial s}W_s^\lambda(x,y) &= \frac{\partial}{\partial s}\left(\frac{1}{\sqrt{2s}}e^{-\frac{(x-y)^2}{4s}}\right)e^{-\frac{xy}{2s}}\sqrt{\frac{xy}{2s}}I_{\lambda-\frac{1}{2}}\left(\frac{xy}{2s}\right)\\ &+ \frac{1}{\sqrt{2s}}e^{-\frac{(x-y)^2}{4s}}\frac{\partial}{\partial s}\left(e^{-\frac{xy}{2s}}\left(\frac{xy}{2s}\right)^\lambda\left(\frac{xy}{2s}\right)^{-\lambda+\frac{1}{2}}I_{\lambda-\frac{1}{2}}\left(\frac{xy}{2s}\right)\right)\\ &= \frac{\partial}{\partial s}\left(\frac{1}{\sqrt{2s}}e^{-\frac{(x-y)^2}{4s}}\right)e^{-\frac{xy}{2s}}\sqrt{\frac{xy}{2s}}I_{\lambda-\frac{1}{2}}\left(\frac{xy}{2s}\right)\\ &+ \frac{xy}{2\sqrt{2}s^{\frac{5}{2}}}e^{-\frac{(x-y)^2}{4s}}\left(e^{-\frac{xy}{2s}}\sqrt{\frac{xy}{2s}}I_{\lambda-\frac{1}{2}}\left(\frac{xy}{2s}\right)\right.\\ &- e^{-\frac{xy}{2s}}\frac{2\lambda s}{xy}\sqrt{\frac{xy}{2s}}I_{\lambda-\frac{1}{2}}\left(\frac{xy}{2s}\right)\\ &- \left.e^{-\frac{xy}{2s}}\sqrt{\frac{xy}{2s}}I_{\lambda+\frac{1}{2}}\left(\frac{xy}{2s}\right)\right), \quad t,x,y\in(0,\infty).\end{aligned}$$

Then according to [9, 5.11.10] we get

$$\frac{\partial}{\partial s}W_s^\lambda(x,y) = \frac{\partial}{\partial s}\left(\frac{1}{2\sqrt{\pi s}}e^{-\frac{(x-y)^2}{4s}}\right) + e^{-\frac{(x-y)^2}{8s}}O\left(\frac{1}{xy\sqrt{s}}\right), \tag{2.10}$$

provided that $s,x,y\in(0,\infty)$ and $xy\geq s$. Hence we deduce that

$$\left|\frac{\partial}{\partial s}W_s^\lambda(x,y)\right| \leq C\frac{e^{-\frac{(x-y)^2}{8s}}}{s^{\frac{3}{2}}}, \quad s,x,y\in(0,\infty) \text{ and } xy\geq s.$$

Thus, if $t,x\in(0,\infty)$,

$$\begin{aligned}\left|\int_{\max\{\frac{3x}{2},\frac{t^2}{x}\}}^\infty \frac{\partial}{\partial s}W_s^\lambda(x,y)_{|s=t^2}dy\right| &\leq \frac{C}{t^3}\int_{\max\{\frac{3x}{2},\frac{t^2}{x}\}}^\infty e^{-\frac{y^2}{72t^2}}dy \leq C\frac{e^{-\frac{x^2}{64t^2}}}{t^3}\int_{\max\{\frac{3x}{2},\frac{t^2}{x}\}}^\infty e^{-\frac{y^2}{144t^2}}dy\\ &\leq C\frac{e^{-\frac{x^2}{64t^2}}}{t^3}\int_0^\infty e^{-\frac{y^2}{144t^2}}dy \leq C\frac{e^{-\frac{x^2}{64t^2}}}{t^2},\end{aligned} \tag{2.11}$$

and,

$$\left|\int_{\min\{\frac{t^2}{x},\frac{x}{2}\}}^{\frac{x}{2}} \frac{\partial}{\partial s}W_s^\lambda(x,y)_{|s=t^2}dy\right| \leq \frac{C}{t^3}\int_0^{\frac{x}{2}} e^{-\frac{(x-y)^2}{8t^2}}dy \leq C\frac{xe^{-\frac{x^2}{32t^2}}}{t^3} \leq C\frac{e^{-\frac{x^2}{64t^2}}}{t^2}. \tag{2.12}$$

By combining (2.11) and (2.12) we obtain

$$\frac{1}{|I|}\int_0^{|I|}\int_I\left|t^2\left(\int_{\max\{\frac{t^2}{x},\frac{3x}{2}\}}^\infty + \int_{\min\{\frac{t^2}{x},\frac{x}{2}\}}^{\frac{x}{2}}\right)\frac{\partial}{\partial s}W_s^\lambda(x,y)_{|s=t^2}dy\right|^2\frac{dxdt}{t}$$

$$\leq \frac{C}{|I|}\int_0^{|I|}\int_I e^{-\frac{x^2}{32t^2}}\frac{dxdt}{t} \leq \frac{C}{|I|}\int_0^{|I|}\frac{1}{t}\int_0^\infty e^{-\frac{x^2}{32t^2}}dxdt \leq C,$$

and also, when $x_I > |I|$, we can write

$$\frac{1}{|I|}\int_0^{|I|}\int_I \left| t^2 \left(\int_{\max\{\frac{t^2}{x},\frac{3x}{2}\}}^\infty + \int_{\min\{\frac{t^2}{x},\frac{x}{2}\}}^{\frac{x}{2}} \right) \frac{\partial}{\partial s} W_s^\lambda(x,y)_{|s=t^2} dy \right|^2 \frac{dxdt}{t}$$

$$\leq \frac{C}{|I|}\int_0^{|I|}\int_I e^{-\frac{x^2}{32t^2}} \frac{dxdt}{t} \leq \frac{C}{|I|}\int_I \frac{1}{x^3}dx \int_0^{|I|} t^2 dt \leq C|I|^3 \frac{x_I}{(4x_I^2-|I|^2)^2}.$$

By (2.6) and proceeding as above we conclude that

$$\frac{|f_{2I}|^2}{|I|}\int_0^{|I|}\int_I \left| t^2 \left(\int_{\max\{\frac{t^2}{x},\frac{3x}{2}\}}^\infty + \int_{\min\{\frac{t^2}{x},\frac{x}{2}\}}^{\frac{x}{2}} \right) \frac{\partial}{\partial s} W_s^\lambda(x,y)_{\big|s=t^2} dy \right|^2 \frac{dxdt}{t} \leq C||f||^2_{BMO_+}. \tag{2.13}$$

We now prove that

$$\frac{|f_{2I}|^2}{|I|}\int_0^{|I|}\int_I \left| t^2 \int_{\max\{\frac{t^2}{x},\frac{x}{2}\}}^{\max\{\frac{t^2}{x},\frac{3x}{2}\}} \frac{\partial}{\partial s} W_s^\lambda(x,y)_{\big|s=t^2} dy \right|^2 \frac{dxdt}{t} \leq C||f||^2_{BMO_+}. \tag{2.14}$$

Equality (2.10) suggests us to write, for each $t, x \in (0,\infty)$,

$$\begin{aligned}\int_{\max\{\frac{t^2}{x},\frac{x}{2}\}}^{\max\{\frac{t^2}{x},\frac{3x}{2}\}} \frac{\partial}{\partial s} W_s^\lambda(x,y)_{\big|s=t^2} dy &= \int_{(\frac{x}{2},\frac{3x}{2})\cap(\frac{t^2}{x},\infty)} \frac{\partial}{\partial s}\left[W_s^\lambda(x,y) - W_s(x,y)\right]_{\big|s=t^2} dy \\ &+ \frac{\partial}{\partial s}\left(\int_{(\frac{x}{2},\frac{3x}{2})\cap(\frac{t^2}{x},\infty)} W_s(x,y)dy \right)_{\big|s=t^2},\end{aligned}$$

where $W_s(x,y) = \frac{e^{-\frac{(x-y)^2}{4s}}}{2\sqrt{\pi s}}$, $s, x, y \in (0,\infty)$.

We note that

$$\begin{aligned}\int_{\frac{x}{2}}^{\frac{3x}{2}} W_s(x,y)dy &= \int_{\frac{x}{2}}^{\frac{3x}{2}} \frac{e^{-\frac{(x-y)^2}{4s}}}{2\sqrt{\pi s}} dy = \frac{1}{2\sqrt{\pi}}\int_{-\frac{x}{2}}^{\frac{x}{2}} \frac{e^{-\frac{u^2}{4s}}}{\sqrt{s}} du = \frac{1}{\sqrt{\pi}}\int_0^{\frac{x}{2}} \frac{e^{-\frac{u^2}{4s}}}{\sqrt{s}} du \\ &= \frac{1}{\sqrt{\pi}}\int_0^\infty \frac{e^{-\frac{u^2}{4s}}}{\sqrt{s}} du - \frac{1}{\sqrt{\pi}}\int_{\frac{x}{2}}^\infty \frac{e^{-\frac{u^2}{4s}}}{\sqrt{s}} du \\ &= 1 - \frac{1}{\sqrt{\pi}}\int_{\frac{x}{2}}^\infty \frac{e^{-\frac{u^2}{4s}}}{\sqrt{s}} du, \quad s, x \in (0,\infty).\end{aligned}$$

Then

$$\frac{\partial}{\partial s}\int_{\frac{x}{2}}^{\frac{3x}{2}} W_s(x,y)dy = -\frac{1}{\sqrt{\pi}}\int_{\frac{x}{2}}^\infty \frac{\partial}{\partial s}\left(\frac{e^{-\frac{u^2}{4s}}}{\sqrt{s}} \right) du, \quad s, x \in (0,\infty),$$

and, for every $t,x\in(0,\infty)$,

$$\left|\frac{\partial}{\partial s}\left(\int_{\frac{x}{2}}^{\frac{3x}{2}}W_s(x,y)dy\right)_{\big|s=t^2}\right|\le C\int_{\frac{x}{2}}^{\infty}\frac{e^{-\frac{u^2}{8t^2}}}{t^3}du\le C\frac{e^{-\frac{x^2}{64t^2}}}{t^3}\int_0^{\infty}e^{-\frac{u^2}{16t^2}}du\le C\frac{e^{-\frac{x^2}{64t^2}}}{t^2}.$$

Moreover, for each $t,x\in(0,\infty)$,

$$\int_{(\frac{x}{2},\frac{3x}{2})\cap(0,\frac{t^2}{x})}\left|\frac{\partial}{\partial s}W_s(x,y)_{\big|s=t^2}\right|dy\le C\int_0^{\frac{t^2}{x}}\frac{e^{-\frac{(x-y)^2}{8t^2}}}{t^3}dy\le C\int_0^{\frac{t^2}{x}}\frac{e^{-\frac{x^2+y^2}{8t^2}}}{t^3}dy\le C\frac{e^{-\frac{x^2}{8t^2}}}{t^2}.$$

By proceeding as in the proof of (2.13) we obtain

$$\frac{|f_{2I}|^2}{|I|}\int_0^{|I|}\int_I\left|t^2\frac{\partial}{\partial s}\left(\int_{(\frac{x}{2},\frac{3x}{2})\cap(\frac{t^2}{x},\infty)}W_s(x,y)dy\right)_{\big|s=t^2}\right|^2\frac{dxdt}{t}\le C||f||^2_{BMO_+}. \tag{2.15}$$

On the other hand, equation (2.10) leads to

$$\begin{aligned}\left|\int_{(\frac{x}{2},\frac{3x}{2})\cap(\frac{t^2}{x},\infty)}\frac{\partial}{\partial s}\Big[W_s^\lambda(x,y)-W_s(x,y)\Big]_{\big|s=t^2}dy\right| &\le \frac{C}{xt}\int_{\frac{x}{2}}^{\frac{3x}{2}}e^{-\frac{(x-y)^2}{8t^2}}\frac{dy}{y}\\ &\le \frac{C}{x^2t}\int_{-\infty}^{+\infty}e^{-\frac{(x-y)^2}{8t^2}}dy\le\frac{C}{x^2},\end{aligned}$$

for each $t,x\in(0,\infty)$.

Hence,

$$\frac{1}{|I|}\int_0^{|I|}\int_I\left|t^2\int_{(\frac{x}{2},\frac{3x}{2})\cap(\frac{t^2}{x},\infty)}\frac{\partial}{\partial s}\Big[W_s^\lambda(x,y)-W_s(x,y)\Big]_{\big|s=t^2}dy\right|^2\frac{dxdt}{t}\le\frac{C}{|I|}\int_0^{|I|}t^3\int_{\sqrt{\frac{2}{3}}t}^{\infty}\frac{1}{x^4}dxdt\le C,$$

and if $x_I>|I|$, we can also write

$$\begin{aligned}&\frac{1}{|I|}\int_0^{|I|}\int_I\left|t^2\int_{(\frac{x}{2},\frac{3x}{2})\cap(\frac{t^2}{x},\infty)}\frac{\partial}{\partial s}\Big[W_s^\lambda(x,y)-W_s(x,y)\Big]_{\big|s=t^2}dy\right|^2\frac{dxdt}{t}\\ &\le\frac{C}{|I|}\int_0^{|I|}t^3dt\int_I\frac{1}{x^4}dx\le C|I|^3\frac{(2x_I+|I|)^3-(2x_I-|I|)^3}{(4x_I^2-|I|^2)^3}.\end{aligned}$$

These estimates and (2.6) allow us to obtain, when $x_I<|I|$,

$$\frac{|f_{2I}|^2}{|I|}\int_0^{|I|}\int_I\left|t^2\int_{(\frac{x}{2},\frac{3x}{2})\cap(\frac{t^2}{x},\infty)}\frac{\partial}{\partial s}\Big[W_s^\lambda(x,y)-W_s(x,y)\Big]_{\big|s=t^2}dy\right|^2\frac{dxdt}{t}\le C||f||^2_{BMO_+},$$

and, in the case that $x_I>|I|$,

$$\begin{aligned}&\frac{|f_{2I}|^2}{|I|}\int_0^{|I|}\int_I\left|t^2\int_{(\frac{x}{2},\frac{3x}{2})\cap(\frac{t^2}{x},\infty)}\frac{\partial}{\partial s}\Big[W_s^\lambda(x,y)-W_s(x,y)\Big]_{\big|s=t^2}dy\right|^2\frac{dxdt}{t}\\ &\le C\frac{|I|}{x_I+|I|}\frac{(2x_I+|I|)^3-(2x_I-|I|)^3}{(2x_I-|I|)^3}||f||^2_{BMO_+}\le C||f||^2_{BMO_+}.\end{aligned}$$

Thus we have that

$$\frac{|f_{2I}|^2}{|I|}\int_0^{|I|}\int_I\left|t^2\int_{(\frac{x}{2},\frac{3x}{2})\cap(\frac{t^2}{x},\infty)}\frac{\partial}{\partial s}\Big[W_s^\lambda(x,y)-W_s(x,y)\Big]_{\big|s=t^2}dy\right|^2\frac{dxdt}{t}\le C||f||^2_{BMO_+},$$

that, jointly (2.15), leads to (2.14).

By (2.9), (2.13) and (2.14) we obtain

$$\frac{1}{|I|}\int_0^{|I|}\int_I|Q_t^\lambda(f_3)(x)|^2\frac{dxdt}{t}\le C||f||^2_{BMO_+}. \tag{2.16}$$

Hence, by taking into account estimations (2.4), (2.5) and (2.16) we establish (2.3) and the proof is thus finished.

3. Proof of $(ii)\implies(iii)$ in Theorem 1.5

Let $\lambda>0$ and f a measurable function on $(0,\infty)$ such that $(1+x^2)^{-1}f\in L^1(0,\infty)$. Assume that $d\mu_f$ is a Carleson measure.

By using subordination formula we have that

$$P_t^\lambda(f)(x)=\frac{1}{\sqrt{\pi}}\int_0^\infty\frac{e^{-u}}{\sqrt{u}}W^\lambda_{\frac{t^2}{4u}}(f)(x)du,\quad t,x\in(0,\infty).$$

Then,

$$t\frac{\partial}{\partial t}P_t^\lambda(f)(x)=\frac{1}{\sqrt{\pi}}\int_0^\infty\frac{t^2}{2u}\frac{e^{-u}}{\sqrt{u}}\frac{\partial}{\partial s}W_s^\lambda(f)(x)_{\big|s=\frac{t^2}{4u}}du,\quad t,x\in(0,\infty).$$

Minkowski inequality leads to

$$\begin{aligned}
&\left\{\frac{1}{|I|}\int_0^{|I|}\int_I\left|t\frac{\partial}{\partial t}P_t^\lambda(f)(x)\right|^2\frac{dxdt}{t}\right\}^{\frac12}\\
&\le\frac{1}{\sqrt{\pi}}\int_0^\infty\frac{e^{-u}}{\sqrt{u}}\left\{\frac{1}{|I|}\int_0^{|I|}\int_I\left|\frac{t^2}{2u}\frac{\partial}{\partial s}W_s^\lambda(f)(x)_{\big|s=\frac{t^2}{4u}}\right|^2\frac{dxdt}{t}\right\}^{\frac12}du\\
&=\frac{1}{\sqrt{\pi}}\int_0^\infty\frac{e^{-u}}{\sqrt{u}}\left\{\frac{1}{|I|}\int_0^{\frac{|I|}{2\sqrt{u}}}\int_I\left|2v^2\frac{\partial}{\partial s}W_s^\lambda(f)(x)_{|u=v^2}\right|^2\frac{dxdv}{v}\right\}^{\frac12}du\\
&\le C\left(\int_{2\sqrt{u}\ge1}\frac{e^{-u}}{\sqrt{u}}\left\{\frac{1}{|I|}\int_0^{|I|}\int_I\left|2v^2\frac{\partial}{\partial s}W_s^\lambda(f)(x)_{|u=v^2}\right|^2\frac{dxdv}{v}\right\}^{\frac12}du\right.\\
&\left.+\int_{2\sqrt{u}<1}\frac{e^{-u}}{u^{\frac34}}\left\{\frac{1}{|J|}\int_0^{|J|}\int_J\left|2v^2\frac{\partial}{\partial s}W_s^\lambda(f)(x)_{|u=v^2}\right|^2\frac{dxdv}{v}\right\}^{\frac12}du\right)
\end{aligned}$$

$$\leq C(||\mu_f||_{\mathcal{C}})^{\frac{1}{2}} \int_0^\infty \frac{e^{-\frac{u}{2}}}{u^{\frac{3}{4}}} du \leq C(||\mu_f||_{\mathcal{C}})^{\frac{1}{2}},$$

for every $I \subset (0,\infty)$. Hence $d\gamma_f$ is a Carleson measure and $||\gamma_f||_{\mathcal{C}} \leq C||\mu_f||_{\mathcal{C}}$.

4. Proof of $(iii) \Longrightarrow (i)$ in Theorem 1.5

In order to prove that $(iii) \Rightarrow (i)$ in Theorem 1.5 we previously need to show several results. First, we establish new characterizations of the space $H^1_o(\mathbb{R})$ by using nontangencial g-functions associated with the Poisson semigroup for the Bessel operators Δ_λ.

We denote by $P_t(x)$ the classical Poisson kernel, that is,

$$P_t(x) = \frac{1}{\pi}\frac{t}{t^2+x^2}, \quad t \in (0,\infty), \;\; x \in \mathbb{R}.$$

The classical Poisson integral of f is defined by

$$P_t(f)(x) = \int_{\mathbb{R}} P_t(y-x) f(y) dy, \quad t \in (0,\infty), \;\; x \in \mathbb{R}.$$

We consider the sets

$$\Gamma(x) = \{(y,t) \in \mathbb{R} \times (0,\infty) : |x-y| < t\}, \quad x \in \mathbb{R},$$

and

$$\Gamma_+(x) = \{(y,t) \in (0,\infty) \times (0,\infty) : |x-y| < t\}, \quad x \in (0,\infty).$$

In the next proposition we use the nontangencial g-functions defined as follows:

$$g(f)(x) = \left\{ \int_{\Gamma(x)} \left| t \frac{\partial}{\partial t} P_t(f)(y) \right|^2 \frac{dtdy}{t^2} \right\}^{\frac{1}{2}}, \quad x \in \mathbb{R},$$

$$g_+(f)(x) = \left\{ \int_{\Gamma_+(x)} \left| t \frac{\partial}{\partial t} P_t(f)(y) \right|^2 \frac{dtdy}{t^2} \right\}^{\frac{1}{2}}, \quad x \in (0,\infty),$$

and

$$g_\lambda(f)(x) = \left\{ \int_{\Gamma_+(x)} \left| t \frac{\partial}{\partial t} P_t^\lambda(f)(y) \right|^2 \frac{dtdy}{t^2} \right\}^{\frac{1}{2}}, \quad x \in (0,\infty).$$

Proposition 4.1. *Let $\lambda > 0$. Suppose that $f \in L^1(\mathbb{R})$ and f is odd. Then the following assertions are equivalent:*

(a) $f \in H^1(\mathbb{R})$.

(b) $g(f) \in L^1(\mathbb{R})$.

(c) $g_+(f) \in L^1(0,\infty)$.

(d) $g_\lambda(f) \in L^1(0,\infty)$.

Moreover, the quantities $||f||_{L^1(\mathbb{R})} + ||g(f)||_{L^1(\mathbb{R})}$, $||f||_{L^1(0,\infty)} + ||g_+(f)||_{L^1(0,\infty)}$ *and* $||f||_{L^1(0,\infty)} + ||g_\lambda(f)||_{L^1(0,\infty)}$ *are equivalent to* $||f||_{H^1(\mathbb{R})}$.

Proof. $(a) \Leftrightarrow (b)$. It is a well-known result ([11, Proposition 4, p. 124]).

$(b) \Leftrightarrow (c)$. It is clear that $(b) \Rightarrow (c)$. On the other hand, since f is odd, we can write, for every $y \in \mathbb{R}$ and $t > 0$,

$$P_t(f)(y) = \int_{-\infty}^{+\infty} P_t(z-y)f(z)dz = \int_0^\infty (P_t(z-y) - P_t(z+y))f(z)dz.$$

Hence $P_t(f)$, $t > 0$, is an odd function. Then, for every $t, x \in (0,\infty)$, we can write

$$\begin{aligned}\int_{\{y\in\mathbb{R}:(y,t)\in\Gamma(-x)\}} \left|t\frac{\partial}{\partial t}P_t(f)(y)\right|^2 dy &= \int_{-x-t}^{-x+t} \left|t\frac{\partial}{\partial t}P_t(f)(y)\right|^2 dy = \int_{x-t}^{x+t} \left|t\frac{\partial}{\partial t}P_t(f)(y)\right|^2 dy \\ &= \int_{\{y\in\mathbb{R}:(y,t)\in\Gamma(x)\}} \left|t\frac{\partial}{\partial t}P_t(f)(y)\right|^2 dy,\end{aligned}$$

and also, if $0 < x < t$,

$$\begin{aligned}\int_{\{y\in\mathbb{R}:(y,t)\in\Gamma(x)\}} \left|t\frac{\partial}{\partial t}P_t(f)(y)\right|^2 dy &= \left(\int_{x-t}^0 + \int_0^{x+t}\right) \left|t\frac{\partial}{\partial t}P_t(f)(y)\right|^2 dy \leq 2\int_0^{x+t} \left|t\frac{\partial}{\partial t}P_t(f)(y)\right|^2 dy \\ &\leq 2\int_{\{y\in(0,\infty):(y,t)\in\Gamma_+(x)\}} \left|t\frac{\partial}{\partial t}P_t(f)(y)\right|^2 dy.\end{aligned}$$

Thus, we get that $g(f)$ is an even function verifying that $g(f)(x) \leq \sqrt{2}g_+(f)(|x|)$, $x \in \mathbb{R}$, and we conclude that $(c) \Rightarrow (b)$.

Let us prove $(c) \Leftrightarrow (d)$. First we show that (c) is equivalent to the following property:

(c') $g_{+,\text{loc}}(f)(x) = \left\{\int_{\Gamma_+(x)} \left|t\frac{\partial}{\partial t}\int_{\frac{x}{2}}^{2x} P_t(z-y)f(z)dz\right|^2 \frac{dtdy}{t^2}\right\}^{\frac{1}{2}} \in L^1(0,\infty)$.

Note that, for every $x \in (0,\infty)$,

$$|g_+(f)(x) - g_{+,\text{loc}}(f)(x)| \leq \left\{\int_{\Gamma_+(x)} \left|t\frac{\partial}{\partial t}\left[P_t(f)(y) - \int_{\frac{x}{2}}^{2x} P_t(z-y)f(z)dz\right]\right|^2 \frac{dtdy}{t^2}\right\}^{\frac{1}{2}}$$

$$= \left\{\int_{\Gamma_+(x)} \left|t\frac{\partial}{\partial t}\left[\int_0^\infty (P_t(z-y) - P_t(z+y))f(z)dz - \int_{\frac{x}{2}}^{2x} P_t(z-y)f(z)dz\right]\right|^2 \frac{dtdy}{t^2}\right\}^{\frac{1}{2}}$$

$$= \left\{ \int_{\Gamma_+(x)} \left| t\frac{\partial}{\partial t} \int_{(0,\infty)\setminus(\frac{x}{2},2x)} P_t(z-y)f(z)dz - t\frac{\partial}{\partial t}\int_0^\infty P_t(z+y)f(z)dz \right|^2 \frac{dtdy}{t^2} \right\}^{\frac{1}{2}}.$$

By using Minkowski inequality, it gets, for every $x \in (0,\infty)$,

$$|g_+(f)(x) - g_{+,\mathrm{loc}}(f)(x)| \leq \mathcal{H}_1(f)(x) + \mathcal{H}_2(f)(x),$$

where

$$\mathcal{H}_1(f)(x) = \int_{(0,\infty)\setminus(\frac{x}{2},2x)} |f(z)| \left\{ \int_{\Gamma_+(x)} \left| t\frac{\partial}{\partial t}\Big[P_t(z-y) - P_t(z+y)\Big] \right|^2 \frac{dtdy}{t^2} \right\}^{\frac{1}{2}} dz,$$

and

$$\mathcal{H}_2(f)(x) = \int_{\frac{x}{2}}^{2x} |f(z)| \left\{ \int_{\Gamma_+(x)} \left| t\frac{\partial}{\partial t} P_t(z+y) \right|^2 \frac{dtdy}{t^2} \right\}^{\frac{1}{2}} dz.$$

Then the equivalence between (c) and (c') will be established when we see that $\mathcal{H}_i(f)$, $i = 1, 2$, belongs to $L^1(0,\infty)$.

We begin analyzing $\mathcal{H}_2$. We can write, for every $t, z, y \in (0,\infty)$,

$$\left| t\frac{\partial}{\partial t} P_t(z+y) \right| = |(1 - 2\pi t P_t(z+y))P_t(z+y)| \leq C P_t(z+y) \leq C\frac{t}{(z+y+t)^2}.$$

Hence, for each $x, z \in (0,\infty)$,

$$\begin{aligned}\int_{\Gamma_+(x)} \left| t\frac{\partial}{\partial t} P_t(z+y) \right|^2 \frac{dtdy}{t^2} &\leq C\int_{\Gamma_+(x)} \frac{dtdy}{(z+y+t)^4} \leq C\int_0^\infty \frac{dy}{(z+y+|x-y|)^3} \\ &\leq C\int_0^\infty \frac{1}{(z+y)^3} dy \leq \frac{C}{z^2},\end{aligned}$$

and then,

$$\int_0^\infty |\mathcal{H}_2(f)(x)| dx \leq C\int_0^\infty \int_{\frac{x}{2}}^{2x} \frac{|f(z)|}{z} dzdx \leq C||f||_{L^1(0,\infty)}.$$

On the other hand, it has

$$\frac{\partial}{\partial t}\left[P_t(z-y) - P_t(z+y)\right] = \frac{4zy}{\pi((z-y)^2+t^2)((z+y)^2+t^2)} - \frac{16zyt^2(z^2+y^2+t^2)}{\pi((z-y)^2+t^2)^2((z+y)^2+t^2)^2},$$

for each $z, y, t \in (0,\infty)$.

Then,

$$\left| \frac{\partial}{\partial t}\left[P_t(z-y) - P_t(z+y)\right] \right| \leq C\frac{zy}{((z-y)^2+t^2)((z+y)^2+t^2)}$$

$$\leq \; C\frac{\sqrt{z}}{(|z-y|+t)^2\sqrt{z+y+t}} \leq \frac{\sqrt{z}}{(|z-y|+t)^{\frac{5}{2}}}, \quad z,y,t\in(0,\infty).$$

Hence, for $z,x\in(0,\infty)$,

$$\begin{aligned}\int_{\Gamma_+(x)}\Big|t\frac{\partial}{\partial t}\Big[P_t(z-y)-P_t(z+y)\Big]\Big|^2\frac{dtdy}{t^2} &\leq C\int_{\Gamma_+(x)}\frac{z}{(|z-y|+t)^5}dtdy \leq C\int_0^\infty \frac{z}{(|z-y|+|x-y|)^4}dy\\ (4.2)\quad &\leq C\left[\int_{\min\{x,z\}}^{\max\{x,z\}}\frac{z}{(x-z)^4}dy+\Big(\int_0^{\min\{x,z\}}+\int_{\max\{x,z\}}^\infty\Big)\frac{z}{(x+z-2y)^4}dy\right]\leq C\frac{z}{|x-z|^3},\end{aligned}$$

and we get

$$\int_0^\infty |\mathcal{H}_1(f)(x)|dx \leq C\int_0^\infty |f(z)|\sqrt{z}\left(\int_0^{\frac{z}{2}}+\int_{2z}^\infty\right)\frac{1}{|x-z|^{\frac{3}{2}}}dxdz \leq C||f||_{L^1(0,\infty)}.$$

Thus we have proved that $||g_+(f)-g_{+,\mathrm{loc}}(f)||_{L^1(0,\infty)} \leq ||f||_{L^1(0,\infty)}$ which implies that (c) is equivalent to (c').

We now show that $(c') \Leftrightarrow (d)$. The Poisson kernel P_t^λ given by (1.4) is splited into two parts as follows

$$\begin{aligned}P_t^\lambda(y,z) &= \frac{2\lambda(yz)^\lambda t}{\pi}\left(\int_0^{\frac{\pi}{2}}+\int_{\frac{\pi}{2}}^\pi\right)\frac{(\sin\theta)^{2\lambda-1}}{((z-y)^2+t^2+2zy(1-\cos\theta))^{\lambda+1}}d\theta\\ &= P_{t,1}^\lambda(y,z)+P_{t,2}^\lambda(y,z), \quad t,y,z\in(0,\infty).\end{aligned}$$

We observe that, for each $x\in(0,\infty)$,

$$|g_\lambda(f)(x)-g_{+,\mathrm{loc}}(f)(x)| \leq \left\{\int_{\Gamma_+(x)}\left|t\frac{\partial}{\partial t}\left[P_t^\lambda(f)(y)-\int_{\frac{x}{2}}^{2x}P_t(z-y)f(z)dz\right]\right|^2\frac{dtdy}{t^2}\right\}^{\frac{1}{2}}.$$

Then, Minkowski inequality leads to

$$\begin{aligned}|g_\lambda(f)(x)-g_{+,\mathrm{loc}}(f)(x)| &\leq \int_0^\infty |f(z)|\left\{\int_{\Gamma_+(x)}\left|t\frac{\partial}{\partial t}P_{t,2}^\lambda(y,z)\right|^2\frac{dtdy}{t^2}\right\}^{\frac{1}{2}}dz\\ &+ \int_{(0,\infty)\setminus(\frac{x}{2},2x)}|f(z)|\left\{\int_{\Gamma_+(x)}\left|t\frac{\partial}{\partial t}P_{t,1}^\lambda(y,z)\right|^2\frac{dtdy}{t^2}\right\}^{\frac{1}{2}}dz\\ &+ \int_{\frac{x}{2}}^{2x}|f(z)|\left\{\int_{\Gamma_+(x)}\left|t\frac{\partial}{\partial t}\left[P_{t,1}^\lambda(y,z)-P_t(z-y)\right]\right|^2\frac{dtdy}{t^2}\right\}^{\frac{1}{2}}dz = \sum_{j=1}^3\mathcal{K}_j(f)(x), \quad x\in(0,\infty).\end{aligned}$$

To see that (c') is equivalent to (d) it is sufficient to show that $\mathcal{K}_j(f)\in L^1(0,\infty)$, $j=1,2,3$.

Note firstly that, since $\lambda > 0$,

$$\begin{aligned}\left|\frac{\partial}{\partial t}P^{\lambda}_{t,2}(y,z)\right| &\leq C(yz)^{\lambda}\left[\int_{\frac{\pi}{2}}^{\pi}\frac{(\sin\theta)^{2\lambda-1}}{(z^2+y^2+t^2-2zy\cos\theta)^{\lambda+1}}d\theta+\int_{\frac{\pi}{2}}^{\pi}\frac{t^2(\sin\theta)^{2\lambda-1}}{(z^2+y^2+t^2-2zy\cos\theta)^{\lambda+2}}d\theta\right]\\ &\leq C\frac{(yz)^{\lambda}}{(z^2+y^2+t^2)^{\lambda+1}}\leq C\frac{z^{\lambda}}{(z+y+t)^{\lambda+2}},\quad t,y,z\in(0,\infty).\end{aligned}$$

Hence, for every $z,x\in(0,\infty)$,

$$\begin{aligned}\int_{\Gamma_+(x)}\left|t\frac{\partial}{\partial t}P^{\lambda}_{t,2}(y,z)\right|^2\frac{dtdy}{t^2} &\leq C\int_0^{\infty}\int_{|x-y|}^{\infty}\frac{z^{2\lambda}}{(z+y+t)^{2\lambda+4}}dtdy\leq C\int_0^{\infty}\frac{z^{2\lambda}}{(z+y+|x-y|)^{2\lambda+3}}dy\\ &\leq Cz^{2\lambda}\left(\int_0^x\frac{1}{(z+x)^{2\lambda+3}}dy+\int_x^{\infty}\frac{1}{(z-x+2y)^{2\lambda+3}}dy\right)\leq C\frac{z^{2\lambda}}{(z+x)^{2\lambda+2}}.\end{aligned}$$

Then,

$$\int_0^{\infty}|\mathcal{K}_1(f)(x)|dx\leq C\int_0^{\infty}|f(z)|\int_0^{\infty}\frac{z^{\lambda}}{(z+x)^{\lambda+1}}dxdz\leq C||f||_{L^1(0,\infty)}.$$

Let us now see that $\mathcal{K}_2(f)\in L^1(0,\infty)$. Since $\sin\theta\sim\theta$ and $2(1-\cos\theta)\sim\theta^2$, $\theta\in[0,\frac{\pi}{2}]$, by considering separately $z>\frac{y}{2}$ and $0<z\leq\frac{y}{2}$, we have that

$$\begin{aligned}\left|\frac{\partial}{\partial t}P^{\lambda}_{t,1}(y,z)\right| &\leq C\int_0^{\frac{\pi}{2}}\frac{(yz)^{\lambda}(\sin\theta)^{2\lambda-1}}{(z^2+y^2+t^2-2zy\cos\theta)^{\lambda+1}}d\theta\leq C\int_0^{\frac{\pi}{2}}\frac{(yz)^{\lambda}\theta^{2\lambda-1}}{((z-y)^2+t^2+zy\theta^2)^{\lambda+1}}d\theta\\ &\leq Cz^{\lambda}\int_0^{\frac{\pi}{2}}\frac{\theta^{\lambda-1}}{((z-y)^2+t^2+yz\theta^2)^{\frac{\lambda}{2}+1}}d\theta\leq C\frac{z^{\lambda}}{(|z-y|+t)^{\lambda+2}},\quad t,y,z\in(0,\infty).\end{aligned}$$

Hence, by proceeding as in (4.2), it follows that, for each $x,z\in(0,\infty)$,

$$\int_{\Gamma_+(x)}\left|t\frac{\partial}{\partial t}P^{\lambda}_{t,1}(y,z)\right|^2\frac{dtdy}{t^2}\leq C\int_0^{\infty}\frac{z^{2\lambda}}{(|z-y|+|x-y|)^{2\lambda+3}}dy\leq C\frac{z^{2\lambda}}{|x-z|^{2\lambda+2}},$$

and

$$\int_0^{\infty}|\mathcal{K}_2(f)(x)|dx\leq C\int_0^{\infty}|f(z)|z^{\lambda}\left(\int_0^{\frac{z}{2}}+\int_{2z}^{\infty}\right)\frac{1}{|x-z|^{\lambda+1}}dxdz\leq C||f||_{L^1(0,\infty)}.$$

Finally, we are going to see that $\mathcal{K}_3(f)\in L^1(0,\infty)$. The proof of this fact is divided in several parts.

In a first step we will prove that, if $0<\frac{x}{2}<z<2x$, then

(4.3)

$$\left\{\int_{\Gamma_+(x)}\left|t\frac{\partial}{\partial t}\int_0^{\frac{\pi}{2}}\frac{(zy)^{\lambda}t((\sin\theta)^{2\lambda-1}-\theta^{2\lambda-1})}{((z-y)^2+t^2+2zy(1-\cos\theta))^{\lambda+1}}d\theta\right|^2\frac{dtdy}{t^2}\right\}^{\frac{1}{2}}\leq\frac{C}{z}\left(1+\log_+\frac{z^2}{|x-z|^2}\right).$$

Since $\sin\theta \sim \theta$, and $2(1-\cos\theta) \sim \theta^2$, as $\theta \in [0, \frac{\pi}{2}]$, the mean value theorem leads to

$$\begin{aligned}
\left| \frac{\partial}{\partial t} \int_0^{\frac{\pi}{2}} \frac{t((\sin\theta)^{2\lambda-1} - \theta^{2\lambda-1})}{((z-y)^2+t^2+2zy(1-\cos\theta))^{\lambda+1}} d\theta \right| &\leq C \int_0^{\frac{\pi}{2}} \frac{|(\sin\theta)^{2\lambda-1} - \theta^{2\lambda-1}|}{((z-y)^2+t^2+2zy(1-\cos\theta))^{\lambda+1}} d\theta \\
&\leq C \int_0^{\frac{\pi}{2}} \frac{\theta^{2\lambda+1}}{((z-y)^2+t^2+zy\theta^2)^{\lambda+1}} d\theta, \quad t,z,y \in (0,\infty). \qquad (4.4)
\end{aligned}$$

According to [10, p. 61] and making the change of variables $u = \theta\sqrt{\frac{zy}{(z-y)^2+t^2}}$, we get

$$\begin{aligned}
\int_0^{\frac{\pi}{2}} \frac{\theta^{2\lambda+1}}{((z-y)^2+t^2+zy\theta^2)^{\lambda+1}} d\theta &\leq \frac{C}{(zy)^{\lambda+1}} \int_0^{\frac{\pi}{2}\sqrt{\frac{zy}{(z-y)^2+t^2}}} \frac{u^{2\lambda+1}}{(1+u^2)^{\lambda+1}} du \\
&\leq \frac{C}{(zy)^{\lambda+1}} \left(\frac{\sqrt{zy/((z-y)^2+t^2)}}{1+\sqrt{zy/((z-y)^2+t^2)}} \right)^{2\lambda+2} \left[1 + \log_+ \left(\frac{zy}{(z-y)^2+t^2} \right) \right] \\
&\leq \frac{C}{(zy+(z-y)^2+t^2)^{\lambda+1}} \left[1 + \log_+ \left(\frac{zy}{(z-y)^2+t^2} \right) \right], \\
&\leq C \begin{cases} \dfrac{1}{(zy+(z-y)^2+t^2)^{\lambda+1}}, & 0 < y < \dfrac{z}{2} \text{ or } y > 2z > 0, t > 0, \\[2ex] \dfrac{1}{(zy+(z-y)^2+t^2)^{\lambda+1}} \left[1 + \log_+ \left(\dfrac{zy}{(z-y)^2+t^2} \right) \right], & \dfrac{z}{2} < y < 2z, t > 0. \end{cases}
\end{aligned}$$

In the last inequality we have taken into account that, when $0 < 2z < y$, it has that $|z-y|^2+t^2 \geq |z-y|^2 \geq \frac{zy}{2}$.

Thus we can write

$$\begin{aligned}
&\int_{\Gamma_+(x)} \left| t \frac{\partial}{\partial t} \int_0^{\frac{\pi}{2}} \frac{(zy)^\lambda t((\sin\theta)^{2\lambda-1} - \theta^{2\lambda-1})}{((z-y)^2+t^2+2zy(1-\cos\theta))^{\lambda+1}} d\theta \right|^2 \frac{dtdy}{t^2} \\
&= \left(\int_{(0,\infty)\setminus(\frac{z}{2},2z)} + \int_{\frac{z}{2}}^{2z} \right) \int_{|x-y|}^\infty \left| \frac{\partial}{\partial t} \int_0^{\frac{\pi}{2}} \frac{(zy)^\lambda t((\sin\theta)^{2\lambda-1} - \theta^{2\lambda-1})}{((z-y)^2+t^2+2zy(1-\cos\theta))^{\lambda+1}} d\theta \right|^2 dtdy \\
&= \sum_{j=1}^2 I_j(x,z), \quad x,z \in (0,\infty).
\end{aligned}$$

If $0 < \frac{x}{2} < z < 2x$ we have that

$$\begin{aligned}
I_1(x,z) &\leq C \left(\int_0^{\frac{z}{2}} + \int_{2z}^\infty \right) \int_{|x-y|}^\infty \frac{(zy)^{2\lambda} dtdy}{(zy+(z-y)^2+t^2)^{2\lambda+2}} \leq C \left(\int_0^{\frac{z}{2}} + \int_{2z}^\infty \right) \int_{|x-y|}^\infty \frac{dtdy}{(|z-y|+t)^4} \\
&\leq C \left(\int_0^{\frac{z}{2}} + \int_{2z}^\infty \right) \frac{1}{(|z-y|+|y-x|)^3} dy = C \left(\int_0^{\frac{z}{2}} + \int_{2z}^\infty \right) \frac{1}{|z+x-2y|^3} dy \leq \frac{C}{z^2}.
\end{aligned}$$

On the other hand, if $|x-y|<t$ and $z\in(0,\infty)$, then $\log_+\frac{z^2}{|z-y|^2+t^2}\leq\log_+\frac{z^2}{|z-y|^2+|x-y|^2}\leq\log_+\frac{2z^2}{|x-z|^2}$. Hence, we get, for $x,z\in(0,\infty)$,

$$\begin{aligned}I_2(x,z) &\leq C\int_{\frac{z}{2}}^{2z}\int_{|x-y|}^{\infty}\frac{z^{4\lambda}}{(z^2+(z-y)^2+t^2)^{2\lambda+2}}\left(1+\log_+\frac{z^2}{|z-y|^2+t^2}\right)^2 dtdy\\ &\leq C\left(1+\log_+\frac{2z^2}{|x-z|^2}\right)^2\int_{\frac{z}{2}}^{2z}\int_{|x-y|}^{\infty}\frac{1}{(z+|z-y|+t)^4}dtdy\\ &\leq C\left(1+\log_+\frac{z^2}{|x-z|^2}\right)^2\int_{\frac{z}{2}}^{2z}\frac{1}{(z+|z-y|+|y-x|)^3}dy\leq\frac{C}{z^2}\left(1+\log_+\frac{z^2}{|x-z|^2}\right)^2,\end{aligned}$$

and the proof of (4.3) is thus finished.

In the second step, our objective is to establish that

$$\left\{\int_{\Gamma_+(x)}\left|t\frac{\partial}{\partial t}\int_0^{\frac{\pi}{2}}\left[\frac{(zy)^\lambda t\theta^{2\lambda-1}}{((z-y)^2+t^2+2zy(1-\cos\theta))^{\lambda+1}}-\frac{(zy)^\lambda t\theta^{2\lambda-1}}{((z-y)^2+t^2+zy\theta^2)^{\lambda+1}}\right]d\theta\right|^2\frac{dtdy}{t^2}\right\}^{\frac{1}{2}}$$

$$\leq\frac{C}{z}\left(1+\log_+\frac{z^2}{(x-z)^2}\right),\quad 0<\frac{x}{2}<z<2x. \tag{4.5}$$

By using that $2(1-\cos\theta)\sim\theta^2$, as $\theta\in[0,\frac{\pi}{2}]$, and the mean value theorem, we can write

$$\begin{aligned}J(z,t,y) &= \left|\frac{\partial}{\partial t}\int_0^{\frac{\pi}{2}}\left[\frac{t\theta^{2\lambda-1}}{((z-y)^2+t^2+2zy(1-\cos\theta))^{\lambda+1}}-\frac{t\theta^{2\lambda-1}}{((z-y)^2+t^2+zy\theta^2)^{\lambda+1}}\right]d\theta\right|\\ &\leq \int_0^{\frac{\pi}{2}}\theta^{2\lambda-1}\left|\frac{1}{((z-y)^2+t^2+2zy(1-\cos\theta))^{\lambda+1}}-\frac{1}{((z-y)^2+t^2+zy\theta^2)^{\lambda+1}}\right|d\theta\\ &+ 2(\lambda+1)\int_0^{\frac{\pi}{2}}t^2\theta^{2\lambda-1}\left|\frac{1}{((z-y)^2+t^2+2zy(1-\cos\theta))^{\lambda+2}}-\frac{1}{((z-y)^2+t^2+zy\theta^2)^{\lambda+2}}\right|d\theta\\ &\leq C\int_0^{\frac{\pi}{2}}\frac{zy\theta^{2\lambda+3}}{((z-y)^2+t^2+zy\theta^2)^{\lambda+2}}d\theta\leq C\int_0^{\frac{\pi}{2}}\frac{\theta^{2\lambda+1}}{((z-y)^2+t^2+zy\theta^2)^{\lambda+1}}d\theta,\quad z,t,y\in(0,\infty).\end{aligned}$$

The last term is the same one that appears in the last term in (4.4). Then (4.5) is established.

In the third step we will prove that, for $x,z\in(0,\infty)$,

$$\left\{\int_{\Gamma_+(x)}\left|t\frac{\partial}{\partial t}\left[\frac{2\lambda}{\pi}\int_0^{\frac{\pi}{2}}\frac{(zy)^\lambda t\theta^{2\lambda-1}}{((z-y)^2+t^2+yz\theta^2)^{\lambda+1}}d\theta-P_t(z-y)\right]\right|^2\frac{dtdy}{t^2}\right\}^{\frac{1}{2}}\leq\frac{C}{z}. \tag{4.6}$$

We first write the following equality:

$$\int_0^{\frac{\pi}{2}}\frac{t\theta^{2\lambda-1}}{((z-y)^2+t^2+zy\theta^2)^{\lambda+1}}d\theta=\left(\int_0^\infty-\int_{\frac{\pi}{2}}^\infty\right)\frac{t\theta^{2\lambda-1}}{((z-y)^2+t^2+zy\theta^2)^{\lambda+1}}d\theta,\ t,z,y\in(0,\infty).$$

Moreover, by making the change of variable $u=\theta\sqrt{\frac{zy}{(z-y)^2+t^2}}$ we obtain

$$\begin{aligned}\int_0^\infty \frac{t\theta^{2\lambda-1}}{((z-y)^2+t^2+zy\theta^2)^{\lambda+1}}d\theta &= \frac{1}{(zy)^\lambda}\frac{t}{(z-y)^2+t^2}\int_0^\infty \frac{u^{2\lambda-1}}{(1+u^2)^{\lambda+1}}du\\ &= \frac{\pi}{2\lambda}\frac{1}{(zy)^\lambda}P_t(z-y),\ \ t,z,y\in(0,\infty).\end{aligned}$$

Then, it has, for every $t,z,y\in(0,\infty)$,

$$\frac{2\lambda}{\pi}\int_0^{\frac{\pi}{2}}\frac{(zy)^\lambda t\theta^{2\lambda-1}}{((z-y)^2+t^2+zy\theta^2)^{\lambda+1}}d\theta-P_t(z-y)=-\frac{2\lambda}{\pi}\int_{\frac{\pi}{2}}^\infty\frac{(zy)^\lambda t\theta^{2\lambda-1}}{((z-y)^2+t^2+zy\theta^2)^{\lambda+1}}d\theta. \tag{4.7}$$

We now analyze

$$\mathcal{J}(x,z)=\left\{\int_{\Gamma_+(x)}\left|t\frac{\partial}{\partial t}\int_{\frac{\pi}{2}}^\infty\frac{(zy)^\lambda t\theta^{2\lambda-1}}{((z-y)^2+t^2+zy\theta^2)^{\lambda+1}}d\theta\right|^2\frac{dtdy}{t^2}\right\}^{\frac12},\quad x,z\in(0,\infty).$$

Minkowski inequality leads to

$$\begin{aligned}\mathcal{J}(x,z) &\leq C\left\{\int_{\Gamma_+(x)}\left|\int_{\frac{\pi}{2}}^\infty\frac{(zy)^\lambda\theta^{2\lambda-1}}{((z-y)^2+t^2+zy\theta^2)^{\lambda+1}}d\theta\right|^2dtdy\right\}^{\frac12}\\ &\leq C\int_{\frac{\pi}{2}}^\infty\left\{\int_{\Gamma_+(x)}\frac{(zy)^{2\lambda}\theta^{4\lambda-2}}{((z-y)^2+t^2+zy\theta^2)^{2\lambda+2}}dtdy\right\}^{\frac12}d\theta\\ &\leq C\int_{\frac{\pi}{2}}^\infty\frac{1}{\theta}\left\{\int_{\Gamma_+(x)}\frac{1}{(|z-y|+t+\theta\sqrt{zy})^4}dtdy\right\}^{\frac12}d\theta\\ &\leq C\int_{\frac{\pi}{2}}^\infty\frac{1}{\theta}\left\{\int_0^\infty\frac{1}{(|z-y|+|y-x|+\theta\sqrt{zy})^3}dy\right\}^{\frac12}d\theta\\ &\leq C\int_{\frac{\pi}{2}}^\infty\frac{1}{\theta}\left\{\left(\int_0^{\frac{z}{2}}+\int_{\frac{z}{2}}^\infty\right)\frac{1}{(|z-y|+\theta\sqrt{zy})^3}dy\right\}^{\frac12}d\theta,\quad x,z\in(0,\infty).\end{aligned}$$

Moreover, by using Holder inequality we get, for $x,z\in(0,\infty)$ and $\theta\in(\frac{\pi}{2},\infty)$,

$$\int_0^{\frac{z}{2}}\frac{1}{(|z-y|+\theta\sqrt{zy})^3}dy\leq C\int_0^{\frac{z}{2}}\frac{1}{|z-y|^{\frac52}(\theta\sqrt{zy})^{\frac12}}dy\leq C\left\{\int_0^{\frac{z}{2}}\frac{1}{\theta\sqrt{zy}}dy\right\}^{\frac12}\left\{\int_0^{\frac{z}{2}}\frac{dy}{|z-y|^5}\right\}^{\frac12}\leq\frac{C}{\sqrt{\theta}z^2}.$$

On the other hand,

$$\int_{\frac{z}{2}}^\infty\frac{1}{(|z-y|+\theta\sqrt{zy})^3}dy\ \ \leq\ \ C\int_{\frac{z}{2}}^\infty\frac{1}{(\theta\sqrt{zy})^3}dy\leq\frac{C}{\theta^3z^2}.$$

Then, we obtain that

$$\mathcal{J}(x,z) \leq \frac{C}{z}\int_{\frac{\pi}{2}}^{\infty}\frac{d\theta}{\theta^{\frac{5}{4}}} \leq \frac{C}{z}, \quad x,z \in (0,\infty).$$

From (4.7) we get then (4.6).

By combining (4.3), (4.5) and (4.6) it follows that

$$\left\{\int_{\Gamma_+(x)}\left|t\frac{\partial}{\partial t}\left[P_{t,1}^{\lambda}(z,y)-P_t(z-y)\right]\right|^2\frac{dtdy}{t^2}\right\}^{\frac{1}{2}} \leq \frac{C}{z}\left(1+\log_+\frac{z^2}{|z-x|^2}\right), \quad 0<\frac{x}{2}<z<2x.$$

Hence

$$\begin{aligned}\int_0^{\infty}|\mathcal{K}_3(f)(x)|dx &\leq C\int_0^{\infty}\int_{\frac{x}{2}}^{2x}\left(1+\log_+\frac{z^2}{|z-x|^2}\right)\frac{|f(z)|}{z}dzdx\\ &\leq C\int_0^{\infty}|f(z)|\int_{\frac{z}{2}}^{2z}\frac{1}{x}\left(1+\log_+\frac{z^2}{|z-x|^2}\right)dxdz\\ &\leq C\int_0^{\infty}|f(z)|\int_{\frac{1}{2}}^{2}\frac{1}{u}\left(1+\log_+\frac{1}{|1-u|}\right)dudz \leq C||f||_{L^1(0,\infty)}.\end{aligned}$$

Then we conclude that $||g_\lambda(f)-g_{+,\mathrm{loc}}(f)||_{L^1(0,\infty)} \leq ||f||_{L^1(0,\infty)}$, which leads to $(c') \Leftrightarrow (d)$.

Finally, from the estimations established above and the fact that $||f||_{H^1(\mathbb{R})} \sim ||f||_{L^1(\mathbb{R})} + ||g(f)||_{L^1(\mathbb{R})}$, we deduce that

$$||f||_{H^1(\mathbb{R})} \sim ||f||_{L^1((0,\infty)} + ||g_+(f)||_{L^1(0,\infty)} \sim ||f||_{L^1((0,\infty)} + ||g_\lambda(f)||_{L^1(0,\infty)}.$$

Thus the proof of this Proposition is finished. □

Remark 4.8. *Since the operator $f \longrightarrow g(f)$ is bounded from $L^p(0,\infty)$ into itself, when $1<p<\infty$, and from $L^1(0,\infty)$ into $L^{1,\infty}(0,\infty)$, the estimates established in the proof of Proposition 4.1 allow us to conclude that, for every $\lambda>0$, the operator $f \longrightarrow g_\lambda(f)$ is also bounded from $L^p(0,\infty)$ into itself, when $1<p<\infty$, and from $L^1(0,\infty)$ into $L^{1,\infty}(0,\infty)$. As far as we know, this result is new.*

We now present other useful results in order to establish $(iii) \Rightarrow (i)$.

If F is a measurable function on $(0,\infty)\times(0,\infty)$, we define $\widetilde{F}$, $\Phi(F)$ and $\Psi(F)$ as follows:

$$\widetilde{F}(y,t) = \begin{cases} F(y,t), & (y,t)\in(0,\infty)\times(0,\infty),\\ F(-y,t) & (y,t)\in(-\infty,0)\times(0,\infty),\end{cases}$$

$$\Phi(F)(x) = \sup_{I\subset(0,\infty), I\ni x}\left(\frac{1}{|I|}\int_0^{|I|}\int_I |F(y,t)|^2\frac{dydt}{t}\right)^{\frac{1}{2}}, \quad x\in(0,\infty),$$

and

$$\Psi(F)(x) = \left(\int_{\Gamma_+(x)} |F(y,t)|^2 \frac{dydt}{t^2} \right)^{\frac{1}{2}}, \quad x \in (0,\infty).$$

From a well-known result about tent spaces (see [11, p. 162]) we can deduce the following.

Proposition 4.9. *Let F and G be measurable functions on $(0,\infty)\times(0,\infty)$. Suppose that $\Phi(F) \in L^\infty(0,\infty)$ and $\Psi(G) \in L^1(0,\infty)$. Then*

$$\int_0^\infty \int_0^\infty |F(y,t)G(y,t)| \frac{dydt}{t} \leq C \int_0^\infty \Phi(F)(x)\Psi(G)(x)dx.$$

Proof. According to [11, p. 162] we have

$$\begin{aligned} \int_0^\infty \int_0^\infty |F(y,t)G(y,t)| \frac{dydt}{t} &= 2\int_0^\infty \int_{-\infty}^\infty |\widetilde{F}(y,t)\widetilde{G}(y,t)| \frac{dydt}{t} \\ &\leq C \int_{-\infty}^{+\infty} \Upsilon(\widetilde{F})(x)\Omega(\widetilde{G})(x)dx, \end{aligned} \tag{4.10}$$

where $\Omega(\widetilde{G})(x) = \left\{ \int_{\Gamma(x)} |\widetilde{G}(y,t)|^2 \frac{dydt}{t^2} \right\}^{\frac{1}{2}}$, $x \in \mathbb{R}$, and

$$\Upsilon(\widetilde{F})(x) = \sup_{I\subset\mathbb{R}, I\ni x} \left(\frac{1}{|I|} \int_0^{|I|} \int_I |\widetilde{F}(y,t)|^2 \frac{dydt}{t} \right)^{\frac{1}{2}}, \quad x \in \mathbb{R}.$$

Let $I = (a,b)$ with $-\infty < a < b < +\infty$. We define

$$\mathbb{I} = \begin{cases} I, & a \geq 0 \\ (-b,-a), & b \leq 0 \\ (0,|I|), & a < 0 < b. \end{cases}$$

Then, since $\widetilde{F}(y,t) = \widetilde{F}(-y,t)$, $t \in (0,\infty)$ and $y \in \mathbb{R}$, it gets, for each interval I in $\mathbb{R}$,

$$\frac{1}{|I|} \int_0^{|I|} \int_I |\widetilde{F}(y,t)|^2 \frac{dydt}{t} \leq \frac{1}{|\mathbb{I}|} \int_0^{|\mathbb{I}|} \int_{\mathbb{I}} |F(y,t)|^2 \frac{dydt}{t}.$$

Hence, $\Upsilon(\widetilde{F})(x) = \Phi(F)(x)$, $x \in (0,\infty)$, and $\Upsilon(\widetilde{F})(x) = \Phi(F)(-x)$, $x \in (-\infty,0)$.

On the other hand, since $\widetilde{G}(y,t) = \widetilde{G}(-y,t)$, $t \in (0,\infty)$ and $y \in \mathbb{R}$, we have

$$\int_{\Gamma(x)} |\widetilde{G}(y,t)|^2 \frac{dydt}{t^2} = \int_{\Gamma(-x)} |\widetilde{G}(y,t)|^2 \frac{dydt}{t^2} \leq 2\int_{\Gamma_+(x)} |G(y,t)|^2 \frac{dydt}{t^2}, \quad x \in (0,\infty).$$

Then $\Omega(\widetilde{G})(x) = \Omega(\widetilde{G})(-x)$, $x \in \mathbb{R}$, and $\Omega(\widetilde{G})(x) \leq \sqrt{2}\Psi(G)(x)$, $x \in (0,\infty)$.

From (4.10) it deduces that

$$\int_0^\infty \int_0^\infty |F(y,t)G(y,t)\frac{dydt}{t} \leq C\int_0^\infty \Phi(F)(x)\Psi(G)(x)dx.$$

□

Proposition 4.11. *Suppose that f is a measurable function on $(0,\infty)$ such that $(1+x^2)^{-1}f \in L^1(0,\infty)$ and $d\gamma_f$ is a Carleson measure on $(0,\infty)$. Then*

$$\frac{1}{4}\int_0^\infty f(x)a(x)dx = \int_0^\infty \int_0^\infty t\frac{\partial}{\partial t}P_t^\lambda(f)(y)t\frac{\partial}{\partial t}P_t^\lambda(a)(y)\frac{dydt}{t},$$

provided that a is an (odd)-atom.

Proof. According to Propositions 4.1 and 4.9 we have that

$$\int_0^\infty \int_0^\infty \left|t\frac{\partial}{\partial t}P_t^\lambda(f)(y)t\frac{\partial}{\partial t}P_t^\lambda(a)(y)\right|\frac{dydt}{t} < \infty.$$

Then

$$\int_0^\infty \int_0^\infty t\frac{\partial}{\partial t}P_t^\lambda(f)(y)t\frac{\partial}{\partial t}P_t^\lambda(a)(y)\frac{dydt}{t} = \lim_{\varepsilon\to 0, N\to\infty} H(\varepsilon,N),$$

where, for $0<\varepsilon<N<\infty$,

$$H(\varepsilon,N) = \int_\varepsilon^N \int_0^\infty t\frac{\partial}{\partial t}P_t^\lambda(f)(y)\frac{\partial}{\partial t}P_t^\lambda(a)(y)dydt.$$

Our next objective is to prove that

$$\int_0^\infty t\frac{\partial}{\partial t}P_t^\lambda(f)(y)\frac{\partial}{\partial t}P_t^\lambda(a)(y)dy = \int_0^\infty f(z)\int_0^\infty t\frac{\partial}{\partial t}P_t^\lambda(y,z)\frac{\partial}{\partial t}P_t^\lambda(a)(y)dydz, \quad t>0. \tag{4.12}$$

To see (4.12) it is sufficient to show that

$$\int_0^\infty \left|t\frac{\partial}{\partial t}P_t^\lambda(y,z)\frac{\partial}{\partial t}P_t^\lambda(a)(y)\right|dy \leq \frac{C}{1+z^2}, \quad t,z\in(0,\infty).$$

Since $a\in L^2(0,\infty)$, according to [3, proof of Proposition 6.1] it has that

$$\frac{\partial}{\partial t}P_t^\lambda(a)(y) = \frac{\partial}{\partial t}[h_\lambda(e^{-tu}h_\lambda(a)(u)](y) = -h_\lambda(ue^{-tu}h_\lambda(a)(u))(y), \quad t,y\in(0,\infty). \tag{4.13}$$

By using (1.3) and well-known properties of Bessel functions ([9, §5.3]) we get

$$\begin{aligned}\frac{\partial}{\partial t}P_t^\lambda(a)(y) &= -\frac{1}{1+y^2}\int_0^\infty (1+\Delta_{\lambda,u})(\sqrt{yu}J_{\lambda-\frac{1}{2}}(yu))ue^{-tu}h_\lambda(a)(u)du \\ &= -\frac{1}{1+y^2}h_\lambda[(1+\Delta_{\lambda,u})(ue^{-tu}h_\lambda(a)(u))](y), \quad t,y\in(0,\infty).\end{aligned}$$

Moreover, we can write

$$\begin{aligned}\Delta_{\lambda,u}(ue^{-tu}h_\lambda(a)(u)) &= -ue^{-tu}h_{\lambda+2}(x^2a(x))(u)+(2\lambda+3-2tu)e^{-tu}h_{\lambda+1}(xa(x))(u)\\ &- \left(ut^2-(2\lambda+2)t+\frac{2\lambda}{u}\right)e^{-tu}h_\lambda(a)(u), \quad t,u\in(0,\infty).\end{aligned}$$

Note also that, since a is an (odd)-atom and $z^{-\nu}J_\nu(z)$ is a bounded function on $(0,\infty)$, when $\nu>-1/2$, it has

$$|h_\nu(x^na(x))(u)|\le Cu^\nu, \quad u\in(0,\infty), n\in\mathbb{N},$$

provided that $\nu>0$. Then we deduce that

$$|(1+\Delta_{\lambda,u})(ue^{-tu}h_\lambda(a)(u))|\le Ce^{-tu}p(t)q(u)u^{\lambda-1}, \quad t,u\in(0,\infty),$$

where p and q are polynomials. Hence, since $\sqrt{z}J_{\lambda-\frac{1}{2}}(z)$ is a bounded function on $(0,\infty)$, and $\lambda>0$, it follows that

$$\left|\frac{\partial}{\partial t}P_t^\lambda(a)(y)\right|\le\frac{A(t)}{1+y^2}, \quad t,y\in(0,\infty). \tag{4.14}$$

Here A denotes a continuous function on $(0,\infty)$.

On the other hand, according to [10, (b) p. 86] it has that

$$\left|\frac{\partial}{\partial t}P_t^\lambda(y,z)\right|\le C\frac{t}{(z-y)^2+t^2}, \quad t,z,y\in(0,\infty). \tag{4.15}$$

Estimations (4.14) and (4.15) lead to

$$\begin{aligned}\int_0^\infty\left|t\frac{\partial}{\partial t}P_t^\lambda(y,z)\frac{\partial}{\partial t}P_t^\lambda(a)(y)\right|dy &\le A(t)\left(\int_0^{\frac{z}{2}}+\int_{\frac{z}{2}}^\infty\right)\frac{t}{((z-y)^2+t^2)(1+y^2)}dy\\ &\le \frac{A(t)}{1+z^2}\left(\int_0^\infty\frac{1}{1+y^2}dy+\int_0^\infty\frac{t}{(y-z)^2+t^2}dy\right)\\ &\le \frac{A(t)}{1+z^2}, \quad t,z\in(0,\infty),\end{aligned}$$

and (4.12) is thus proved. In the last chain of inequalities A represents a continuous function on $(0,\infty)$ not necessarily the same in each ocurrence.

Also we have that

$$H(\varepsilon,N)=\int_0^\infty f(z)\int_\varepsilon^N\int_0^\infty t\frac{\partial}{\partial t}P_t^\lambda(z,y)\frac{\partial}{\partial t}P_t^\lambda(a)(y)dydtdz, \quad 0<\varepsilon<N<\infty.$$

Moreover, by taking into account that h_λ is an isometry in $L^2(0,\infty)$ and using (4.13) it follows that

$$\begin{aligned}\int_0^\infty t\frac{\partial}{\partial t}P_t^\lambda(z,y)\frac{\partial}{\partial t}P_t^\lambda(a)(y)dy &= t\frac{\partial}{\partial t}\left(\int_0^\infty P_t^\lambda(z,y)\frac{\partial}{\partial t}P_t^\lambda(a)(y)dy\right)-\int_0^\infty tP_t^\lambda(z,y)\frac{\partial^2}{\partial t^2}P_t^\lambda(a)(y)dy\\ &= -t\frac{\partial}{\partial t}h_\lambda(ue^{-2tu}h_\lambda(a))(z)-th_\lambda(u^2e^{-2tu}h_\lambda(a))(z)\\ &= h_\lambda(tu^2e^{-2tu}h_\lambda(a))(z),\quad t,z\in(0,\infty).\end{aligned}$$

Since the function $\sqrt{z}J_{\lambda-\frac{1}{2}}(z)$ is bounded in $(0,\infty)$, Fubini theorem allows us to write

$$\int_\varepsilon^N\int_0^\infty t\frac{\partial}{\partial t}P_t^\lambda(z,y)\frac{\partial}{\partial t}P_t^\lambda(a)(y)dydt = h_\lambda\left(\int_\varepsilon^N te^{-2tu}dtu^2h_\lambda(a)(u)\right)(z)$$

$$= -\frac{1}{4}h_\lambda\left((2Nu+1)e^{-2Nu}h_\lambda(a)(u)\right)(z)+\frac{1}{4}h_\lambda\left((2\varepsilon u+1)e^{-2\varepsilon u}h_\lambda(a)(u)\right)(z) \tag{4.16}$$

$$= M_N^\lambda(a)(z)-M_\varepsilon^\lambda(a)(z),\quad z\in(0,\infty),$$

where

$$M_t^\lambda(a)=\frac{1}{4}\left(t\frac{\partial}{\partial v}P_{2v}^\lambda(a)_{|v=t}-P_{2t}^\lambda(a)\right),\quad t>0.$$

A straightforward manipulation leads to

$$t\frac{\partial}{\partial v}P_{2v}^\lambda(z,y)_{|v=t}-P_{2t}^\lambda(z,y)=-\frac{32\lambda(\lambda+1)}{\pi}(zy)^\lambda\int_0^\pi\frac{t^2(\sin\theta)^{2\lambda-1}}{((z-y)^2+4t^2+2zy(1-\cos\theta))^{\lambda+2}}d\theta,$$

for every $t,z,y\in(0,\infty)$.

Then it has,

$$\left|t\frac{\partial}{\partial v}P_{2v}^\lambda(z,y)_{|v=t}-P_{2t}^\lambda(z,y)\right|\leq C\frac{(zy)^\lambda}{|z-y|^{2\lambda+2}},\quad t,z,y\in(0,\infty). \tag{4.17}$$

Also, by [10, (b) p. 86], we get

$$\left|t\frac{\partial}{\partial v}P_{2v}^\lambda(z,y)_{|v=t}-P_{2t}^\lambda(z,y)\right|\leq C\frac{t}{(z-y)^2+t^2},\quad t,z,y\in(0,\infty).$$

Let $\alpha>0$ such that supp $a\subset[0,\alpha]$. It follows that

$$|M_t^\lambda(a)(z)|\leq C||a||_{L^\infty(0,\infty)}\int_0^\infty\frac{t}{(z-y)^2+t^2}dy\leq||a||_{L^\infty(0,\infty)}\leq\frac{C}{1+z^2},\quad 0<z\leq 2\alpha,t>0.$$

Moreover (4.17) implies that

$$|M_t^\lambda(a)(z)|\leq C\int_0^\alpha|a(y)|\frac{(zy)^\lambda}{|z-y|^{2\lambda+2}}dy\leq\frac{C}{z^{\lambda+2}}\int_0^\alpha|a(y)|y^\lambda dy\leq\frac{C}{1+z^2},\quad z\geq 2\alpha,t>0.$$

Hence we conclude that

$$\sup_{t>0} |M_t^\lambda(a)(z)| \leq \frac{C}{1+z^2}, \quad z \in (0,\infty).$$

From here it deduces that

$$\sup_{0<\varepsilon<N<\infty} \left| \int_\varepsilon^N \int_0^\infty t \frac{\partial}{\partial t} P_t^\lambda(z,y) \frac{\partial}{\partial t} P_t^\lambda(a)(y) dy dt \right| \leq \frac{C}{1+z^2}, \quad z \in (0,\infty).$$

Moreover, by taking into account this estimate and that $(1+x^2)^{-1} f \in L^1(0,\infty)$, the proof will be completed if we show that

$$\lim_{k\to\infty} \int_{\frac{1}{N_k}}^{N_k} \int_0^\infty t \frac{\partial}{\partial t} P_t^\lambda(z,y) \frac{\partial}{\partial t} P_t^\lambda(a)(y) dy dt = \frac{a(z)}{4}, \quad \text{a.e. } z \in (0,\infty), \tag{4.18}$$

for some increasing sequence $\{N_k\}_{k\in\mathbb{N}}$ of nonnegative integers. To see this, it is sufficient to prove that

$$\lim_{\varepsilon\to 0, N\to\infty} \int_\varepsilon^N \int_0^\infty t \frac{\partial}{\partial t} P_t^\lambda(z,y) \frac{\partial}{\partial t} P_t^\lambda(a)(y) dy dt = \frac{a(z)}{4} \quad \text{in } L^2(0,\infty). \tag{4.19}$$

Note that (4.18) and (4.19) can be seen as Calderon reproducing formulas in our setting.

Since h_λ is an isometry in $L^2(0,\infty)$([13, p. 473 (1)]), by (4.16) we have that (4.19) is equivalent to

$$\lim_{\varepsilon\to 0, N\to\infty} (2Nu+1)e^{-2Nu} h_\lambda(a)(u) + (2\varepsilon u+1)e^{-2\varepsilon u} h_\lambda(a)(u) = h_\lambda(a)(u), \quad \text{in } L^2(0,\infty). \tag{4.20}$$

We obtain (4.20) as a consequence of the dominated convergence theorem because $h_\lambda(a) \in L^2(0,\infty)$.

Thus the proof of this proposition is finished. □

Proof of $(iii) \Rightarrow (i)$.

Suppose that a is a finite combination of (odd)-atoms. Since γ_f is a Carleson measure, from Propositions 4.1, 4.9 and 4.11 we deduce

$$\begin{aligned} \left| \int_0^\infty f(x) a(x) dx \right| &\leq 4 \int_0^\infty \int_0^\infty \left| t \frac{\partial}{\partial t} P_t^\lambda(f)(y) t \frac{\partial}{\partial t} P_t^\lambda(a)(y) \right| \frac{dy dt}{t} \\ &\leq C \left\| \Phi \left(t \frac{\partial}{\partial t} P_t^\lambda(f) \right) \right\|_{L^\infty(0,\infty)} \left\| \Psi \left(t \frac{\partial}{\partial t} P_t^\lambda(a) \right) \right\|_{L^1(0,\infty)} \leq C(||\gamma_f||_{\mathcal{C}})^{\frac{1}{2}} ||a_{\rm o}||_{H^1(\mathbb{R})}. \end{aligned}$$

Here $a_{\rm o}$ denotes the odd extension of a to $\mathbb{R}$. By remembering that $(H^1_{\rm o}(\mathbb{R}))' = BMO_{\rm o}(\mathbb{R})$, we conclude that $f \in BMO_+$ and $||f||^2_{BMO_+} \leq C||\gamma_f||_{\mathcal{C}}$.

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