MEASURING THE LEVEL SETS OF GENERALIZED HOMOGENEOUS FUNCTIONS

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Abstract. In this note we prove a formula for the volume of level sets of generalized homogeneous functions in terms of measures supported on the level surfaces. We relate the results to some well known mean value formulas for solutions of PDE.

1. Introduction and statement of the result

The well known mean value formulas for harmonic functions on $\mathbb{R}^n$ are

$$u(x) = \frac{1}{|B(x,r)|}\int_{B(x,r)} u(y)dy,$$

and

$$u(x) = \frac{1}{\sigma(\partial B(x,r))}\int_{\partial B(x,r)} u(y)d\sigma(y),$$

where $B(x,r)$ denotes the open ball centered at x with radious $r > 0$; $\partial B(x,r)$ is its boundary and σ is the surface area.

The rotational invariance of the Laplacian, reflects in the fundamental solution for $\Delta u = \delta$, which is given essentially by $\Gamma(x) = |x|^{2-n}$, $n \geq 3$. A little less standard, but equally well known is the case of the mean values for the solutions of the heat equation. Now, the fundamental solution is given by

$$\Gamma(x,t) = \begin{cases} (\sqrt{4\pi t})^{-n} e^{-\frac{|x|^2}{4t}} & t > 0 \\ 0 & t \leq 0 \end{cases} \tag{1.1}$$

in $\mathbb{R}^{n+1} = \{(x,t) : x \in \mathbb{R}^n; t \in \mathbb{R}\}$. Following [2] (see also [3] and [5]) let us define the heat balls $E((x,t),r)$ as the set of all the points $(y,s) \in \mathbb{R}^{n+1}$ for which $\Gamma(x-y,t-s) > r^{-n}$. The corresponding mean value for a temperature $u(x,t)$ takes now the following form

$$u(x,t) = \frac{1}{4r^n}\iint_{E((x,t),r)} u(y,s)\frac{|x-y|^2}{|t-s|^2}dyds. \tag{1.2}$$

For the parabolic case a corresponding mean value on the boundaries of the heat balls is also available (see [3] and [5]). Precisely,

$$u(x,t) = \frac{1}{4r^n}\int_{\partial E((x,t),r)} u(y,s)\frac{|x-y|^2}{\sqrt{4|x-y|^2(t-s)^2 + (|x-y|^2 - 2n(t-s))^2}}d\sigma(y,s)$$

where $\partial E((x,t),r)$, the topological boundary of the heat ball, is called the heat sphere or parabolic sphere.

The elliptic and parabolic situations briefly described above share a common pattern. In fact both, $\Gamma(x)$ and $\Gamma(x,t)$ are homogeneous functions. Of course not with respect to the same dilations. While $\Gamma(x)$ is homogeneous of degree $-n+2$ with respect to the usual dilations in $\mathbb{R}^n$: $\Gamma(\lambda x) = \lambda^{-n+2}\Gamma(x)$ for every $x \in \mathbb{R}^n - \{0\}$ and $\lambda > 0$; the function $\Gamma(x,t)$ is parabolically homogeneous of degree $-n$. Precisely $\Gamma(\lambda x, \lambda^2 t) = \lambda^{-n}\Gamma(x,t)$ for every $x \in \mathbb{R}^n$, $t \in \mathbb{R}$ and $\lambda > 0$.

In both cases, elliptic and parabolic, the mean value formulas on the solid balls are equivalent to the corresponding mean value formulas on the spherical shells. We would like to mention also the results in [6] and [4] where mean values for more general linear hypoelliptic PDEs are considered.

In the applications to some problems in PDE (see [1]), sometimes it is important to have smooth versions of the mean value formulas. Smoothness here means that the indicator functions of $B(x,r)$ or of $E((x,t),r)$ can be substituted by compactly supported $\mathscr{C}^\infty$ functions whose level surfaces are $\partial B(x,\rho)$ and $\partial E((x,t),\rho)$. In order to get these smooth formulas all we need are the mean value identities on the surfaces ∂B and ∂E.

In this note we show that this behavior is a general fact concerning the computation of the volume of a level set of a generalized homogeneous function in terms of the radial integral of some specific measures supported on the corresponding level surfaces.

We shall consider generalized nonisotropic dilations induced by an $n \times n$ diagonal matrix A with eigenvalues $1 \le a_1 \le a_2 \le \ldots \le a_n$ of the type

$$T_\lambda = \begin{pmatrix} \lambda^{a_1} & & 0 \\ & \ddots & \\ 0 & & \lambda^{a_n} \end{pmatrix}$$

for $\lambda > 0$. We say that a function $\Gamma : \mathbb{R}^n - \{0\} \to \mathbb{R}^+ \cup \{0\}$ is A-homogeneous of negative degree m if the identity $\Gamma(T_\lambda x) = \lambda^m \Gamma(x)$ holds for each $x \in \mathbb{R}^n - \{0\}$ and each $\lambda > 0$. Let us define the set $E(\alpha)$ of the function Γ as $E(\alpha) = \{x \in \mathbb{R}^n - \{0\} : \Gamma(x) > \alpha^m\}$ for $\alpha > 0$. With ∂_α we shall denote the topological boundary of the closure in $\mathbb{R}^n$ of $E(\alpha)$.

The main result of this note is contained in the next statement.

Theorem 1.1. *Let Γ be a nontrivial and nonnegative A-homogeneous function of degree $m < 0$. Assume that Γ is $\mathscr{C}^1$ on S^{n-1}, the unit sphere of $\mathbb{R}^n$. Then for each $r > 0$ there exists a regular, finite and positive Borel measure μ_r supported on ∂_r such that the identity*

$$\iint_{\mathbb{R}^n} \psi(x)dx = \int_0^\infty \left(\int_{\partial_r} \psi(y) d\mu_r(y) \right) dr \tag{1.3}$$

holds for every ψ smooth and compactly supported. Moreover, μ_r is absolutely continuous with respect to the surface area measure σ_r on ∂_r and the Radon-Nikodym derivative is given by $\frac{d\mu_r}{d\sigma_r}(z) = \frac{Az \cdot \vec{n_r}(z)}{r}$, where $\vec{n_r}$ is the unit outer normal vector for ∂_r.

Let us notice that for the parabolic case, regarding $\mathbb{R}^n$ as $\{(x,t) : x \in \mathbb{R}^{n-1}; t \in \mathbb{R}\}$ we have that $\Gamma(x,t)$ defined as in (1.1) is A-homogeneous of degree $-n$ with the matrix

$$A = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \\ & & & 2 \end{pmatrix}.$$

If $u(x,t)$ is a solution of the heat equation $\frac{\partial u}{\partial t} = \Delta u = \sum_{i=1}^{n-1} \frac{\partial^2 u}{\partial x_i^2}$, then from (1.2) and (1.3) we see that

$$u(x,t) = \frac{1}{R^{n-1}} \int_0^R \left(\int_{\partial E(r)} u(y,s) \frac{|x-y|^2}{(t-s)^2} d\mu_r(y,s) \right) dr.$$

By taking the derivative with respect to R in the above formula, we get

$$0 = (-n+1)R^{-1} u(x,t) + \frac{1}{R^{n-1}} \int_{\partial E(R)} u(y,s) \frac{|x-y|^2}{(t-s)^2} d\mu_R(y,s).$$

On the other hand, Theorem 1.1 shows that each μ_R is given by the surface measure on $\partial E(R)$, the boundary of the heat ball. Hence, for each $r > 0$

$$u(x,t) = \frac{1}{(n-1)r^{n-1}} \int_{\partial E(r)} u(y,s) \frac{|x-y|^2}{(t-s)^2} (y, 2s) \cdot \vec{n_r}(y,s)\, d\sigma_r(y,s). \tag{1.4}$$

Actually the basic result in the proof of Theorem 1.1 is contained in the next statement.

Theorem 1.2. *Let Γ be a nonnegative A-homogeneous function of degree $m < 0$. Assume that Γ is $\mathscr{C}^1$ on the unit sphere of $I\!R^n$. Then the function $\eta_\psi(\alpha) = \iint_{E(\alpha)} \psi\, dx$ is $\mathscr{C}^1(I\!R)$ for every $\psi \in \mathscr{C}_0^\infty(I\!R^n)$ and for any $r > 0$ the linear functional which to each $\psi \in \mathscr{C}_0^\infty(I\!R^n)$ assigns the real number $\frac{d\eta_\psi}{d\alpha}(r)$ defines a positive distribution in $\mathcal{D}'(I\!R^n)$ of order zero. Precisely,*

$$\frac{d\eta_\psi}{d\alpha}(r) = \frac{1}{r} \int_{\partial_r} \psi(y)\, Ay \cdot \vec{n_r}(y)\, d\sigma_r(y). \tag{1.5}$$

From this result the proof of Theorem 1.1 is fairly easy. In fact, we only have to integrate both sides of (1.5) with respect to r.

We observe that $\mathrm{vol}(E(\alpha))$ is an increasing function. In fact, for $0 < s < 5r$ and $\psi \equiv 1$ on $E(5r)$, we see that

$$\mathrm{vol}(E(s)) = \int_{E(s)} \psi\, dx = \eta_\psi(s).$$

Hence $\frac{d}{ds}\mathrm{vol}(E(s)) = \frac{d\eta_\psi}{ds}(s)$ which is positive from (1.5).

2. Some properties of generalized homogeneous function

For a given diagonal matrix A with eigenvalues $1 \le a_1 \le \ldots \le a_n$, we have an associated polar description of each point $x \in I\!R^n - \{0\}$. In fact, it is easy to see that given such point x there exists only one $\lambda > 0$ and only one $y \in S^{n-1}$ such that $T_\lambda y = x$. In fact, take $y_i = \frac{x_i}{\lambda^{a_i}}$. The function $\sum_{i=1}^n \left(\frac{x_i}{\lambda^{a_i}}\right)^2$ as a function of λ is continuos and decreasing, then there exists only one $\lambda > 0$ such that it takes the value 1.

The next lemma collects some elementary properties of the level sets of A-homogeneous functions, which shall be used in the proof of Theorem 1.2.

Lemma 2.1. *Assume that A and Γ satisfy the condition of Theorem 1.1. Set $E(\alpha) = \{x \in \mathbb{R}^n - \{0\} : \Gamma(x) > \alpha^m\}$ for $\alpha > 0$. Then*

(2.1.i) each $E(\alpha)$ is a nonempty bounded and measurable subset of $\mathbb{R}^n$;
(2.1.ii) if $\alpha < \beta$, then $E(\alpha) \subsetneq E(\beta)$;
(2.1.iii) for each $\alpha > 0$ we have that $E(\alpha) = T_\alpha(E(1))$;
(2.1.iv) $|E(\alpha)| = \alpha^\tau |E(1)|$.

Proof. Property *(2.1.iii)* follows from the homogeneity of Γ,

$$\begin{aligned} T_\alpha(E(1)) &= \{T_\alpha x : \Gamma(x) > 1\} \\ &= \left\{y : \Gamma\left((T_\alpha)^{-1}(y)\right) > 1\right\} \\ &= \{y : \Gamma(T_{\alpha^{-1}}(y)) > 1\} \\ &= \left\{y : \frac{1}{\alpha^m}\Gamma(y) > 1\right\} \\ &= E(\alpha) \end{aligned}$$

for $\alpha > 0$. So that to prove *(2.1.i)* it shall be enough to show that $E(1)$ is bounded. Take $x \in E(1)$. As we observe above there exists $y_x \in S^{n-1}$ and $\lambda(x) > 0$ such that $x = T_{\lambda(x)}(y(x))$. Applying the homogeneity of Γ we have $\lambda(x)^m\Gamma(y) = \Gamma(x) > 1$. Since Γ is $\mathscr{C}^1$ function and since $m < 0$ we get that $0 < \lambda(x)^{|m|} < \Gamma(y) \leq \kappa$ for some κ. Then

$$|x|^2 = \sum_{i=1}^n x_i^2 = \sum_{i=1}^n \lambda(x)^{2a_i} y(x)_i^2 \leq \max\{1, \kappa^{\frac{2a_n}{|m|}}\}\, |y|^2 .$$

In order to prove *(2.1.ii)*, notice first that for $\alpha < \beta$ the inclusion $E(\alpha) \subset E(\beta)$ holds. Let us show that $E(\alpha)$ and $E(\beta)$ can not coincide. Since Γ in nontrivial, then for some point $\xi \neq 0$ we must have $\Gamma(\xi) > 0$. Hence $\Gamma(T_s(\xi)) = s^m\Gamma(\xi)$ a function of s is one to one and onto $\mathbb{R}^+$. To prove that $E(\alpha) \neq E(\beta)$ it is enough to take $s = \frac{\alpha+\beta}{2(\Gamma(\xi))^{1/m}}$. The formula in *(2.1.iv)* follows directly from *(2.1.iii)*. □

Notice that for the parabolic case considered in the introduction $\Gamma(x,t)$ is smooth on S^{n-1}, but it vanishes exactly on the whole hemisphere $\{(x,t) \in S^{n-1} : t \leq 0\}$. Moreover, each level surface $\Sigma_\alpha = \{(x,t) \in \mathbb{R}^n - \{0\} : \Gamma(x,t) = \alpha^m\}$, $\alpha > 0$, has a limit point at the origin 0 of $\mathbb{R}^n$. So that in general the topological boundary ∂_α of $E(\alpha)$ is not exactly the level surface Σ_α.

Lemma 2.2. *Let Γ be as in Theorem 1.1, then*

(1) Γ is positive if and only if $d(\Sigma_1, 0) > 0$;
(2) $\partial_1 = \Sigma_1$ in $\mathbb{R}^n - \{0\}$;
(3) Γ is positive if and only if $d(\Sigma_\alpha, 0) > 0$ for each $\alpha > 0$;
(4) $\partial_\alpha = \Sigma_\alpha$ in $\mathbb{R}^n - \{0\}$.

Proof. Let us start by noticing that for each $\alpha > 0$ we have $\Sigma_\alpha = T_\alpha(\Sigma_1)$ and $\partial_\alpha = T_\alpha(\partial_1)$. Hence (3) and (4) are consequences of (1) and (2).

To prove (1) let us start by assuming that $\Gamma > 0$ everywhere. Take $x \in \Sigma_1$. Then there exist $y(x) \in S^{n-1}$ and $\lambda(x) > 0$ such that $x = T_{\lambda(x)}(y(x))$. Hence $1 = \Gamma(x) = \Gamma(T_{\lambda(x)}(y(x))) = \lambda(x)^m\Gamma(y(x))$. Since Γ is positive and continuous on the unit sphere of $\mathbb{R}^n$ we have that $0 < A \leq \Gamma(y(x)) \leq B$ for some constants A

and B which do not depend on x. So that $0 < B^{-\frac{1}{m}} \leq \lambda(x) \leq A^{-\frac{1}{m}} < \infty$ for every $x \in \Sigma_1$. Then

$$|x|^2 = \sum_{i=1}^{n} x_i^2 = \sum_{i=1}^{n} \lambda^{2a_i}(x) y_i^2(x) \geq \gamma \, |y|^2 = \gamma > 0$$

where γ is a positive lower bound for $\lambda(x)^{2a_i}$. This proves that $d(\Sigma_1, 0) > 0$. Assume now that there exists $y \in S^{n-1}$ such that $\Gamma(y) = 0$ and that $d(\Sigma_1, 0) > 0$. Take $\varepsilon > 0$ such that $B(0,\varepsilon) \cap \Sigma_1 = \varnothing$. Let x be a point in Σ_1. For small λ we have that $T_\lambda x$ and $T_\lambda y$ belong to $B(0,\varepsilon)$. But $\Gamma(T_\lambda x) = \lambda^m \Gamma(x) = \lambda^m > 1$ and $\Gamma(T_\lambda y) = 0$. Hence there must be a point ξ in $B(0,\varepsilon)$ such that $\Gamma(\xi) = 1$. This contradicts the choice of ε. Then Γ is positive on the unit sphere. So it is positive on the whole $\mathbb{R}^n$.

To prove (2) let us take $x \in \Sigma_1$. Then $x \neq 0$ and $\Gamma(x) = 1$. We need to prove that x is a limit point of $E(1)$ and of $E(1)^c$. To see this, notice that x can be approximated by $T_\lambda(x)$ for $\lambda \nearrow 1$ and $\lambda \searrow 1$. The first gives points in $E(1)$ and the second in its complement.

Finally, given $x \in \partial_1$, $x \neq 0$, there exist two sequences $\{x_k\}$ in $E(1)$ and $\{y_k\}$ in $E(1)^c$, both converging to x. Since Γ is continuous on $\mathbb{R}^n - \{0\}$ we have that $\Gamma(x_k) \to \Gamma(x)$ and $\Gamma(y_k) \to \Gamma(x)$. But $\Gamma(x_k) > 1$ and $\Gamma(y_k) \leq 1$. Hence $\Gamma(x)$ must be 1. In other words $x \in \Sigma_1$. □

Lemma 2.3. *Let Γ be as in Theorem 1.1. Then Γ vanishes at some point in $\mathbb{R}^n - \{0\}$ if and only if 0 is a limit point of Σ_1.*

Proof. Assume 0 is not a limit point of Σ_1. Then, for some positive δ we have $d(0, \Sigma_1) > \delta$. Since Γ vanishes at some point $\xi \in \mathbb{R}^n - \{0\}$, then Γ vanishes along the whole curve $T_s(\xi)$. In particular Γ vanishes on some $y \in S^{n-1}$.

Let us consider, as before, the curves $T_\lambda y$ and $T_\lambda x$ for some $x \in \Sigma_1$ and λ small. Then from continuity of Γ we find a point η in $B(0,\delta)$ with $\Gamma(\eta) = 1$ which is impossible. □

Corollary 2.4. *Let Γ be as above. If $\Gamma > 0$, then $\partial E(1)$ is a compact set on $\mathbb{R}^n - \{0\}$ and $d(\partial E(1), 0) > 0$.*

Proof. We have that $\Sigma_1 = \{x \in \mathbb{R}^n - \{0\} : \Gamma(x) = 1\} = \Gamma^{-1}(\{1\}) = F$. Since Γ is continuous F is closed $\mathbb{R}^n - \{0\}$, then Σ_1 is closed. □

Lemma 2.5. *Let Γ be as before. Then the following statements are equivalent*

(2.5.a) $\Gamma > 0$ *for every point of* $\mathbb{R}^n - \{0\}$;
(2.5.b) $\Gamma > 0$ *on* S^{n-1};
(2.5.c) $0 \notin \partial_1$;
(2.5.d) $0 \notin \partial_\alpha$ *for any* $\alpha > 0$.

Proof. The equivalence of *(2.5.a)* and *(2.5.b)* follows directly from the homogeneity of Γ. Notice that *(2.5.b)* implies *(2.5.c)* follows from Corollary 2.4.

Properties *(2.5.c)* and *(2.5.d)* are equivalent from the homogeneity of Γ. From Lemma 2.2 we see that *(2.5.c)* implies *(2.5.a)*.

□

In the following lemma we give a generalized version of Euler's formula for generalized homogeneous functions.

Lemma 2.6. *Let A and Γ be as before. The identity*

$$m\Gamma(x) = \nabla\Gamma(x) \cdot A(x) \tag{2.1}$$

holds for every $x \in \mathbb{R}^n - \{0\}$.

Proof. From the homogeneity of Γ we have that $\lambda^m\Gamma(x) = \Gamma(T_\lambda x)$ for every $\lambda > 0$. Taking derivative with respect to λ for fixed x we have

$$\begin{aligned} m\lambda^{m-1}\Gamma(x) = \frac{d}{d\lambda}\left(\Gamma(T_\lambda x)\right) &= \frac{d}{d\lambda}\left(\Gamma(e^{A\log\lambda}x)\right) \\ &= \sum_{i=1}^{n} D_i\Gamma(e^{A\log\lambda}x) \cdot \left(\frac{A}{\lambda}e^{A\log\lambda}x\right)_i \\ &= \nabla\Gamma(e^{A\log\lambda}x) \cdot \left(\frac{A}{\lambda}e^{A\log\lambda}\right)(x). \end{aligned}$$

With $\lambda = 1$ we get (2.1). □

In the next lemmas we show that the level sets $E(\alpha)$ are adequate for the application of Gauss divergence theorem.

Lemma 2.7. *Let A and Γ be as in Theorem 1.1. Then*
(2.7.A) the level set Σ_1 is a $\mathscr{C}^1$ surface contained in $\mathbb{R}^n - \{0\}$;
(2.7.B) for each $\alpha > 0$ each Σ_α is a $\mathscr{C}^1$ surface contained in $\mathbb{R}^n - \{0\}$.

Proof. From the implicit function theorem we only here to prove that for each $x \in \Sigma_1$ the gradient of Γ at x does not vanish. Applying the generalized Euler formula (2.1) we have that

$$0 > m = \nabla\Gamma(x) \cdot A(x).$$

Hence $\nabla\Gamma(x) \neq 0$. □

Lemma 2.8. *(a) If $\Gamma > 0$, then $E(1) \cup \{0\}$ is a Gauss domain in $\mathbb{R}^n$.*
(b) If Γ vanishes at some points, then $E(1)$ is a Gauss domain in $\mathbb{R}^n$.

3. Proof of Theorem 1.2

Let u_r be a linear functional which to each $\psi \in \mathscr{C}_0^\infty(\mathbb{R}^n)$ assigns the real numbers $\frac{d\eta_\psi}{d\alpha}(r)$. It is immediate that u_r is well defined and linear. We shall prove that u_r is a Schwartz distribution od order zero and positive. Actually we shall obtain an implicit formula for the underlying measure. In order to do this, let us start by computing the derivative of $\eta_\psi(\alpha)$. From *(2.1.iii)* we have that

$$\begin{aligned} \eta_\psi(\alpha) = \iint_{E(\alpha)} \psi(x)dx &= \iint_{x \in T_\alpha(E(1))} \psi(x)dx \\ &= \alpha^\tau \iint_{y\in E(1)} \psi\left(T_\alpha y\right)dy \end{aligned} \tag{3.1}$$

For an α, ν positive from *(2.1.iii)* we get $E(\nu\alpha) = T_\nu(E(\alpha))$. So that

$$\begin{aligned}\frac{\eta_\psi(\nu\alpha)-\eta_\psi(\alpha)}{(\nu-1)\alpha} &= \frac{1}{(\nu-1)\alpha}\left(\iint_{E(\alpha\nu)}\psi - \iint_{E(\alpha)}\psi\right)\\ &= \frac{1}{\alpha}\iint_{E(\alpha)}\frac{\nu^\tau\psi\left(T_\nu y\right)-\psi(y)}{\nu-1}dy.\end{aligned}$$

Taking limit for $\nu\to 1$, we get

$$\frac{d\eta_\psi}{d\alpha}(r) = \frac{1}{r}\iint_{E(r)}\frac{d}{d\lambda}\left(\lambda^\tau\psi\left(T_\lambda y\right)\right)_{|\lambda=1}dy. \tag{3.2}$$

Since $\psi\left(T_\lambda y\right) = \psi(\lambda^{a_1}y_1,\dots,\lambda^{a_n}y_n)$, its derivative with respect to λ is given by

$$\begin{aligned}\frac{d}{d\lambda}\psi\left(T_\lambda y\right) &= \nabla\psi\left(T_\lambda y\right)\cdot\left(a_1\lambda^{a_1-1}y_1,\dots,a_n\lambda^{a_n-1}y_n\right)\\ &= \frac{1}{\lambda}\nabla\psi\left(T_\lambda y\right)\cdot A\left(T_\lambda y\right) \qquad (3.3)\end{aligned}$$

If $\vec{F}(z) = \psi(z)Az$ we have that

$$\nabla\psi\left(T_\lambda y\right)\cdot A\left(T_\lambda y\right) = \mathrm{div}\vec{F}\left(T_\lambda y\right) - \tau\psi\left(T_\lambda y\right). \tag{3.4}$$

From (3.3) and (3.4) we obtain

$$\begin{aligned}\frac{d}{d\lambda}\left(\lambda^\tau\psi\left(T_\lambda y\right)\right) &= \tau\lambda^{\tau-1}\psi\left(T_\lambda y\right)+\lambda^\tau\frac{d}{d\lambda}\left(\psi\left(T_\lambda y\right)\right)\\ &= \tau\lambda^{\tau-1}\psi\left(T_\lambda y\right)+\frac{\lambda^\tau}{\lambda}\left[\mathrm{div}\vec{F}\left(T_\lambda y\right)-\tau\psi\left(T_\lambda y\right)\right]\\ &= \lambda^{\tau-1}\,\mathrm{div}\vec{F}\left(T_\lambda y\right),\end{aligned}$$

which for $\lambda = 1$ gives

$$\frac{d}{d\lambda}\left(\lambda^\tau\psi\left(T_\lambda y\right)\right)_{|\lambda=1} = \mathrm{div}\vec{F}(y).$$

By substitution in (3.2) we get

$$\frac{d\eta_\psi}{d\alpha}(r) = \frac{1}{r}\iint_{E(r)}\mathrm{div}\vec{F}(y)\,dy.$$

Since, from Lemma 2.8, each $E(r)$ is a Gauss domain in $I\!R^n$, we have that

$$\iint_{E(r)}\mathrm{div}\vec{F}(y)\,dy = \int_{\partial E(r)}\psi(y)Ay\cdot\vec{n_r}(y)\,d\sigma_r(y).$$

From which (1.5) follows immediately.

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