

# GLOBAL SATURATION OF REGULARIZATION METHODS FOR INVERSE ILL-POSED PROBLEMS \*

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**Abstract.** In this article the concept of saturation of an arbitrary regularization method is formalized based upon the original idea of saturation for spectral regularization methods introduced by Neubauer [6]. Necessary and sufficient conditions for a regularization method to have global saturation are provided. It is shown that for a method to have global saturation the total error must be optimal in two senses, namely as optimal order of convergence over a certain set which at the same time, must be optimal (in a very precise sense) with respect to the error. Finally, two converse results are proved and the theory is applied to find sufficient conditions which ensure the existence of global saturation for spectral methods with classical qualification of finite positive order and for methods with maximal qualification. Finally, several examples of regularization methods possessing global saturation are shown.

**Key words.** Ill-posed, inverse problem, qualification, saturation.

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**1. Introduction.** Let  $X, Y$  be infinite dimensional Hilbert spaces and  $T : X \rightarrow Y$  a bounded linear operator such that  $\mathcal{R}(T)$  is not closed. It is well known that under these conditions, the linear operator equation

$$Tx = y \tag{1.1}$$

is ill-posed, in the sense that  $T^\dagger$ , the Moore-Penrose generalized inverse of  $T$ , is not bounded [1]. The Moore-Penrose generalized inverse is strongly related to the least squares solutions of (1.1). In fact this equation has a least squares solution if and only if  $y \in \mathcal{D}(T^\dagger) \doteq \mathcal{R}(T) \oplus \mathcal{R}(T)^\perp$ . In that case,  $x^\dagger \doteq T^\dagger y$  is the least squares solution of minimum norm and the set of all least-squares solutions of (1.1) is given by  $x^\dagger + \mathcal{N}(T)$ . If the problem is ill-posed then  $x^\dagger$  does not depend continuously on the data  $y$ . Therefore, if instead of the exact data  $y$ , a noisy observation  $y^\delta$  is available, with  $\|y - y^\delta\| \leq \delta$ , where  $\delta > 0$  is small, then it is possible that  $T^\dagger y^\delta$  does not even exist and if it does, it will not necessarily be a good approximation of  $x^\dagger$ . This instability becomes evident when trying to approximate  $x^\dagger$  by traditional numerical methods and procedures. Thus, for instance, it is possible that the application of the standard least squares approximating procedure on an increasing sequence of finite-dimensional subspaces  $\{X_n\}$  of  $X$  whose union is dense in  $X$ , result in a sequence  $\{x_n\}$  of least squares solutions that does not converge to  $x^\dagger$  (see [8]) or, even worst, that they diverge from  $x^\dagger$  with speed arbitrarily large (see [9]).

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Ill-posed problems must be first regularized if one wants to successfully attack the task of numerically approximating their solutions. Regularizing an ill-posed problem such as (1.1) essentially means approximating the operator  $T^\dagger$  by a parametric family of bounded operators  $\{R_\alpha\}$ , where  $\alpha$  is a regularization parameter. If  $y \in \mathcal{D}(T^\dagger)$ , then the best approximate solution  $x^\dagger$  of (1.1) can be written as  $x^\dagger = \int_0^{\|T\|^2 + \frac{1}{\lambda}} dE_\lambda T^* y$  where  $\{E_\lambda\}$  is the spectral family associated to the operator  $T^*T$  (see [1]). This is mainly why many regularization methods are based on spectral theory and consist in defining  $R_\alpha \doteq \int_0^{\|T\|^2 + \frac{1}{\lambda}} g_\alpha(\lambda) dE_\lambda T^*$  where  $\{g_\alpha\}$  is a family of functions appropriately chosen such that for every  $\lambda \in (0, \|T\|^2]$  there holds  $\lim_{\alpha \rightarrow 0^+} g_\alpha(\lambda) = \frac{1}{\lambda}$ .

However, it is important to emphasize that no mathematical trick can make stable a problem that is intrinsically unstable. In any case there is always loss of information. All a regularization method can do is to recover the largest possible amount of information about the solution of the problem, maintaining stability. It is often said that the art of applying regularization methods consist always in maintaining an adequate balance between accuracy and stability. In 1994, however, Neubauer ([6]) showed that certain spectral regularization methods “*saturate*”, that is, they become unable to continue extracting additional information about the exact solution even upon increasing regularity assumptions on it. In his article, Neubauer introduced for the first time the idea of the concept of “*saturation*” of regularization methods. This idea referred to the best order of convergence that a method can achieve independently of the smoothness assumptions on the exact solution and on the selection of the parameter choice rule. Later on, in 1997, Neubauer ([7]) showed that this saturation phenomenon occurs in particular in the classical Tikhonov-Phillips method. Saturation is however a rather subtle and complex issue in the study of regularization methods for inverse ill-posed problems and the concept has always escaped rigorous formalization in a general context.

In 2001, Mathé and Pereverzev ([4]) used Hilbert scales to study the efficiency of approximating solutions based on observations with noise (stochastic or deterministic). In this context it is possible to quantify the degree of ill-posedness and to obtain general conditions on projection methods so that they attain optimal order of convergence. These concepts were later extended by the same authors ([5]) who studied the optimal convergence problem in variable Hilbert scales. In their article they showed that there is a close relationship between the optimal convergence of a method and the “*a-priori*” regularity (in terms of source sets) for spectral methods possessing qualification of finite order. In 2009 Herdman et al. ([2]) introduced an extension of the concept of qualification and introduced three different levels: weak, strong and optimal. It was shown that weak qualification extends the definition introduced by Mathé and Pereverzev ([5]), in the sense that the functions associated to orders of convergence and source sets need not be the same.

In 2004, Mathé ([3]) proposed general definitions of the concepts of qualification and saturation for spectral regularization methods. However, the concept of saturation defined by Mathé is not applicable to general regularization methods and it is not fully compatible with the original idea of saturation proposed by Neubauer in [6]. In particular, for instance, the definition of saturation given in [3] does not imply uniqueness and therefore, neither a best global order of convergence.

In this article a general theory of global saturation for arbitrary regularization methods is developed. It is shown that saturation involves two aspects: on one hand (just like in Neubauer’s original idea) the characterization of the best global order of

convergence of the method, and on the other hand, the description of the source set on which such a best global order of convergence is achieved. Also, necessary and sufficient conditions are found for a regularization method to have global saturation. In particular, it is shown that for a method to have saturation, it is necessary that the total error be optimal in two senses, namely as optimal order of convergence over a certain set which at the same time, must satisfy a certain optimality condition with respect to the error. Moreover, an explicit form for the global saturation is given in terms of the family of regularization operators and the operator associated to the problem. Lastly, sufficient conditions are provided for spectral methods with qualification of positive finite order and for spectral methods with maximal qualification to have global saturation.

The organization of the paper is as follows. In Section 2 convergence bounds for regularization methods are defined and an appropriate framework for their comparison is developed. In Section 3 the concept of global saturation is introduced, its relation with the total error and with convergence bounds is shown and necessary and sufficient conditions for the existence of global saturation are provided. In Section 4, a few converse results are proved which, together with the results of Section 3, are used to derive sufficient conditions for the existence of global saturation for certain spectral regularization methods.

**2. Upper bounds of convergence for regularization methods.** In this section we define what we call *upper bounds of convergence* for regularization methods and we develop ways of comparing them on the same as well as on different sets. Although this section may seem a little lengthy and tedious at a first glance, it provides a solid mathematical background on which all subsequent formalization and definitions are based upon.

In sequel and for convenience of notation, unless otherwise specified, we shall assume that all subsets of the Hilbert space  $X$  under consideration are not empty and they do not contain  $x = 0$ . Also, without loss of generality we will assume that the operator  $T$  is invertible (since in the context of inverse problems one always works with the Moore-Penrose generalized inverse of  $T$ , the lack of injectivity is not really a problem). Given  $M \subset X$ , we will denote with  $\mathcal{F}_M$  the collection of the following functions: we will say that  $\psi \in \mathcal{F}_M$  if there exists  $a = a(\psi) > 0$  such that  $\psi$  is defined in  $M \times (0, a)$ , with values in  $(0, \infty)$  and it satisfies the following conditions:

1.  $\lim_{\delta \rightarrow 0^+} \psi(x, \delta) = 0$  for all  $x \in M$ , and
2.  $\psi$  is continuous and increasing as a function of  $\delta$  in  $(0, a)$  for each fixed  $x \in M$ .

Roughly speaking, the collection  $\mathcal{F}_M$  contains all possible  $\delta$ -“orders of convergence” on the set  $M$ .

DEFINITION 2.1. *Let  $M \subset X$  and  $\psi, \tilde{\psi} \in \mathcal{F}_M$ .*

*i) We say that “ $\psi$  precedes  $\tilde{\psi}$  on  $M$ ”, and we denote it  $\psi \preceq^M \tilde{\psi}$ , if there exist a constant  $r > 0$  and  $p : M \rightarrow (0, \infty)$  such that  $\psi(x, \delta) \leq p(x)\tilde{\psi}(x, \delta)$  for all  $x \in M$  and for every  $\delta \in (0, r)$ .*

*ii) We say that “ $\psi, \tilde{\psi}$  are equivalent on  $M$ ”, and we denote it  $\psi \approx^M \tilde{\psi}$ , if  $\psi \preceq^M \tilde{\psi}$  and  $\tilde{\psi} \preceq^M \psi$ .*

*iii) We say that “ $\psi$  strictly precedes  $\tilde{\psi}$  on  $M$ ” and we denote it  $\psi \prec^M \tilde{\psi}$  if  $\psi \preceq^M \tilde{\psi}$  and  $\limsup_{\delta \rightarrow 0^+} \frac{\psi(x, \delta)}{\tilde{\psi}(x, \delta)} = 0$  for every  $x \in M$ .*

The following observations follow immediately from these definitions.

• Given that  $\psi, \tilde{\psi} > 0$ , in *iii*) the condition  $\limsup_{\delta \rightarrow 0^+} \frac{\psi(x, \delta)}{\tilde{\psi}(x, \delta)} = 0$  is equivalent to  $\lim_{\delta \rightarrow 0^+} \frac{\psi(x, \delta)}{\tilde{\psi}(x, \delta)} = 0$ , i.e.,  $\psi(x, \delta) = o(\tilde{\psi}(x, \delta))$  for  $\delta \rightarrow 0^+$ .

• The relation “ $\stackrel{M}{\preceq}$ ” introduces a partial ordering in  $\mathcal{F}_M$  and “ $\stackrel{M}{\approx}$ ” is an equivalence relation in  $\mathcal{F}_M$ .

• If  $\psi \stackrel{M}{\preceq} (\approx, <) \tilde{\psi}$  then  $\psi \stackrel{\tilde{M}}{\preceq} (\approx, <) \tilde{\psi}$  for every  $\tilde{M} \subset M$ . With  $\not\preceq$ ,  $\not\approx$  and  $\not\prec$  we will denote the negation of the relations  $\preceq$ ,  $\approx$  and  $\prec$ , respectively.

LEMMA 2.2. *Let  $M \subset X$  and  $\psi, \tilde{\psi} \in \mathcal{F}_M$ . If  $\psi \prec \tilde{\psi}$  then  $\tilde{\psi} \not\preceq \psi$  for every  $\tilde{M} \subset M$ .*

*Proof.* For the contrareciprocal. Suppose there exists  $\tilde{M} \subset M$  such that  $\tilde{\psi} \stackrel{\tilde{M}}{\preceq} \psi$ . Let  $x_0 \in \tilde{M}$ , then  $\tilde{\psi} \stackrel{\{x_0\}}{\preceq} \psi$ , that is, there exist constants  $0 < p < \infty$  and  $r > 0$  such that  $\sup_{\delta \in (0, r)} \frac{\tilde{\psi}(x_0, \delta)}{\psi(x_0, \delta)} \leq p < \infty$ . Then,

$$\begin{aligned} \limsup_{\delta \rightarrow 0^+} \frac{\psi(x_0, \delta)}{\tilde{\psi}(x_0, \delta)} &\geq \liminf_{\delta \rightarrow 0^+} \frac{\psi(x_0, \delta)}{\tilde{\psi}(x_0, \delta)} \geq \inf_{\delta \in (0, r)} \frac{\psi(x_0, \delta)}{\tilde{\psi}(x_0, \delta)} \\ &= \left( \sup_{\delta \in (0, r)} \frac{\tilde{\psi}(x_0, \delta)}{\psi(x_0, \delta)} \right)^{-1} \geq \frac{1}{p} > 0. \end{aligned}$$

Therefore,  $\psi \not\preceq \tilde{\psi}$ , from which we deduce that  $\psi \not\preceq \tilde{\psi}$ , since  $x_0 \in \tilde{M} \subset M$ .  $\square$

DEFINITION 2.3. *Let  $\{R_\alpha\}_{\alpha \in (0, \alpha_0)}$  be a family of regularization operators for the problem  $Tx = y$ . We define the “total error of  $\{R_\alpha\}_{\alpha \in (0, \alpha_0)}$  at  $x \in X$  for a noise level  $\delta$ ” as*

$$\mathcal{E}_{\{R_\alpha\}}^{\text{tot}}(x, \delta) \doteq \inf_{\alpha \in (0, \alpha_0)} \sup_{y^\delta \in \overline{B_\delta(Tx)}} \|R_\alpha y^\delta - x\|,$$

where  $\overline{B_\delta(Tx)} \doteq \{y \in Y : \|Tx - y\| \leq \delta\}$ .

Note that  $\mathcal{E}_{\{R_\alpha\}}^{\text{tot}}$  is the error in the sense of the largest possible discrepancy that can be obtained for an observation within the noise level  $\delta$ , with any choice of the regularization parameter  $\alpha$ .

REMARK 2.4. *Let  $a > 0$ ,  $M \subset X$  and  $\mathcal{E}_{\{R_\alpha\}}^{\text{tot}} : M \times (0, a) \rightarrow (0, \infty)$  be the total error of  $\{R_\alpha\}$ . Then  $\mathcal{E}_{\{R_\alpha\}}^{\text{tot}} \in \mathcal{F}_M$ . In fact, for each  $x \in M$ ,  $\mathcal{E}_{\{R_\alpha\}}^{\text{tot}}(x, \delta)$  is increasing as a function of  $\delta$ , and given that  $\{R_\alpha\}$  is a family of regularization operators, it follows that  $\mathcal{E}_{\{R_\alpha\}}^{\text{tot}}(x, \delta)$  is continuous as a function of  $\delta$  for each fixed  $x \in M$  and  $\lim_{\delta \rightarrow 0^+} \mathcal{E}_{\{R_\alpha\}}^{\text{tot}}(x, \delta) = 0$  for every  $x \in M$ .*

DEFINITION 2.5. *Let  $\{R_\alpha\}_{\alpha \in (0, \alpha_0)}$  be a family of regularization operators for the problem  $Tx = y$ ,  $M \subset X$  and  $\psi \in \mathcal{F}_M$ .*

*i) We say that  $\psi$  is an “upper bound of convergence for the total error of  $\{R_\alpha\}_{\alpha \in (0, \alpha_0)}$  on  $M$ ” if  $\mathcal{E}_{\{R_\alpha\}}^{\text{tot}} \stackrel{M}{\preceq} \psi$ .*

*ii) We say that  $\psi$  is a “strict upper bound of convergence for the total error of  $\{R_\alpha\}_{\alpha \in (0, \alpha_0)}$  on  $M$ ” if  $\mathcal{E}_{\{R_\alpha\}}^{\text{tot}} \prec \psi$ .*

iii) We say that  $\psi$  is an “optimal upper bound of convergence for the total error of  $\{R_\alpha\}_{\alpha \in (0, \alpha_0)}$  on  $M$ ” if  $\mathcal{E}_{\{R_\alpha\}}^{\text{tot}} \stackrel{M}{\preceq} \psi$  and

$$\limsup_{\delta \rightarrow 0^+} \frac{\mathcal{E}_{\{R_\alpha\}}^{\text{tot}}(x, \delta)}{\psi(x, \delta)} > 0 \quad \text{for every } x \in M,$$

or equivalently, if for every  $x \in M$   $\mathcal{E}_{\{R_\alpha\}}^{\text{tot}}(x, \delta) \neq o(\psi(x, \delta))$  when  $\delta \rightarrow 0^+$ .

We will denote with  $\mathcal{U}_M(\mathcal{E}_{\{R_\alpha\}}^{\text{tot}})$ ,  $\mathcal{U}_M^{\text{str}}(\mathcal{E}_{\{R_\alpha\}}^{\text{tot}})$  and  $\mathcal{U}_M^{\text{opt}}(\mathcal{E}_{\{R_\alpha\}}^{\text{tot}})$  the set of all functions  $\psi \in \mathcal{F}_M$  that are, respectively, upper bounds, strict upper bounds and optimal upper bounds of convergence for the total error of  $\{R_\alpha\}_{\alpha \in (0, \alpha_0)}$  on  $M$ . In view of Remark 2.4, it is clear that  $\mathcal{E}_{\{R_\alpha\}}^{\text{tot}} \in \mathcal{U}_M^{\text{opt}}(\mathcal{E}_{\{R_\alpha\}}^{\text{tot}})$  for every  $M \subset X$ .

The observations below follow immediately from the previous definitions.

- If  $\psi \in \mathcal{F}_M$ , then  $\psi \in \mathcal{U}_M(\mathcal{E}_{\{R_\alpha\}}^{\text{tot}})$  if (and only if)  $\mathcal{E}_{\{R_\alpha\}}^{\text{tot}}(x, \delta) = O(\psi(x, \delta))$  as  $\delta \rightarrow 0^+$  for every  $x \in M$ . Moreover,  $\mathcal{U}_M^{\text{str}}(\mathcal{E}_{\{R_\alpha\}}^{\text{tot}})$  and  $\mathcal{U}_M^{\text{opt}}(\mathcal{E}_{\{R_\alpha\}}^{\text{tot}})$  are disjoint subsets of  $\mathcal{U}_M(\mathcal{E}_{\{R_\alpha\}}^{\text{tot}})$ , although their union is not all of  $\mathcal{U}_M(\mathcal{E}_{\{R_\alpha\}}^{\text{tot}})$  (except when  $M$  consists of just one element).
- If  $\tilde{M} \subset M$ , then  $\mathcal{U}_M(\mathcal{E}_{\{R_\alpha\}}^{\text{tot}}) \subset \mathcal{U}_{\tilde{M}}(\mathcal{E}_{\{R_\alpha\}}^{\text{tot}})$ ,  $\mathcal{U}_M^{\text{opt}}(\mathcal{E}_{\{R_\alpha\}}^{\text{tot}}) \subset \mathcal{U}_{\tilde{M}}^{\text{opt}}(\mathcal{E}_{\{R_\alpha\}}^{\text{tot}})$  and  $\mathcal{U}_M^{\text{str}}(\mathcal{E}_{\{R_\alpha\}}^{\text{tot}}) \subset \mathcal{U}_{\tilde{M}}^{\text{str}}(\mathcal{E}_{\{R_\alpha\}}^{\text{tot}})$ .
- If  $\psi \in \mathcal{U}_M(\mathcal{E}_{\{R_\alpha\}}^{\text{tot}})$ ,  $\tilde{\psi} \in \mathcal{F}_M$  and  $\psi \stackrel{M}{\preceq} \tilde{\psi}$ , then  $\tilde{\psi} \in \mathcal{U}_M(\mathcal{E}_{\{R_\alpha\}}^{\text{tot}})$ .
- If  $\psi \in \mathcal{U}_M^{\text{opt}}(\mathcal{E}_{\{R_\alpha\}}^{\text{tot}})$ ,  $\tilde{\psi} \in \mathcal{U}_M(\mathcal{E}_{\{R_\alpha\}}^{\text{tot}})$  and  $\tilde{\psi} \stackrel{M}{\preceq} \psi$ , then  $\tilde{\psi} \in \mathcal{U}_M^{\text{opt}}(\mathcal{E}_{\{R_\alpha\}}^{\text{tot}})$ .
- If  $\psi \in \mathcal{U}_M^{\text{str}}(\mathcal{E}_{\{R_\alpha\}}^{\text{tot}})$ ,  $\tilde{\psi} \in \mathcal{F}_M$  and  $\psi \stackrel{M}{\preceq} \tilde{\psi}$ , then  $\tilde{\psi} \in \mathcal{U}_M^{\text{str}}(\mathcal{E}_{\{R_\alpha\}}^{\text{tot}})$ .

DEFINITION 2.6. Let  $\psi, \tilde{\psi} \in \mathcal{F}_M$ . We say that “ $\psi$  and  $\tilde{\psi}$  are comparable on  $M$ ” if they verify  $\psi \stackrel{M}{\preceq} \tilde{\psi}$  or  $\tilde{\psi} \stackrel{M}{\preceq} \psi$  (or both).

DEFINITION 2.7. Let  $\mathcal{A} \subset \mathcal{F}_M$  and  $\psi^* \in \mathcal{A}$ . We say that “ $\psi^*$  is a minimal element of  $\left(\mathcal{A}, \stackrel{M}{\preceq}\right)$ ” if  $\psi^* \stackrel{M}{\preceq} \psi$  for every  $\psi \in \mathcal{A}$  comparable with  $\psi^*$  on  $M$ . Equivalently,  $\psi^*$  is minimal element of  $\left(\mathcal{A}, \stackrel{M}{\preceq}\right)$  if for every  $\psi \in \mathcal{A}$ , the condition  $\psi \stackrel{M}{\preceq} \psi^*$  implies  $\psi^* \stackrel{M}{\preceq} \psi$ .

LEMMA 2.8. Let  $\mathcal{A} \subset \mathcal{F}_M$ ,  $\psi, \psi^* \in \mathcal{A}$  and  $\psi, \psi^*$  be comparable on  $M$ . If there exists  $M_0 \subset M$  such that  $\psi \stackrel{M_0}{\prec} \psi^*$  then  $\psi^*$  is not a minimal element of  $\left(\mathcal{A}, \stackrel{M}{\preceq}\right)$ .

*Proof.* Let  $\mathcal{A} \subset \mathcal{F}_M$  and  $\psi, \psi^* \in \mathcal{A}$  be comparable on  $M$ . Let us suppose that there exists  $M_0 \subset M$  such that  $\psi \stackrel{M_0}{\prec} \psi^*$ , then it follows from Lemma 2.2 that  $\psi^* \not\stackrel{M_0}{\preceq} \psi$ . Thus  $\psi^* \not\stackrel{M}{\preceq} \psi$  and since  $\psi, \psi^* \in \mathcal{A}$  are comparable on  $M$ , it follows from Definition 2.7 that  $\psi^*$  cannot be a minimal element of  $\left(\mathcal{A}, \stackrel{M}{\preceq}\right)$ .  $\square$

COROLLARY 2.9. If  $\psi^* \in \mathcal{U}_M(\mathcal{E}_{\{R_\alpha\}}^{\text{tot}})$  and there exist  $x_0 \in M$  and  $\psi_0 \in \mathcal{U}_{\{x_0\}}(\mathcal{E}_{\{R_\alpha\}}^{\text{tot}})$  such that  $\psi_0 \stackrel{\{x_0\}}{\prec} \psi^*$  then  $\psi^*$  is not a minimal element of  $\left(\mathcal{U}_M(\mathcal{E}_{\{R_\alpha\}}^{\text{tot}}), \stackrel{M}{\preceq}\right)$ .

*Proof.* This corollary is an immediate consequence of the previous lemma with

$\mathcal{A} = \mathcal{U}_M(\mathcal{E}_{\{R_\alpha\}}^{\text{tot}})$ ,  $M_0 = \{x_0\}$  and

$$\psi(x, \delta) \doteq \begin{cases} \psi_0(x_0, \delta), & \text{if } x = x_0 \\ \psi^*(x, \delta), & \text{if } x \neq x_0. \end{cases}$$

Note that this function  $\psi$  so defined is in  $\mathcal{U}_M(\mathcal{E}_{\{R_\alpha\}}^{\text{tot}})$  and it is comparable with  $\psi^*$  on  $M$  (moreover  $\psi \stackrel{M}{\preceq} \psi^*$ ).  $\square$

Next we will show that the optimal upper bounds of convergence for the total error of  $\{R_\alpha\}_{\alpha \in (0, \alpha_0)}$  on  $M$  are characterized by being minimal elements of the partially ordered set  $\left(\mathcal{U}_M(\mathcal{E}_{\{R_\alpha\}}^{\text{tot}}), \stackrel{M}{\preceq}\right)$ . More precisely, we have the following result.

**THEOREM 2.10.** *Let  $\psi \in \mathcal{U}_M(\mathcal{E}_{\{R_\alpha\}}^{\text{tot}})$ . Then  $\psi \in \mathcal{U}_M^{\text{opt}}(\mathcal{E}_{\{R_\alpha\}}^{\text{tot}})$  if and only if  $\psi$  is a minimal element of  $\left(\mathcal{U}_M(\mathcal{E}_{\{R_\alpha\}}^{\text{tot}}), \stackrel{M}{\preceq}\right)$ .*

*Proof.* Let  $\psi \in \mathcal{U}_M^{\text{opt}}(\mathcal{E}_{\{R_\alpha\}}^{\text{tot}})$  and suppose that  $\psi$  is not a minimal element of  $\left(\mathcal{U}_M(\mathcal{E}_{\{R_\alpha\}}^{\text{tot}}), \stackrel{M}{\preceq}\right)$ . Then there exists  $\psi_c \in \mathcal{U}_M(\mathcal{E}_{\{R_\alpha\}}^{\text{tot}})$  comparable with  $\psi$  on  $M$  for which it is not true that  $\psi \stackrel{M}{\preceq} \psi_c$ . Then, there exists  $x_0 \in M$  such that

$$\limsup_{\delta \rightarrow 0^+} \frac{\psi(x_0, \delta)}{\psi_c(x_0, \delta)} = \infty. \quad (2.1)$$

Now, since  $\psi \in \mathcal{U}_M^{\text{opt}}(\mathcal{E}_{\{R_\alpha\}}^{\text{tot}})$  and  $x_0 \in M$ , we have that

$$\limsup_{\delta \rightarrow 0^+} \frac{\mathcal{E}_{\{R_\alpha\}}^{\text{tot}}(x_0, \delta)}{\psi(x_0, \delta)} > 0. \quad (2.2)$$

Thus

$$\limsup_{\delta \rightarrow 0^+} \frac{\mathcal{E}_{\{R_\alpha\}}^{\text{tot}}(x_0, \delta)}{\psi_c(x_0, \delta)} = \limsup_{\delta \rightarrow 0^+} \frac{\mathcal{E}_{\{R_\alpha\}}^{\text{tot}}(x_0, \delta)}{\psi(x_0, \delta)} \frac{\psi(x_0, \delta)}{\psi_c(x_0, \delta)} = \infty$$

which implies that  $\psi_c \notin \mathcal{U}_{\{x_0\}}(\mathcal{E}_{\{R_\alpha\}}^{\text{tot}})$ . This contradicts the fact that  $\psi_c \in \mathcal{U}_M(\mathcal{E}_{\{R_\alpha\}}^{\text{tot}})$ .

Therefore,  $\psi$  must be a minimal element of  $\left(\mathcal{U}_M(\mathcal{E}_{\{R_\alpha\}}^{\text{tot}}), \stackrel{M}{\preceq}\right)$ .

Conversely, assume that  $\psi \in \mathcal{U}_M(\mathcal{E}_{\{R_\alpha\}}^{\text{tot}})$  and  $\psi \notin \mathcal{U}_M^{\text{opt}}(\mathcal{E}_{\{R_\alpha\}}^{\text{tot}})$ . Then there exists  $x_0 \in M$  such that  $\psi \in \mathcal{U}_{\{x_0\}}^{\text{str}}(\mathcal{E}_{\{R_\alpha\}}^{\text{tot}})$ , which implies that  $\mathcal{E}_{\{R_\alpha\}}^{\text{tot}} \stackrel{\{x_0\}}{\prec} \psi$ . Lemma 2.8 then implies that  $\psi$  is not a minimal element of  $\left(\mathcal{U}_M(\mathcal{E}_{\{R_\alpha\}}^{\text{tot}}), \stackrel{M}{\preceq}\right)$ .  $\square$

From the proof of Theorem 2.10 it follows immediately that  $\psi$  is a minimal element of  $\left(\mathcal{U}_M(\mathcal{E}_{\{R_\alpha\}}^{\text{tot}}), \stackrel{M}{\preceq}\right)$  if and only if it is minimal of  $\left(\mathcal{U}_{M^*}(\mathcal{E}_{\{R_\alpha\}}^{\text{tot}}), \stackrel{M^*}{\preceq}\right)$  for every  $M^* \subset M$ . Also, as a consequence of Theorem 2.10 one has that all optimal upper bounds of convergence for the total error must be equivalent in the sense of Definition 2.1-ii. More precisely we have the following

**COROLLARY 2.11.** *Let  $\mathcal{U}_M^{\text{opt}}(\mathcal{E}_{\{R_\alpha\}}^{\text{tot}})$  and  $\mathcal{E}_{\{R_\alpha\}}^{\text{tot}}$  be as before*

- i) *If  $\psi \in \mathcal{U}_M^{\text{opt}}(\mathcal{E}_{\{R_\alpha\}}^{\text{tot}})$  then  $\psi \stackrel{M}{\approx} \mathcal{E}_{\{R_\alpha\}}^{\text{tot}}$ .*

ii) If  $\psi, \tilde{\psi} \in \mathcal{U}_M^{\text{opt}}(\mathcal{E}_{\{R_\alpha\}}^{\text{tot}})$  then  $\psi \stackrel{M}{\approx} \tilde{\psi}$ .

*Proof.*

i) If  $\psi \in \mathcal{U}_M^{\text{opt}}(\mathcal{E}_{\{R_\alpha\}}^{\text{tot}})$  then  $\mathcal{E}_{\{R_\alpha\}}^{\text{tot}} \stackrel{M}{\preceq} \psi$ , from which it follows that  $\mathcal{E}_{\{R_\alpha\}}^{\text{tot}}$  and  $\psi$  are comparable on  $M$ . Then, since  $\mathcal{E}_{\{R_\alpha\}}^{\text{tot}} \in \mathcal{U}_M(\mathcal{E}_{\{R_\alpha\}}^{\text{tot}})$  and by Theorem 2.10  $\psi$  is a minimal element of  $\left(\mathcal{U}_M(\mathcal{E}_{\{R_\alpha\}}^{\text{tot}}), \stackrel{M}{\preceq}\right)$ , we have that  $\psi \stackrel{M}{\preceq} \mathcal{E}_{\{R_\alpha\}}^{\text{tot}}$ . Hence,  $\psi \stackrel{M}{\approx} \mathcal{E}_{\{R_\alpha\}}^{\text{tot}}$ .

ii) This is an immediate consequence of i) and the transitivity and reflexivity of the equivalence relation “ $\stackrel{M}{\approx}$ ”, because by i) every  $\psi \in \mathcal{U}_M^{\text{opt}}(\mathcal{E}_{\{R_\alpha\}}^{\text{tot}})$  is equivalent to  $\mathcal{E}_{\{R_\alpha\}}^{\text{tot}}$  on  $M$ .  $\square$

This result says that if  $\psi$  is an optimal upper bound of convergence on  $M$  for the total error of a regularization method, then at every point of  $M$ ,  $\psi$  tends to zero, as the noise level tends to zero, exactly with the “*same speed*” with which the total error does.

In order to introduce the concept of saturation in the next section, we will previously need a few more definitions and tools that will allow us to compare bounds of convergence on different sets of  $X$ .

DEFINITION 2.12. Let  $M, \tilde{M} \subset X$ ,  $\psi \in \mathcal{F}_M$  and  $\tilde{\psi} \in \mathcal{F}_{\tilde{M}}$ .

i) We say that “ $\psi$  on  $M$  precedes  $\tilde{\psi}$  on  $\tilde{M}$ ”, and we denote it with  $\psi \stackrel{M, \tilde{M}}{\preceq} \tilde{\psi}$ , if there exist a constant  $d > 0$  and a function  $k : M \times \tilde{M} \rightarrow (0, \infty)$  such that  $\psi(x, \delta) \leq k(x, \tilde{x}) \tilde{\psi}(\tilde{x}, \delta)$  for every  $x \in M$ , for every  $\tilde{x} \in \tilde{M}$  and for every  $\delta \in (0, d)$ .

ii) We say that “ $\psi$  on  $M$  is equivalent to  $\tilde{\psi}$  on  $\tilde{M}$ ”, and we denote it with  $\psi \stackrel{M, \tilde{M}}{\approx} \tilde{\psi}$ , if  $\psi \stackrel{M, \tilde{M}}{\preceq} \tilde{\psi}$  and  $\tilde{\psi} \stackrel{\tilde{M}, M}{\preceq} \psi$ .

iii) We say that “ $\psi$  on  $M$  strictly precedes  $\tilde{\psi}$  on  $\tilde{M}$ ”, and we denote it with  $\psi \stackrel{M, \tilde{M}}{\prec} \tilde{\psi}$ , if  $\psi \stackrel{M, \tilde{M}}{\preceq} \tilde{\psi}$  and  $\limsup_{\delta \rightarrow 0^+} \frac{\psi(x, \delta)}{\tilde{\psi}(\tilde{x}, \delta)} = 0$  for every  $x \in M$ ,  $\tilde{x} \in \tilde{M}$ .

REMARK 2.13. In a certain sense, when  $M = \tilde{M}$ , the previous definitions generalize (although they are slightly stronger than) the relations introduced in Definition 2.1. Note for instance that if  $\psi \stackrel{M, M}{\prec} \tilde{\psi}$  then  $\psi \stackrel{M}{\prec} \tilde{\psi}$ , although the converse, in general, is not true.

It follows immediately from Definition 2.12 that if  $\psi \stackrel{M, N}{\preceq} \tilde{\psi}$  then  $\psi \stackrel{\tilde{M}, \tilde{N}}{\preceq} \tilde{\psi}$  for every  $\tilde{M} \subset M$  and for every  $\tilde{N} \subset N$ . The same happens for the relations “ $\stackrel{M, N}{\approx}$ ” and “ $\stackrel{M, N}{\prec}$ ”.

Next, we also need to extend the notion of “comparability” given in Definition 2.6, to this case.

DEFINITION 2.14. Let  $M, \tilde{M} \subset X$ ,  $\psi \in \mathcal{F}_M$  and  $\tilde{\psi} \in \mathcal{F}_{\tilde{M}}$ .

i) We say that “ $\psi$  on  $M$  is comparable with  $\tilde{\psi}$  on  $\tilde{M}$ ” if  $\psi \stackrel{M, \tilde{M}}{\preceq} \tilde{\psi}$  or  $\tilde{\psi} \stackrel{\tilde{M}, M}{\preceq} \psi$ .

ii) We say that “ $\psi$  is invariant over  $M$ ” if  $\psi \stackrel{M, M}{\approx} \psi$ .

REMARK 2.15. It is immediate that the condition  $\psi \stackrel{M, M}{\approx} \psi$  is equivalent to  $\psi \stackrel{M, M}{\preceq} \psi$ .

This last notion of “invariance”, which will play an important roll in the characterization of saturation, roughly speaking establishes that if  $\psi$  is invariant over  $M$

then the orders of convergence of  $\psi$  as a function of  $\delta$  when  $\delta \rightarrow 0^+$ , in any two points of  $M$ , are equivalent.

The following result is related to a certain transitivity property of this invariance relation.

LEMMA 2.16. *Let  $M \subset X$ ,  $\psi, \tilde{\psi} \in \mathcal{F}_M$  be such that  $\tilde{\psi} \stackrel{M}{\approx} \psi$  and  $\psi \stackrel{M,M}{\approx} \tilde{\psi}$ . Then:*

- i)  $\tilde{\psi} \stackrel{M,M}{\approx} \psi$  and
- ii)  $\tilde{\psi} \stackrel{M,M}{\approx} \tilde{\psi}$  (i.e.  $\tilde{\psi}$  is also invariant over  $M$ ).

*Proof.*

Let  $M \subset X$ ,  $\psi, \tilde{\psi} \in \mathcal{F}_M$ ,  $x, \tilde{x} \in M$  and suppose that  $\psi \stackrel{M,M}{\approx} \tilde{\psi}$  and  $\tilde{\psi} \stackrel{M}{\approx} \psi$ .

i) Since  $\tilde{\psi} \stackrel{M}{\approx} \psi$ , there exist positive constants  $d, k_x$  and  $k_{\tilde{x}}$  such that for every  $\delta \in (0, d)$ ,

$$\tilde{\psi}(x, \delta) \leq k_x \psi(x, \delta) \quad \text{and} \quad \psi(\tilde{x}, \delta) \leq k_{\tilde{x}} \tilde{\psi}(\tilde{x}, \delta). \quad (2.3)$$

On the other hand, from the invariance of  $\psi$  over  $M$  it follows that there exist positive constants  $d^*$  y  $k_{x, \tilde{x}}^*$  such that  $\psi(x, \delta) \leq k_{x, \tilde{x}}^* \psi(\tilde{x}, \delta)$  for every  $\delta \in (0, d^*)$ , which together with (2.3) implies that for every  $\delta \in (0, \min\{d, d^*\})$ ,

$$\tilde{\psi}(x, \delta) \leq k_x \psi(x, \delta) \leq k_x k_{x, \tilde{x}}^* \psi(\tilde{x}, \delta) \quad \text{and} \quad \psi(x, \delta) \leq k_{x, \tilde{x}}^* \psi(\tilde{x}, \delta) \leq k_{x, \tilde{x}}^* k_{\tilde{x}} \tilde{\psi}(\tilde{x}, \delta). \quad (2.4)$$

Since  $x, \tilde{x} \in M$  are arbitrary, it follows that  $\tilde{\psi} \stackrel{M,M}{\preceq} \psi$  and  $\psi \stackrel{M,M}{\preceq} \tilde{\psi}$ , that is,  $\tilde{\psi} \stackrel{M,M}{\approx} \psi$ .

ii) From the first inequality in (2.4) and from the second inequality in (2.3) it follows immediately that  $\tilde{\psi} \stackrel{M,M}{\preceq} \tilde{\psi}$  and therefore by Remark 2.15,  $\tilde{\psi}$  is invariant over  $M$ .  $\square$

The following result is analogous to Lemma 2.2 for this case of comparison of convergence bounds on different sets.

LEMMA 2.17. *Let  $M, N \subset X$ ,  $\psi \in \mathcal{F}_M$  and  $\tilde{\psi} \in \mathcal{F}_N$ . If  $\psi \stackrel{M,N}{\prec} \tilde{\psi}$  then  $\forall \tilde{M} \subset M$ ,  $\forall \tilde{N} \subset N$  we have that  $\tilde{\psi} \stackrel{\tilde{N}, \tilde{M}}{\not\preceq} \psi$ .*

*Proof.* By the contrareciprocal. Suppose that there exist  $\tilde{M} \subset M$  and  $\tilde{N} \subset N$  such that  $\tilde{\psi} \stackrel{\tilde{N}, \tilde{M}}{\preceq} \psi$ . Then there exist a constant  $d > 0$  and  $k : \tilde{N} \times \tilde{M} \rightarrow (0, \infty)$  such that  $\tilde{\psi}(\tilde{x}, \delta) \leq k(\tilde{x}, x) \psi(x, \delta)$  for every  $\tilde{x} \in \tilde{N}$ ,  $x \in \tilde{M}$  and  $\delta \in (0, d)$ . Let  $x_0 \in \tilde{M}$  and  $\tilde{x}_0 \in \tilde{N}$ , then  $\tilde{\psi}(\tilde{x}_0, \delta) \leq k(\tilde{x}_0, x_0) \psi(x_0, \delta)$  for every  $\delta \in (0, d)$ . Thus,  $\sup_{\delta \in (0, d)} \frac{\tilde{\psi}(\tilde{x}_0, \delta)}{\psi(x_0, \delta)} \leq k(\tilde{x}_0, x_0) < \infty$ . Then,

$$\begin{aligned} \limsup_{\delta \rightarrow 0^+} \frac{\psi(x_0, \delta)}{\tilde{\psi}(\tilde{x}_0, \delta)} &\geq \liminf_{\delta \rightarrow 0^+} \frac{\psi(x_0, \delta)}{\tilde{\psi}(\tilde{x}_0, \delta)} \geq \inf_{\delta \in (0, d)} \frac{\psi(x_0, \delta)}{\tilde{\psi}(\tilde{x}_0, \delta)} \\ &= \left( \sup_{\delta \in (0, d)} \frac{\tilde{\psi}(\tilde{x}_0, \delta)}{\psi(x_0, \delta)} \right)^{-1} \geq \frac{1}{k(\tilde{x}_0, x_0)} > 0. \end{aligned}$$

Hence,  $\psi \stackrel{\{x_0\}, \{\tilde{x}_0\}}{\not\preceq} \tilde{\psi}$ , from which it follows that  $\psi \stackrel{M,N}{\not\preceq} \tilde{\psi}$ , since  $x_0 \in M$  and  $\tilde{x}_0 \in N$ .  $\square$

**3. Global Saturation.** We will now proceed to formalize the concept of global saturation.

DEFINITION 3.1. Let  $M_S \subset X$  and  $\psi_S \in \mathcal{U}_{M_S}(\mathcal{E}_{\{R_\alpha\}}^{\text{tot}})$ . We say that  $\psi_S$  is a “global saturation function of  $\{R_\alpha\}$  over  $M_S$ ” if  $\psi_S$  satisfies the following three conditions:

S1. For every  $x^* \in X$ ,  $x^* \neq 0$ ,  $x \in M_S$ ,  $\limsup_{\delta \rightarrow 0^+} \frac{\mathcal{E}_{\{R_\alpha\}}^{\text{tot}}(x^*, \delta)}{\psi_S(x, \delta)} > 0$ .

S2.  $\psi_S$  is invariant over  $M_S$ .

S3. There is no upper bound of convergence for the total error of  $\{R_\alpha\}$  that is a proper extension of  $\psi_S$  (in the variable  $x$ ) and satisfies S1 and S2, that is, there exist no  $\tilde{M} \supsetneq M_S$  and  $\tilde{\psi} \in \mathcal{U}_{\tilde{M}}(\mathcal{E}_{\{R_\alpha\}}^{\text{tot}})$  such that  $\tilde{\psi}$  satisfies S1 and S2 with  $M_S$  replaced by  $\tilde{M}$  and  $\psi_S$  replaced by  $\tilde{\psi}$ .

We shall refer to  $\psi_S$  and  $M_S$  as the saturation function and the saturation set, respectively.

REMARK 3.2. Note that condition S1 implies that for every  $M \subset X$  and for every  $\psi \in \mathcal{U}_M(\mathcal{E}_{\{R_\alpha\}}^{\text{tot}})$ ,  $\limsup_{\delta \rightarrow 0^+} \frac{\psi(x^*, \delta)}{\psi_S(x, \delta)} > 0$  for every  $x^* \in M$ ,  $x \in M_S$  (this is an immediate

consequence of S1 and the fact that  $\mathcal{E}_{\{R_\alpha\}}^{\text{tot}} \stackrel{M}{\preceq} \psi \forall \psi \in \mathcal{U}_M(\mathcal{E}_{\{R_\alpha\}}^{\text{tot}})$ ). Therefore, it

cannot happen that  $\psi \stackrel{M, M_S}{\prec} \psi_S$ . On the other hand, if  $\psi \in \mathcal{U}_M(\mathcal{E}_{\{R_\alpha\}}^{\text{tot}})$  then it is not necessarily true that  $\psi_S \stackrel{M_S, M}{\preceq} \psi$  even if  $\psi$  on  $M$  is comparable to  $\psi_S$  on  $M_S$ , because in this case it can happen that  $\liminf_{\delta \rightarrow 0^+} \frac{\psi(x, \delta)}{\psi_S(x_S, \delta)} = 0$  for some  $x \in M$  and some  $x_S \in$

$M_S$  (which obviously implies that  $\psi_S \not\preceq \psi$ ), and still have  $\limsup_{\delta \rightarrow 0^+} \frac{\psi(x, \delta)}{\psi_S(x_S, \delta)} > 0$ .

However, if  $\psi$  on  $M$  is comparable with  $\psi_S$  on  $M_S$  and there exists  $\lim_{\delta \rightarrow 0^+} \frac{\psi(x, \delta)}{\psi_S(x_S, \delta)}$  for every  $x \in M$  and for every  $x_S \in M_S$ , then it is in fact true that  $\psi_S \stackrel{M_S, M}{\preceq} \psi$ . Note also that condition S1 can be replaced by

$$\limsup_{\delta \rightarrow 0^+} \frac{\psi(x^*, \delta)}{\psi_S(x, \delta)} > 0 \quad \forall \psi \in \mathcal{U}_{\{x^*\}}(\mathcal{E}_{\{R_\alpha\}}^{\text{tot}}), \forall x^* \in X, x^* \neq 0, x \in M_S.$$

This conception of global saturation essentially establishes that in no point  $x^* \in X$ ,  $x^* \neq 0$ , can exist an upper bound of convergence for the total error of the regularization method that is “strictly better” than the saturation function  $\psi_S$  at any point of the saturation set  $M_S$ .

Next we show that any function satisfying condition S1, in particular any saturation function, is always an optimal upper bound of convergence.

LEMMA 3.3. Let  $\psi_S \in \mathcal{U}_{M_S}(\mathcal{E}_{\{R_\alpha\}}^{\text{tot}})$ . If  $\psi_S$  satisfies the condition S1 on  $M_S$ , then  $\psi_S \in \mathcal{U}_{M_S}^{\text{opt}}(\mathcal{E}_{\{R_\alpha\}}^{\text{tot}})$ .

*Proof.* The condition S1 implies in particular that  $\limsup_{\delta \rightarrow 0^+} \frac{\mathcal{E}_{\{R_\alpha\}}^{\text{tot}}(x, \delta)}{\psi_S(x, \delta)} > 0$  for every  $x \in M_S$ . Since also by definition  $\psi_S \in \mathcal{U}_{M_S}(\mathcal{E}_{\{R_\alpha\}}^{\text{tot}})$  it follows that  $\psi_S$  is an optimal upper bound of convergence for the total error of  $\{R_\alpha\}$ , i.e.  $\psi_S \in \mathcal{U}_{M_S}^{\text{opt}}(\mathcal{E}_{\{R_\alpha\}}^{\text{tot}})$ .  $\square$

An immediate consequence of this lemma is the equivalence between the saturation function and the total error on the saturation set.

COROLLARY 3.4. If  $\psi_S$  is a saturation function of  $\{R_\alpha\}$  on  $M_S$  then  $\psi_S \stackrel{M_S}{\approx} \mathcal{E}_{\{R_\alpha\}}^{\text{tot}}$ . Moreover, we have the stronger equivalence  $\psi_S \stackrel{M_S, M_S}{\approx} \mathcal{E}_{\{R_\alpha\}}^{\text{tot}}$ .

*Proof.* The first part of the corollary is an immediate consequence of the previous lemma and of Corollary 2.11 i). The second part follows from the first and the fact that  $\psi_S \stackrel{M_S, M_S}{\approx} \psi_S$ , via Lemma 2.16 i).  $\square$

REMARK 3.5. *A consequence of the first part of this corollary and of Lemma 2.16 ii) is that if  $\psi_S$  is a saturation function of  $\{R_\alpha\}$  on  $M_S$ , then  $\mathcal{E}_{\{R_\alpha\}}^{\text{tot}} \stackrel{M_S, M_S}{\approx} \mathcal{E}_{\{R_\alpha\}}^{\text{tot}}$ , that is, the total error must be invariant over  $M_S$ . We will shed more light on this matter in Theorem 3.8.*

DEFINITION 3.6. *Let  $M \subset X$  and  $\psi \in \mathcal{U}_X(\mathcal{E}_{\{R_\alpha\}}^{\text{tot}})$ . We say that “ $M$  is optimal for  $\psi$ ”, and we denote it with  $M \in \mathcal{O}(\psi)$ , if the following condition holds:*

C2. *For every  $x \in M$ ,  $x_c \in M^c$  neither  $\psi \stackrel{\{x_c\}, \{x\}}{<} \psi$  nor  $\psi \stackrel{\{x_c\}, \{x\}}{\approx} \psi$ .*

That a set  $M$  be optimal for  $\psi$  essentially means that at any point of the complement of  $M$ , the order of convergence of  $\psi$  as a function of  $\delta$ , for  $\delta \rightarrow 0^+$ , cannot be better nor even equivalent to the order of convergence of  $\psi$  at any point outside  $M$ ; that is, at any point outside of  $M$ , the order of convergence of  $\psi$  must be strictly worse than itself at any point of  $M$ . However, we will see next that this optimality condition imposes a very precise restriction. As we shall see later on (Theorem 3.8), it is precisely this property of the total error, together with its invariance on the set  $M_S$ , what will allow us to characterize the regularization methods which do have saturation.

Condition C2 is very precise and gives no room for maneuver. In fact, let  $\psi \in \mathcal{U}_X(\mathcal{E}_{\{R_\alpha\}}^{\text{tot}})$ ,  $M \subset X$  and consider the following conditions:

C1.  $\psi \stackrel{M, M^c}{<} \psi$ .

C3.  $\psi \not\stackrel{M^c, M}{\approx} \psi$  and  $\psi \not\stackrel{M^c, M}{\approx} \psi$ .

Then it follows that condition C2 (of optimal set) is strictly stronger than condition C3, and strictly weaker than condition C1. In fact, if  $M$  is optimal for  $\psi$  in the sense of Definition 3.6, then for every  $x \in M$ ,  $x_c \in M^c$  we have that  $\psi \stackrel{\{x_c\}, \{x\}}{\not\approx} \psi$  and  $\psi \stackrel{\{x_c\}, \{x\}}{\not\approx} \psi$ , from which it follows immediately that  $\psi \stackrel{M^c, M}{\not\approx} \psi$  and  $\psi \stackrel{M^c, M}{\not\approx} \psi$ , that is, C3 holds. However, for condition C3 to hold it is sufficient that there exist  $x \in M$  and  $x_c \in M^c$  such that  $\psi \stackrel{\{x_c\}, \{x\}}{\not\approx} \psi$  and  $\psi \stackrel{\{x_c\}, \{x\}}{\not\approx} \psi$ , which obviously does not imply condition C2. On the other hand if C1 holds, then it follows from Lemma 2.17 that for every  $x \in M$ ,  $x_c \in M^c$ , there holds  $\psi \stackrel{\{x_c\}, \{x\}}{\not\approx} \psi$  and therefore,  $\psi \stackrel{\{x_c\}, \{x\}}{\not\approx} \psi$  and  $\psi \stackrel{\{x_c\}, \{x\}}{\not\approx} \psi$  for every  $x \in M$ ,  $x_c \in M^c$ , that is, condition C2 holds. However, C2 does not imply C1 since it can happen that  $M$  be optimal for  $\psi$  and that there exist  $x \in M$  and  $x_c \in M^c$  such that  $\psi$  on  $\{x\}$  is not comparable with  $\psi$  on  $\{x_c\}$ . This implies in particular that  $\psi \stackrel{\{x\}, \{x_c\}}{\not\approx} \psi$  and therefore,  $\psi \stackrel{M, M^c}{\not\approx} \psi$ .

In order to be able to characterize the regularization methods which do have saturation, we will previously need the following result.

LEMMA 3.7. *Suppose that  $\{R_\alpha\}$  has saturation function on  $M \subset X$  and for every  $x \in M$ ,  $x_c \in M^c$  there holds  $\mathcal{E}_{\{R_\alpha\}}^{\text{tot}} \stackrel{\{x_c\}, \{x\}}{\not\approx} \mathcal{E}_{\{R_\alpha\}}^{\text{tot}}$ . Then  $\mathcal{E}_{\{R_\alpha\}}^{\text{tot}} \stackrel{\{x_c\}, \{x\}}{\not\approx} \mathcal{E}_{\{R_\alpha\}}^{\text{tot}}$  for every  $x \in M$ ,  $x_c \in M^c$ .*

*Proof.* Since  $\{R_\alpha\}$  has saturation function on  $M$ , it follows from Remark 3.5

that  $\mathcal{E}_{\{R_\alpha\}}^{\text{tot}}$  is invariant over  $M$ . Suppose that  $\mathcal{E}_{\{R_\alpha\}}^{\text{tot}} \not\stackrel{\{x_c\},\{x\}}{\approx} \mathcal{E}_{\{R_\alpha\}}^{\text{tot}}$  for every  $x \in M$ ,  $x_c \in M^c$  and that there exist  $\tilde{x} \in M$ ,  $\tilde{x}_c \in M^c$  such that

$$\mathcal{E}_{\{R_\alpha\}}^{\text{tot}} \stackrel{\{\tilde{x}_c\},\{\tilde{x}\}}{\approx} \mathcal{E}_{\{R_\alpha\}}^{\text{tot}}. \quad (3.1)$$

Then,

$$\limsup_{\delta \rightarrow 0^+} \frac{\mathcal{E}_{\{R_\alpha\}}^{\text{tot}}(\tilde{x}, \delta)}{\mathcal{E}_{\{R_\alpha\}}^{\text{tot}}(\tilde{x}_c, \delta)} > 0. \quad (3.2)$$

Define  $\tilde{M} \doteq M \cup \{\tilde{x}_c\}$  and

$$\tilde{\psi}(x, \delta) \doteq \begin{cases} \psi(x, \delta), & \text{if } x \in M \\ \mathcal{E}_{\{R_\alpha\}}^{\text{tot}}(x, \delta), & \text{if } x = \tilde{x}_c, \end{cases}$$

where  $\psi$  is a saturation function of  $\{R_\alpha\}$  on  $M$ . We will show next that  $\tilde{\psi}$  is saturation function on  $\tilde{M}$ . Clearly,  $\tilde{\psi}$  is upper bound of convergence for the total error on  $\tilde{M}$ , i.e.,  $\tilde{\psi} \in \mathcal{U}_{\tilde{M}}(\mathcal{E}_{\{R_\alpha\}}^{\text{tot}})$  and since  $\psi$  is saturation on  $M$ , it follows that  $\tilde{\psi}(x, \delta)$  satisfies condition *S1* for all  $x \in M$ . We will now check that  $\tilde{\psi}(\tilde{x}_c, \delta)$  also satisfies *S1*. Since  $\tilde{x} \in M$  it follows that

$$\limsup_{\delta \rightarrow 0^+} \frac{\mathcal{E}_{\{R_\alpha\}}^{\text{tot}}(x^*, \delta)}{\mathcal{E}_{\{R_\alpha\}}^{\text{tot}}(\tilde{x}, \delta)} > 0 \quad \forall x^* \in X, x^* \neq 0. \quad (3.3)$$

If  $x^* \in M$ , the above inequality follows from the fact that  $\mathcal{E}_{\{R_\alpha\}}^{\text{tot}}$  is invariant over  $M$  and if  $x^* \in M^c$ , it is a consequence of the fact that  $\mathcal{E}_{\{R_\alpha\}}^{\text{tot}} \not\stackrel{\{x^*\},\{\tilde{x}\}}{\approx} \mathcal{E}_{\{R_\alpha\}}^{\text{tot}}$ .

Then, for every  $x^* \in X$ ,  $x^* \neq 0$  we have that

$$\limsup_{\delta \rightarrow 0^+} \frac{\mathcal{E}_{\{R_\alpha\}}^{\text{tot}}(x^*, \delta)}{\tilde{\psi}(\tilde{x}_c, \delta)} = \limsup_{\delta \rightarrow 0^+} \frac{\mathcal{E}_{\{R_\alpha\}}^{\text{tot}}(x^*, \delta)}{\mathcal{E}_{\{R_\alpha\}}^{\text{tot}}(\tilde{x}, \delta)} \frac{\mathcal{E}_{\{R_\alpha\}}^{\text{tot}}(\tilde{x}, \delta)}{\tilde{\psi}(\tilde{x}_c, \delta)} > 0$$

by virtue of (3.2) and (3.3). Thus,  $\tilde{\psi}(x, \delta)$  satisfies *S1* for every  $x \in \tilde{M}$ .

We will now check that  $\tilde{\psi}$  satisfies *S2* on  $\tilde{M}$ . Since  $\psi$  is saturation function of  $\{R_\alpha\}$  on  $M$ , and  $\tilde{\psi}|_M = \psi$  we have that  $\tilde{\psi}$  is invariant over  $M$ . It remains to prove that  $\tilde{\psi} \stackrel{\{\tilde{x}_c\}, M}{\approx} \tilde{\psi}$ , i.e. that  $\mathcal{E}_{\{R_\alpha\}}^{\text{tot}} \stackrel{\{\tilde{x}_c\}, M}{\approx} \tilde{\psi}$ . But this is an immediate consequence of (3.1), of Corollary 3.4 which implies that  $\tilde{\psi} \stackrel{M}{\approx} \mathcal{E}_{\{R_\alpha\}}^{\text{tot}}$  and the fact that  $\psi$  is invariant over  $M$ .

Thus, we have shown that  $\tilde{\psi}$  is a proper extension of  $\psi$  satisfying *S1* and *S2* on  $\tilde{M}$ , which then implies that  $\psi$  does not satisfy condition *S3*. This contradicts the fact that  $\psi$  is saturation function of  $\{R_\alpha\}$  on  $M$ . Therefore, for every  $x \in M$ ,  $x_c \in M^c$  there must hold that  $\mathcal{E}_{\{R_\alpha\}}^{\text{tot}} \stackrel{\{x_c\},\{x\}}{\approx} \mathcal{E}_{\{R_\alpha\}}^{\text{tot}}$ .  $\square$

**THEOREM 3.8.** *(Necessary and sufficient condition for the existence of saturation.) A regularization method  $\{R_\alpha\}$  has saturation function if and only if there exists  $M \subset X$  ( $M \neq \{0\}, M \neq \emptyset$ ) such that  $\mathcal{E}_{\{R_\alpha\}}^{\text{tot}}$  is invariant over  $M$  and  $M$  is optimal for  $\mathcal{E}_{\{R_\alpha\}}^{\text{tot}}$ . In this case  $\mathcal{E}_M^{\text{tot}}(x, \delta) \doteq \mathcal{E}_{\{R_\alpha\}}^{\text{tot}}(x, \delta)$  for  $x \in M$  and  $\delta > 0$  is saturation function of  $\{R_\alpha\}$  on  $M$ .*

*Proof.* Suppose that  $\{R_\alpha\}$  has saturation function  $\psi$  on  $M$ . Then it follows from Remark 3.5 that  $\mathcal{E}_{\{R_\alpha\}}^{\text{tot}}$  is invariant over  $M$ .

Let us now check that  $M$  is optimal for  $\mathcal{E}_{\{R_\alpha\}}^{\text{tot}}$ . Let  $x \in M$  and  $x_c \in M^c$ . We will first show that  $\mathcal{E}_{\{R_\alpha\}}^{\text{tot}} \not\stackrel{\{x_c\},\{x\}}{\approx} \mathcal{E}_{\{R_\alpha\}}^{\text{tot}}$ . Since  $\psi \in \mathcal{U}_M(\mathcal{E}_{\{R_\alpha\}}^{\text{tot}})$  and  $x \in M$ , there exist positive constants  $d$  and  $k_x$  such that  $\mathcal{E}_{\{R_\alpha\}}^{\text{tot}}(x, \delta) \leq k_x \psi(x, \delta)$  for every  $\delta \in (0, d)$ . Then

$$\limsup_{\delta \rightarrow 0^+} \frac{\mathcal{E}_{\{R_\alpha\}}^{\text{tot}}(x_c, \delta)}{\mathcal{E}_{\{R_\alpha\}}^{\text{tot}}(x, \delta)} \geq \limsup_{\delta \rightarrow 0^+} \frac{\mathcal{E}_{\{R_\alpha\}}^{\text{tot}}(x_c, \delta)}{k_x \psi(x, \delta)} > 0,$$

where the last inequality follows from the fact that  $\psi$  satisfies condition  $S1$  on  $M$ .

Therefore  $\forall x \in M, \forall x_c \in M^c, \mathcal{E}_{\{R_\alpha\}}^{\text{tot}} \not\stackrel{\{x_c\},\{x\}}{\approx} \mathcal{E}_{\{R_\alpha\}}^{\text{tot}}$ . This condition together with the fact that  $\mathcal{E}_{\{R_\alpha\}}^{\text{tot}}$  is invariant over  $M$  implies, by virtue of Lemma 3.7, that  $\forall x \in M, \forall x_c \in M^c, \mathcal{E}_{\{R_\alpha\}}^{\text{tot}} \stackrel{\{x_c\},\{x\}}{\approx} \mathcal{E}_{\{R_\alpha\}}^{\text{tot}}$ . We have thus shown that  $M$  is optimal for  $\mathcal{E}_{\{R_\alpha\}}^{\text{tot}}$ .

Conversely, suppose that there exists  $M \subset X$  ( $M \neq \{0\}, M \neq \emptyset$ ) such that  $\mathcal{E}_{\{R_\alpha\}}^{\text{tot}}$  is invariant over  $M$  and  $M$  is optimal for  $\mathcal{E}_{\{R_\alpha\}}^{\text{tot}}$  and define  $\mathcal{E}_M^{\text{tot}}(x, \delta) \doteq \mathcal{E}_{\{R_\alpha\}}^{\text{tot}}(x, \delta)$  for  $x \in M$  and  $\delta > 0$ . We will show that  $\mathcal{E}_M^{\text{tot}}$  is saturation function of  $\{R_\alpha\}$  on  $M$ . Clearly,  $\mathcal{E}_M^{\text{tot}} \in \mathcal{U}_M(\mathcal{E}_{\{R_\alpha\}}^{\text{tot}})$  and since by hypothesis  $\mathcal{E}_M^{\text{tot}}$  is invariant over  $M$ , it only remains to be shown that  $\mathcal{E}_M^{\text{tot}}$  satisfies conditions  $S1$  and  $S3$ .

In order to prove  $S1$ , let  $x^* \in X, x^* \neq 0$  and  $x \in M$ . If  $x^* \in M$ , then the invariance of  $\mathcal{E}_M^{\text{tot}}$  over  $M$  implies that  $\mathcal{E}_M^{\text{tot}} \stackrel{\{x^*\},\{x\}}{\approx} \mathcal{E}_M^{\text{tot}}$  and therefore

$$\limsup_{\delta \rightarrow 0^+} \frac{\mathcal{E}_{\{R_\alpha\}}^{\text{tot}}(x^*, \delta)}{\mathcal{E}_M^{\text{tot}}(x, \delta)} = \limsup_{\delta \rightarrow 0^+} \frac{\mathcal{E}_M^{\text{tot}}(x^*, \delta)}{\mathcal{E}_M^{\text{tot}}(x, \delta)} > 0. \quad (3.4)$$

On the other hand, if  $x^* \in M^c$ , the previous limit is also positive due to the fact that  $\mathcal{E}_{\{R_\alpha\}}^{\text{tot}} \not\stackrel{\{x^*\},\{x\}}{\approx} \mathcal{E}_M^{\text{tot}}$  (by condition  $C2$ ) because  $M$  is optimal for  $\mathcal{E}_{\{R_\alpha\}}^{\text{tot}}$ . Then,  $\mathcal{E}_M^{\text{tot}}$  satisfies condition  $S1$ .

Finally, suppose that  $\mathcal{E}_M^{\text{tot}}$  does not satisfy condition  $S3$ , i.e. there exist  $\tilde{M} \supsetneq M$  and  $\tilde{\psi} \in \mathcal{U}_{\tilde{M}}(\mathcal{E}_{\{R_\alpha\}}^{\text{tot}})$  such that  $\tilde{\psi}$  is a proper extension of  $\mathcal{E}_M^{\text{tot}}$  satisfying conditions  $S1$  and  $S2$  on  $\tilde{M}$ . Let  $\tilde{x} \in \tilde{M} \setminus M$ , then the invariance of  $\tilde{\psi}$  over  $\tilde{M}$  implies that  $\tilde{\psi} \stackrel{\{\tilde{x}\}, M}{\approx} \tilde{\psi}$  and since  $\tilde{\psi}$  coincides with  $\mathcal{E}_M^{\text{tot}}$  on  $M$ , it follows that

$$\tilde{\psi} \stackrel{\{\tilde{x}\}, M}{\approx} \mathcal{E}_M^{\text{tot}}. \quad (3.5)$$

Now since  $\tilde{\psi} \in \mathcal{U}_{\tilde{M}}(\mathcal{E}_{\{R_\alpha\}}^{\text{tot}})$  satisfies  $S1$  on  $\tilde{M}$ , Lemma 3.3 implies that  $\tilde{\psi} \in \mathcal{U}_{\tilde{M}}^{\text{opt}}(\mathcal{E}_{\{R_\alpha\}}^{\text{tot}})$ .

Then, by virtue of Corollary 2.11.i we have that  $\mathcal{E}_{\{R_\alpha\}}^{\text{tot}} \stackrel{\tilde{M}}{\approx} \tilde{\psi}$ . In particular,  $\mathcal{E}_{\{R_\alpha\}}^{\text{tot}} \stackrel{\{\tilde{x}\}}{\approx} \tilde{\psi}$ , which, together with (3.5) imply that  $\mathcal{E}_{\{R_\alpha\}}^{\text{tot}} \stackrel{\{\tilde{x}\}, M}{\approx} \mathcal{E}_M^{\text{tot}}$ , that is,  $\mathcal{E}_{\{R_\alpha\}}^{\text{tot}} \stackrel{\{\tilde{x}\}, M}{\approx} \mathcal{E}_{\{R_\alpha\}}^{\text{tot}}$ . But since  $\tilde{x} \in M^c$ , this equivalence contradicts the fact that  $M$  is optimal for  $\mathcal{E}_{\{R_\alpha\}}^{\text{tot}}$ . Therefore,  $\mathcal{E}_M^{\text{tot}}$  must satisfy condition  $S3$  and, as a consequence, it is saturation function of  $\{R_\alpha\}$  on  $M$ .  $\square$

REMARK 3.9. *From the previous theorem we conclude that a saturation function of a regularization method is an optimal upper bound of convergence for the total error, invariant and without proper extensions.*

Note that a saturation function must be optimal in two senses. In fact, if  $\psi$  is saturation function on  $M$ , then  $M$  is optimal for  $\psi$  and  $\psi$  is optimal (upper bound) for the total error of  $\{R_\alpha\}$  on  $M$ . Moreover,  $M$  and  $\psi$  (modulus  $M, M$  equivalence) are uniquely determined. In fact, if the domain  $M$  is changed, then  $M$  is no longer optimal for  $\psi$  and if the function  $\psi$  is changed, even at a single point of  $M$ , in such a way that  $\psi$  is not invariant on  $M$ , then  $\psi$  it is no longer an optimal upper bound. Suppose that at a point  $x_0 \in M$ , we redefine  $\psi$  as  $\tilde{\psi}(x_0, \delta)$ , where  $\tilde{\psi} \in \mathcal{F}_M$ . If  $\tilde{\psi} \stackrel{\{x_0\}}{<} \psi$ , then  $\psi$  is no longer an upper bound for the total error of  $\{R_\alpha\}$  on  $M$  and if  $\psi \stackrel{\{x_0\}}{<} \tilde{\psi}$  then  $\psi$  is upper bound but it is not optimal. Thus for every  $\tilde{M} \subset M$  and for every  $\tilde{\psi} \in \mathcal{F}_{\tilde{M}}$ , if  $\psi$  and  $\tilde{\psi}$  are comparable on  $\tilde{M}$  then  $\psi \stackrel{\tilde{M}, \tilde{M}}{\approx} \tilde{\psi}$  must hold.

**4. Saturation for spectral regularization methods.** The objective of this section is to apply the theory previously developed to the case of spectral regularization methods. Further, we show that this theory is consistent with previously existing results about optimal convergence of spectral regularization methods.

Let  $\{E_\lambda\}_{\lambda \in \mathbb{R}}$  be the spectral family associated to the linear selfadjoint operator  $T^*T$  and  $\{g_\alpha\}_{\alpha \in (0, \alpha_0)}$  a parametric family of functions  $g_\alpha : [0, \|T\|^2] \rightarrow \mathbb{R}$  for  $\alpha \in (0, \alpha_0)$ , and consider the following standing hypotheses:

H1. For every  $\alpha \in (0, \alpha_0)$  the function  $g_\alpha$  is piecewise continuous on  $[0, \|T\|^2]$ .

H2. There exists a constant  $C > 0$  (independent of  $\alpha$ ) such that  $|\lambda g_\alpha(\lambda)| \leq C$  for every  $\lambda \in [0, \|T\|^2]$ .

H3. For every  $\lambda \in (0, \|T\|^2]$ , there exists  $\lim_{\alpha \rightarrow 0^+} g_\alpha(\lambda) = \frac{1}{\lambda}$ .

H4.  $G_\alpha \doteq \|g_\alpha(\cdot)\|_\infty = O\left(\frac{1}{\sqrt{\alpha}}\right)$  for  $\alpha \rightarrow 0^+$ .

If  $\{g_\alpha\}_{\alpha \in (0, \alpha_0)}$  satisfies hypotheses H1-H3, then (see [1], Theorem 4.1) the collection of operators  $\{R_\alpha\}_{\alpha \in (0, \alpha_0)}$ , where

$$R_\alpha \doteq \int g_\alpha(\lambda) dE_\lambda T^* = g_\alpha(T^*T)T^*, \quad (4.1)$$

is a family of regularization operators for  $T^\dagger$ . In this case we say that  $\{R_\alpha\}_{\alpha \in (0, \alpha_0)}$  is a family of spectral regularization operators for  $Tx = y$ .

Next, we recall the classical definition of qualification for a family of spectral regularization operators.

**DEFINITION 4.1.** Let  $\{R_\alpha\}_{\alpha \in (0, \alpha_0)}$  be the family of spectral regularization operators for  $Tx = y$  generated by the family of functions  $\{g_\alpha\}_{\alpha \in (0, \alpha_0)}$ ,  $r_\alpha(\lambda) \doteq 1 - \lambda g_\alpha(\lambda)$ ,  $0 < \alpha < \alpha_0$ ,  $0 \leq \lambda \leq \|T\|^2$ , and let us denote with  $\mathcal{I}(g_\alpha)$  the set

$$\mathcal{I}(g_\alpha) \doteq \{\mu \geq 0 : \exists k > 0 \text{ and } \lambda^\mu |r_\alpha(\lambda)| \leq k \alpha^\mu \forall \lambda \in [0, \|T\|^2], \forall \alpha \in (0, \alpha_0)\}.$$

The order of the classical qualification of  $\{R_\alpha\}_{\alpha \in (0, \alpha_0)}$  is defined to be  $\mu_0 \doteq \sup_{\mu \in \mathcal{I}(g_\alpha)} \mu$  and we say that  $\{R_\alpha\}_{\alpha \in (0, \alpha_0)}$  has classical qualification of order  $\mu_0$ .

**REMARK 4.2.** Note that by virtue of H2,  $0 \in \mathcal{I}(g_\alpha)$  and the order  $\mu_0$  of the classical qualification of a regularization method is always nonnegative (it can be equal to 0 or  $+\infty$ ).

**4.1. Spectral methods with classical qualification of finite positive order.** We start by considering first the case of spectral methods for which  $0 < \mu_0 < \infty$ . For these methods we will first show the existence of certain upper bounds of convergence and then we will show that they saturate. We will also characterize their saturation functions and saturation sets.

LEMMA 4.3. *Suppose that  $\{g_\alpha\}_{\alpha \in (0, \alpha_0)}$  satisfies the hypotheses H1-H4. If the family of regularization operators  $\{R_\alpha\}_{\alpha \in (0, \alpha_0)}$ , with  $R_\alpha$  defined as in (4.1), has classical qualification of order  $\mu_0$ ,  $0 < \mu_0 < +\infty$ , then  $\psi_{\mu_0}(x, \delta) \doteq \delta^{\frac{2\mu_0}{2\mu_0+1}}$ , for  $x \in X_{\mu_0} \doteq \mathcal{R}((T^*T)^{\mu_0}) \setminus \{0\}$  and  $\delta > 0$ , is upper bound of convergence for the total error of  $\{R_\alpha\}_{\alpha \in (0, \alpha_0)}$  on  $X_{\mu_0}$ , that is,  $\psi_{\mu_0} \in \mathcal{U}_{X_{\mu_0}}(\mathcal{E}_{\{R_\alpha\}}^{\text{tot}})$ .*

*Proof.* Since  $\{g_\alpha\}$  satisfies hypothesis H4, we have that  $G_\alpha = O\left(\frac{1}{\sqrt{\alpha}}\right)$  when  $\alpha \rightarrow 0^+$  and therefore  $G_\alpha = o\left(\frac{1}{\alpha}\right)$  when  $\alpha \rightarrow 0^+$ . From this and from the fact that  $\{g_\alpha\}$  satisfies hypothesis H1-H3 and  $\{R_\alpha\}$  has classical qualification of order  $\mu_0$ ,  $0 < \mu_0 < +\infty$ , it follows that (see [1], Corollary 4.4 and Remark 4.5 therein) there exists an *a-priori* parameter choice rule  $\alpha^* : \mathbb{R}^+ \rightarrow (0, \alpha_0)$  such that the regularization method  $(R_\alpha, \alpha^*)$  is of optimal order on  $X_{\mu_0}$ , that is, for every  $x \in X_{\mu_0}$  there exists  $k(x) > 0$  such that for every  $\delta > 0$ ,

$$\sup_{y^\delta \in \overline{B_\delta(Tx)}} \|R_{\alpha^*(\delta)} y^\delta - x\| \leq k(x) \delta^{\frac{2\mu_0}{2\mu_0+1}}.$$

Then

$$\inf_{\alpha \in (0, \alpha_0)} \sup_{y^\delta \in \overline{B_\delta(Tx)}} \|R_\alpha y^\delta - x\| \leq k(x) \delta^{\frac{2\mu_0}{2\mu_0+1}},$$

that is,  $\mathcal{E}_{\{R_\alpha\}}^{\text{tot}}(x, \delta) \leq k(x) \psi_{\mu_0}(\delta)$ . Thus  $\mathcal{E}_{\{R_\alpha\}}^{\text{tot}} \stackrel{X_{\mu_0}}{\preceq} \psi_{\mu_0}$  and therefore  $\psi_{\mu_0} \in \mathcal{U}_{X_{\mu_0}}(\mathcal{E}_{\{R_\alpha\}}^{\text{tot}})$ .  $\square$

THEOREM 4.4. *(Saturation for families of spectral regularization operators with classical qualification of finite positive order.)*

Suppose that  $\{g_\alpha\}_{\alpha > 0}$  satisfies hypotheses H1-H4 and let  $r_\alpha(\lambda) \doteq 1 - \lambda g_\alpha(\lambda)$ . Suppose further that:

i) The spectrum of  $T^*T$  has  $\lambda = 0$  as accumulation point.

ii) There exist positive constants  $\gamma_1, \gamma_2, \lambda_1, c_1$ , with  $\lambda_1 \leq \|T\|^2$  and  $c_1 > 1$  such that

a)  $0 \leq r_\alpha(\lambda) \leq 1$ ,  $\alpha > 0$ ,  $0 \leq \lambda \leq \lambda_1$ ;

b)  $r_\alpha(\lambda) \geq \gamma_1$ ,  $0 \leq \lambda < \alpha \leq \lambda_1$ ;

c)  $|r_\alpha(\lambda)|$  is monotone increasing with respect to  $\alpha$  for  $\lambda \in (0, \|T\|^2]$ ;

d)  $g_\alpha(c_1\alpha) \geq \frac{\gamma_2}{\alpha}$ ,  $0 < c_1\alpha \leq \lambda_1$  and

e)  $g_\alpha(\lambda) \geq g_\alpha(\tilde{\lambda})$ , for  $0 < \alpha \leq \lambda \leq \tilde{\lambda} \leq \lambda_1$ .

There exist constants  $\gamma, c > 0$  such that:

iii) The family of regularization operators  $\{R_\alpha\}_{\alpha \in (0, \alpha_0)}$  defined by (4.1), where  $\alpha_0 \doteq \min\{\lambda_1, \frac{\lambda_1}{c}\}$ , has classical qualification of order  $\mu_0$ ,  $0 < \mu_0 < +\infty$ .

iv)

$$\left(\frac{\lambda}{\alpha}\right)^{\mu_0} |r_\alpha(\lambda)| \geq \gamma, \quad \text{for every } 0 < c\alpha \leq \lambda \leq \|T\|^2. \quad (4.2)$$

Then  $\psi_{\mu_0}(x, \delta) \doteq \delta^{\frac{2\mu_0}{2\mu_0+1}}$  for  $x \in X_{\mu_0} \doteq \mathcal{R}((T^*T)^{\mu_0}) \setminus \{0\}$  and  $\delta > 0$ , is saturation function of  $\{R_\alpha\}_{\alpha \in (0, \alpha_0)}$  on  $X_{\mu_0}$ .

Note that the hypothesis **i)** is trivially satisfied if  $T$  is compact. To prove this theorem we will need two previous lemmas. In the first one we show that under the hypotheses of Theorem 4.4, for all  $\alpha$  in a right neighborhood of zero one has that  $0 \in \rho(r_\alpha(T^*T))$ , i.e. zero belongs to the resolvent set of the operator  $r_\alpha(T^*T)$ . More precisely we have the following:

LEMMA 4.5. *Suppose that  $\{g_\alpha\}_{\alpha > 0}$  satisfies hypotheses H1-H4 and assume further that hypotheses **ii.b)**, **ii.c)**, **iii)** and **vi)** of Theorem 4.4 hold. Then for every  $\alpha \in (0, \alpha_0)$  the operator  $r_\alpha(T^*T)$  is invertible, where  $\alpha_0 \doteq \min\{\lambda_1, \frac{\lambda_1}{c}\}$ .*

*Proof.* It suffices to show that for every  $\alpha \in (0, \alpha_0)$  and for every  $x \in X$ , the function  $r_\alpha^{-2}(\lambda)$  is integrable with respect to the measure  $d\|E_\lambda x\|^2$ . Let  $\alpha \in (0, \alpha_0)$  be arbitrary but fixed. Since  $\alpha_0 \leq \lambda_1$ , it follows from hypothesis **ii.b)** that  $r_\alpha(\lambda) \geq \gamma_1 > 0$  for every  $\lambda \in [0, \alpha)$ . Then

$$\int_0^\alpha \frac{1}{r_\alpha^2(\lambda)} d\|E_\lambda x\|^2 \leq \frac{\|x\|^2}{\gamma_1^2} < +\infty. \quad (4.3)$$

It remains to prove that  $\int_\alpha^{\|T\|^2} \frac{1}{r_\alpha^2(\lambda)} d\|E_\lambda x\|^2 < +\infty$ . For that we shall consider two cases.

**Case I:**  $c \leq 1$ . In this case, for every  $\lambda \in [\alpha, \|T\|^2]$  we have that  $\lambda \geq \alpha \geq c\alpha > 0$  and from (4.2) it follows that  $|r_\alpha(\lambda)| \geq \gamma \left(\frac{\alpha}{\lambda}\right)^{\mu_0}$  for every  $\lambda \in [\alpha, \|T\|^2]$ . Therefore

$$\int_\alpha^{\|T\|^2} \frac{1}{r_\alpha^2(\lambda)} d\|E_\lambda x\|^2 \leq \int_\alpha^{\|T\|^2} \frac{\lambda^{2\mu_0}}{(\alpha^{\mu_0} \gamma)^2} d\|E_\lambda x\|^2 \leq \frac{\|(T^*T)^{\mu_0} x\|^2}{(\alpha^{\mu_0} \gamma)^2} < +\infty.$$

**Case II:**  $c > 1$ . In this case, since  $c\alpha < \|T\|^2$  we write

$$\int_\alpha^{\|T\|^2} \frac{1}{r_\alpha^2(\lambda)} d\|E_\lambda x\|^2 = \int_\alpha^{c\alpha} \frac{1}{r_\alpha^2(\lambda)} d\|E_\lambda x\|^2 + \int_{c\alpha}^{\|T\|^2} \frac{1}{r_\alpha^2(\lambda)} d\|E_\lambda x\|^2. \quad (4.4)$$

Like in the previous case, by virtue of (4.2), the second integral on the RHS of (4.4) is bounded above by  $\frac{\|(T^*T)^{\mu_0} x\|^2}{(\alpha^{\mu_0} \gamma)^2} < +\infty$ . For the first integral on the RHS of (4.4), by virtue of hypothesis **ii.c)** we have that

$$r_\alpha^2(\lambda) \geq r_{\alpha/c}^2(\lambda), \quad \forall \lambda \in [\alpha, c\alpha] \quad (4.5)$$

because  $\frac{\alpha}{c} < \alpha$ . On the other hand, again by using (4.2), and given that  $0 < c\left(\frac{\alpha}{c}\right) \leq \lambda$  we have that

$$\left(\frac{\lambda}{\alpha/c}\right)^{2\mu_0} r_{\alpha/c}^2(\lambda) \geq \gamma^2. \quad (4.6)$$

From (4.5) and (4.6) we conclude that  $r_\alpha^2(\lambda) \geq \gamma^2 \left(\frac{\alpha}{c\lambda}\right)^{2\mu_0}$  for every  $\lambda \in [\alpha, c\alpha]$ . Thus, for the first integral on the RHS of (4.4) we have the estimate

$$\int_\alpha^{c\alpha} \frac{1}{r_\alpha^2(\lambda)} d\|E_\lambda x\|^2 \leq \int_\alpha^{c\alpha} \frac{c^{2\mu_0}}{\alpha^{2\mu_0} \gamma^2} \lambda^{2\mu_0} d\|E_\lambda x\|^2 \leq \frac{c^{2\mu_0}}{\alpha^{2\mu_0} \gamma^2} \|(T^*T)^{\mu_0} x\|^2 < \infty.$$

Hence  $r_\alpha(T^*T)$  is an invertible operator for every  $\alpha \in (0, \alpha_0)$ .  $\square$

LEMMA 4.6. *Suppose that  $\{g_\alpha\}_{\alpha>0}$  satisfies the hypotheses H1-H4 and suppose further that hypotheses **ii.b**), **ii.c**), **iii**) and **vi**) of Theorem 4.4 hold. Let  $\varphi : [0, \|T\|^2] \rightarrow \mathbb{R}^+$  be a continuous, strictly increasing function satisfying  $\varphi(0) = 0$ . If for some  $x^* \in X$ ,  $x^* \neq 0$  we have that  $\mathcal{E}_{\{R_\alpha\}}^{\text{tot}}(x^*, \delta) = o(\varphi(\delta))$  for  $\delta \rightarrow 0^+$ , then there exists an a-priori parameter choice rule  $\tilde{\alpha}(\delta)$  such that*

$$\sup_{y^\delta \in B_\delta(Tx^*)} \|R_{\tilde{\alpha}(\delta)} y^\delta - x^*\| = o(\varphi(\delta)) \quad \text{for } \delta \rightarrow 0^+.$$

The same remains true if we replace  $o(\varphi(\delta))$  by  $O(\varphi(\delta))$ .

*Proof.* Let  $\varphi$  be as in the hypotheses and suppose that there exists  $x^* \in X$ ,  $x^* \neq 0$  such that  $\mathcal{E}_{\{R_\alpha\}}^{\text{tot}}(x^*, \delta) = o(\varphi(\delta))$  for  $\delta \rightarrow 0^+$ . Then by definition of  $\mathcal{E}_{\{R_\alpha\}}^{\text{tot}}$ ,

$$\lim_{\delta \rightarrow 0^+} \frac{\inf_{\alpha \in (0, \alpha_0)} \sup_{y^\delta \in B_\delta(Tx^*)} \|R_\alpha y^\delta - x^*\|}{\varphi(\delta)} = \lim_{\delta \rightarrow 0^+} \inf_{\alpha \in (0, \alpha_0)} \frac{\sup_{y^\delta \in B_\delta(Tx^*)} \|R_\alpha y^\delta - x^*\|}{\varphi(\delta)} = 0. \quad (4.7)$$

For the sake of simplify we introduce the following notation:

$$f(\alpha, \delta) \doteq \frac{\sup_{y^\delta \in B_\delta(Tx^*)} \|R_\alpha y^\delta - x^*\|}{\varphi(\delta)} \quad \text{and} \quad h(\delta) \doteq \inf_{\alpha \in (0, \alpha_0)} f(\alpha, \delta).$$

Then  $h(\delta) > 0$  for every  $\delta \in (0, \infty)$  and (4.7) can be written simply as  $\lim_{\delta \rightarrow 0^+} h(\delta) = 0$ .

Next, for  $n \in \mathbb{N}$  we define

$$\delta_n \doteq \sup \left\{ \delta > 0 : h(\delta) \leq \frac{1}{n} \right\}.$$

Clearly,  $\delta_n \downarrow 0$  and  $h(\delta) = \inf_{\alpha \in (0, \alpha_0)} f(\alpha, \delta) \leq \frac{1}{n}$  for every  $\delta \in (0, \delta_n]$  for every  $n \in \mathbb{N}$ .

Then, there exists  $\alpha_n = \alpha_n(\delta_n) \in (0, \alpha_0)$  such that

$$f(\alpha_n, \delta) \leq \frac{2}{n} \quad \forall \delta \in (0, \delta_n], \quad \forall n \in \mathbb{N}. \quad (4.8)$$

We then define  $\alpha(\delta) \doteq \alpha_n$  for all  $\delta \in (\delta_{n+1}, \delta_n]$  for every  $n \in \mathbb{N}$ . Then, since  $\delta_n \downarrow 0$  it follows from (4.8) that  $\lim_{\delta \rightarrow 0^+} f(\alpha(\delta), \delta) = \lim_{n \rightarrow +\infty} f(\alpha_n, \delta_n) = 0$ . We could choose  $\alpha$  as the parameter choice rule we are looking for. The problem is that we cannot guarantee the existence of the limit of  $\alpha(\delta)$  for  $\delta \rightarrow 0^+$ . However, we will see next that  $\alpha(\delta)$  can be replaced by a function  $\tilde{\alpha} : \mathbb{R}^+ \rightarrow (0, \alpha_0)$  such that  $\lim_{\delta \rightarrow 0^+} \tilde{\alpha}(\delta) = 0$  (i.e, such that  $\tilde{\alpha}(\delta)$  is an admissible parameter choice rule) maintaining the condition  $\lim_{\delta \rightarrow 0^+} f(\tilde{\alpha}(\delta), \delta) = 0$ . In fact, since  $\{\alpha_n\}_{n \in \mathbb{N}} \subset (0, \alpha_0)$  is a bounded sequence of real numbers, it contains a convergent subsequence  $\{\alpha_{n_k}\}_{k \in \mathbb{N}}$ , with  $\alpha_{n_k} \rightarrow \alpha^*$  for  $k \rightarrow +\infty$ , and some  $\alpha^* \in [0, \alpha_0]$ . We define  $\tilde{\alpha}(\delta) \doteq \alpha_{n_k}$  for all  $\delta \in (\delta_{n_{k+1}}, \delta_{n_k}]$ , for every  $k \in \mathbb{N}$ . Then,

$$\lim_{\delta \rightarrow 0^+} \tilde{\alpha}(\delta) = \lim_{k \rightarrow +\infty} \alpha_{n_k} = \alpha^*. \quad (4.9)$$

Since  $\{\alpha_{n_k}\}_{k \in \mathbb{N}}$  and  $\{\delta_{n_k}\}_{k \in \mathbb{N}}$  are subsequences of  $\{\alpha_n\}_{n \in \mathbb{N}}$  and  $\{\delta_n\}_{n \in \mathbb{N}}$ ,  $\lim_{\delta \rightarrow 0^+} f(\tilde{\alpha}(\delta), \delta) = \lim_{k \rightarrow +\infty} f(\alpha_{n_k}, \delta_{n_k}) = 0$ . Then, by definition of  $f$ ,

$$\lim_{\delta \rightarrow 0^+} \frac{\sup_{y^\delta \in B_\delta(Tx^*)} \|R_{\tilde{\alpha}(\delta)} y^\delta - x^*\|}{\varphi(\delta)} = 0,$$

that is,

$$\sup_{y^\delta \in B_\delta(Tx^*)} \|R_{\tilde{\alpha}(\delta)} y^\delta - x^*\| = o(\varphi(\delta)), \quad \text{as } \delta \rightarrow 0^+. \quad (4.10)$$

It remains to be shown that  $\alpha^* = 0$ . If  $\alpha^* > 0$ , then it follows from (4.9) that there exists  $\delta_0 > 0$  such that  $\tilde{\alpha}(\delta) > \frac{\alpha^*}{2}$  for all  $\delta \in (0, \delta_0)$ . Hypothesis **ii.c**) of Theorem 4.4 implies then that for every  $\delta \in (0, \delta_0)$ ,  $|r_{\tilde{\alpha}(\delta)}(\lambda)| \geq |r_{\frac{\alpha^*}{2}}(\lambda)|$  for all  $\lambda \in (0, \|T\|^2]$ . It follows that for every  $\delta \in (0, \delta_0)$ ,

$$\begin{aligned} \|r_{\tilde{\alpha}(\delta)}(T^*T)x^*\|^2 &= \int_0^{\|T\|^2} r_{\tilde{\alpha}(\delta)}^2(\lambda) d\|E_\lambda x^*\|^2 \\ &\geq \int_0^{\|T\|^2} r_{\frac{\alpha^*}{2}}^2(\lambda) d\|E_\lambda x^*\|^2 \\ &= \left\| r_{\frac{\alpha^*}{2}}(T^*T)x^* \right\|^2. \end{aligned}$$

Then, for all  $\delta \in (0, \delta_0)$ ,

$$\begin{aligned} \sup_{y^\delta \in B_\delta(Tx^*)} \|R_{\tilde{\alpha}(\delta)} y^\delta - x^*\| &\geq \|R_{\tilde{\alpha}(\delta)} Tx^* - x^*\| = \|(I - g_{\tilde{\alpha}(\delta)}(T^*T)T^*T)x^*\| \\ &= \|r_{\tilde{\alpha}(\delta)}(T^*T)x^*\| \geq \left\| r_{\frac{\alpha^*}{2}}(T^*T)x^* \right\|. \end{aligned}$$

Taking limit for  $\delta \rightarrow 0^+$  and using (4.10) we conclude that  $\left\| r_{\frac{\alpha^*}{2}}(T^*T)x^* \right\| = 0$ . But since  $\frac{\alpha^*}{2} < \alpha_0$ , it follows from Lemma 4.5 that  $r_{\frac{\alpha^*}{2}}(T^*T)$  is invertible and therefore  $x^* = 0$ , which is a contradiction since  $x^*$  was not zero to start with. Hence,  $\alpha^*$  must be equal to zero, as wanted.

We proceed now to prove the second part of the Lemma. Suppose that there exists  $x^* \in X$ ,  $x^* \neq 0$  such that  $\mathcal{E}_{\{R_\alpha\}}^{\text{tot}}(x^*, \delta) = O(\varphi(\delta))$  as  $\delta \rightarrow 0^+$ . Then there exist positive constants  $k$  and  $d$  such that  $\inf_{\alpha \in (0, \alpha_0)} f(\alpha, \delta) \leq k$  for every  $\delta \in (0, d)$ , where  $f(\alpha, \delta)$  is as previously defined. Let  $\{\delta_n\}_{n \in \mathbb{N}} \subset (0, d)$  be such that  $\delta_n \downarrow 0$  and  $\alpha_n = \alpha_n(\delta_n) \in (0, \alpha_0)$  such that

$$f(\alpha_n, \delta) \leq k + \delta_n, \quad \forall \delta \in (0, d), \quad \forall n \in \mathbb{N}.$$

We define (just like we did it previously for the “o” case)  $\alpha(\delta) \doteq \alpha_n$  for all  $\delta \in (\delta_{n+1}, \delta_n]$  for every  $n \in \mathbb{N}$ . Since  $\delta_n \downarrow 0$  it follows that  $f(\alpha(\delta), \delta) \leq k + \delta_1$  for every  $\delta \in (0, d)$  and therefore

$$\sup_{y^\delta \in B_\delta(Tx^*)} \|R_{\alpha(\delta)} y^\delta - x^*\| = O(\varphi(\delta)) \quad \text{as } \delta \rightarrow 0^+. \quad (4.11)$$

Exactly in the same way as we proceeded before in the first part of the proof, by defining the function  $\tilde{\alpha}(\delta)$  (from a convergent subsequence of  $\{\alpha_n\}_{n \in \mathbb{N}}$ ), equation (4.11) is proved with  $\tilde{\alpha}(\delta)$  in place of  $\alpha(\delta)$ . Finally, and also by proceeding in an analogous way, it is shown that  $\tilde{\alpha}(\delta)$  converges to zero as  $\delta \rightarrow 0^+$ , i.e. that  $\tilde{\alpha}(\delta)$  is an admissible parameter choice rule. Since the steps are essentially the same we do not give details here.  $\square$

We are now ready to prove Theorem 4.4.

*Proof of Theorem 4.4.* We will show that  $\psi_{\mu_0}(x, \delta) \doteq \delta^{\frac{2\mu_0}{2\mu_0+1}}$  for  $x \in X_{\mu_0}$  and  $\delta > 0$ , is saturation function of  $\{R_\alpha\}_{\alpha \in (0, \alpha_0)}$  on  $X_{\mu_0}$ .

First we note that by virtue of Lemma 4.3,  $\psi_{\mu_0} \in \mathcal{U}_{X_{\mu_0}}(\mathcal{E}_{\{R_\alpha\}}^{\text{tot}})$ . Next we will show that  $\psi_{\mu_0}$  satisfies condition *S1* of saturation on  $X_{\mu_0}$  (see Definition 3.1). Suppose that it is not true, i.e. suppose that there exist  $x^* \in X$ ,  $x^* \neq 0$  and  $x \in X_{\mu_0}$  such that  $\limsup_{\delta \rightarrow 0^+} \frac{\mathcal{E}_{\{R_\alpha\}}^{\text{tot}}(x^*, \delta)}{\psi_{\mu_0}(x, \delta)} = 0$ . Then  $\mathcal{E}_{\{R_\alpha\}}^{\text{tot}}(x^*, \delta) = o\left(\delta^{\frac{2\mu_0}{2\mu_0+1}}\right)$  as  $\delta \rightarrow 0^+$  and from Lemma 4.6 it follows that there exists an *a-priori* admissible parameter choice rule  $\alpha(\delta)$  such that

$$\sup_{y^\delta \in B_\delta(Tx^*)} \|R_{\alpha(\delta)}y^\delta - x^*\| = o\left(\delta^{\frac{2\mu_0}{2\mu_0+1}}\right) \quad \text{for } \delta \rightarrow 0^+.$$

Now note that hypothesis *H4* implies that there exists a finite positive constant  $\beta$  such that  $\sqrt{\lambda}|g_\alpha(\lambda)| \leq \frac{\beta}{\sqrt{\alpha}}$ , for every  $\alpha \in (0, \alpha_0)$  and for every  $\lambda \in [0, \|T\|^2]$ . Since  $\{g_\alpha\}$  satisfies the hypotheses *H1-H4* and *i)-iv)* hold, it follows from Theorem 3.1 of [6] that  $x^* = 0$ , which contradicts the fact that  $x^*$  was different from zero. Hence,  $\psi_{\mu_0}$  satisfies condition *S1* on  $X_{\mu_0}$ . Since  $\psi_{\mu_0}$  does not depend on  $x$ , we further have that  $\psi_{\mu_0}$  is (trivially) invariant over  $X_{\mu_0}$ , i.e., it satisfies condition *S2*.

It only remains to prove that  $\psi_{\mu_0}$  satisfies condition *S3*, that is, that the set  $X_{\mu_0}$  is optimal for  $\psi_{\mu_0}$ . Suppose that is not the case. Then there must exist  $M \supsetneq X_{\mu_0}$  and  $\tilde{\psi} \in \mathcal{U}_M(\mathcal{E}_{\{R_\alpha\}}^{\text{tot}})$  such that  $\tilde{\psi}|_{X_{\mu_0}} = \psi_{\mu_0}$  and  $\tilde{\psi}$  satisfies *S1* and *S2* on  $M$ . Let  $x^* \in M \setminus X_{\mu_0}$ ,  $x^* \neq 0$ . Since  $\tilde{\psi} \in \mathcal{U}_M(\mathcal{E}_{\{R_\alpha\}}^{\text{tot}})$  we have that

$$\mathcal{E}_{\{R_\alpha\}}^{\text{tot}} \stackrel{\{x^*\}}{\preceq} \tilde{\psi}. \quad (4.12)$$

Also, since  $\tilde{\psi}$  is invariant over  $M$ , we have that  $\tilde{\psi} \stackrel{\{x^*\}, X_{\mu_0}}{\preceq} \tilde{\psi}$ , and since  $\tilde{\psi}$  coincides with  $\psi_{\mu_0}$  on  $X_{\mu_0}$ , it follows that  $\tilde{\psi} \stackrel{\{x^*\}, X_{\mu_0}}{\preceq} \psi_{\mu_0}$ . This, together with (4.12) implies that  $\mathcal{E}_{\{R_\alpha\}}^{\text{tot}} \stackrel{\{x^*\}, X_{\mu_0}}{\preceq} \psi_{\mu_0}$  and therefore  $\mathcal{E}_{\{R_\alpha\}}^{\text{tot}}(x^*, \delta) = O\left(\delta^{\frac{2\mu_0}{2\mu_0+1}}\right)$  as  $\delta \rightarrow 0^+$ . Lemma 4.6 then implies that there exists an *a-priori* admissible parameter choice rule  $\alpha(\delta)$  such that

$$\sup_{y^\delta \in B_\delta(Tx^*)} \|R_{\alpha(\delta)}y^\delta - x^*\| = O\left(\delta^{\frac{2\mu_0}{2\mu_0+1}}\right) \quad \text{as } \delta \rightarrow 0^+.$$

Since  $\mu_0 < +\infty$  it follows that  $x^* \in \mathcal{R}((T^*T)^{\mu_0})$  (see [6], Corollary 2.6) and since  $x^* \neq 0$ , we have that  $x^* \in X_{\mu_0}$  which contradicts that  $x^* \in M \setminus X_{\mu_0}$ . Thus,  $\psi_{\mu_0}$  satisfies condition *S3* and  $\psi_{\mu_0}$  is saturation function of  $\{R_\alpha\}$  on  $X_{\mu_0}$ , as we wanted to prove.

**4.2. Spectral Methods with maximal qualification.** The concept of classical qualification is a special case of a more general definition of qualification introduced by Mathé and Pereverzev ([5], [3]).

**DEFINITION 4.7.** Let  $\{R_\alpha\}_{\alpha \in (0, \alpha_0)}$  be a family of spectral regularization operators for  $Tx = y$  generated by the family of functions  $\{g_\alpha\}_{\alpha \in (0, \alpha_0)}$  and let  $r_\alpha(\lambda) \doteq 1 - \lambda g_\alpha(\lambda)$ . A function  $\rho : (0, \|T\|^2] \rightarrow \mathbb{R}^+$  is said to be qualification of  $\{R_\alpha\}_{\alpha \in (0, \alpha_0)}$  if  $\rho$  is increasing and there exists a constant  $\gamma > 0$  such that

$$\sup_{\lambda \in (0, \|T\|^2]} |r_\alpha(\lambda)| \rho(\lambda) \leq \gamma \rho(\alpha) \quad \text{for every } \alpha \in (0, \alpha_0).$$

If, moreover, for every  $\lambda \in (0, \|T\|^2]$  there exists a constant  $c \doteq c(\lambda) > 0$  such that

$$\inf_{\alpha \in (0, \alpha_0)} \frac{|r_\alpha(\lambda)|}{\rho(\alpha)} \geq c$$

then  $\rho$  is said to be maximal qualification of  $\{R_\alpha\}_{\alpha \in (0, \alpha_0)}$ .

Note then that the classical qualification of order  $\mu$  corresponds to the case in which the functions  $\rho$  are restricted to monomials  $\rho(t) = t^\mu$  for  $0 \leq \mu < +\infty$ .

These two definitions of qualification are closely related. For instance, if a spectral regularization method  $\{R_\alpha\}$  possesses classical qualification of order  $\mu_0 < \infty$ , then any increasing function  $\tilde{\rho} : (0, \|T\|^2] \rightarrow \mathbb{R}^+$  satisfying  $\alpha^{\mu_0} \leq k \tilde{\rho}(\alpha)$  for some constant  $k > 0$ , for  $\alpha$  in a neighborhood of  $\alpha = 0$ , is also qualification of  $\{R_\alpha\}$ . Also, if  $\alpha^{\mu_0}$  and  $\tilde{\rho}(\alpha)$  are two maximal qualifications then they are necessarily equivalent in the sense that there exist constants  $k, \tilde{k} > 0$  such that  $k \alpha^{\mu_0} \leq \tilde{\rho}(\alpha) \leq \tilde{k} \alpha^{\mu_0}$  for every  $\alpha \in (0, \alpha_0)$ . On the other hand, if a spectral regularization method  $\{R_\alpha\}$  has classical qualification of infinite order, then it does not necessarily have maximal qualification.

Next, we will show that under certain general hypotheses, it is also possible to characterize the saturation of spectral regularization methods possessing maximal qualification. For that we will previously need the following definition.

**DEFINITION 4.8.** Let  $\rho : (0, a] \rightarrow (0, +\infty)$  be a continuous non-decreasing function such that  $\lim_{t \rightarrow 0^+} \rho(t) = 0$  and  $\beta \in \mathbb{R}$ ,  $\beta \geq 0$ . We say that  $\rho$  is of local upper type  $\beta$  if there exists a positive constant  $d$  such that  $\rho(t) \leq d (\frac{1}{s})^\beta \rho(st)$  for every  $s \in (0, 1]$ ,  $t \in (0, a]$ .

A function of finite upper type is also said to satisfy a  $\Delta_\beta$  condition.

**THEOREM 4.9.** (Saturation for families of spectral regularization operators with maximal qualification.)

Let  $T$  be a compact linear operator. Suppose that  $\{g_\alpha\}_{\alpha \in (0, \alpha_0)}$  satisfies hypotheses H1-H4 and let  $\{R_\alpha\}_{\alpha \in (0, \alpha_0)}$  be as defined by (4.1). Suppose further that the following hypotheses are satisfied:

**M1:** There exist  $\{\tilde{\lambda}_n\}_{n=1}^\infty \subset \sigma_p(TT^*)$  and  $c \geq 1$  such that  $\tilde{\lambda}_n \downarrow 0$  and  $\frac{\tilde{\lambda}_n}{\tilde{\lambda}_{n+1}} \leq c$  for every  $n \in \mathbb{N}$ .

**M2:** There exist positive constants  $\lambda_1 \leq \|T\|^2$ ,  $\gamma_1, \gamma_2$  and  $c_1 > 1$  such that

- a)  $0 \leq r_\alpha(\lambda) \leq 1$ ,  $\alpha > 0$ ,  $0 \leq \lambda \leq \lambda_1$ ;
- b)  $r_\alpha(\lambda) \geq \gamma_1$ ,  $0 \leq \lambda < \alpha \leq \lambda_1$ ;
- c)  $|r_\alpha(\lambda)|$  is monotone increasing as a function of  $\alpha$  for each  $\lambda \in (0, \|T\|^2]$ ;
- d)  $g_\alpha(c_1 \alpha) \geq \frac{\gamma_2}{\alpha}$ ,  $0 < c_1 \alpha \leq \lambda_1$  and
- e)  $g_\alpha(\lambda) \geq g_\alpha(\tilde{\lambda})$ , for  $0 < \alpha \leq \lambda \leq \tilde{\lambda} \leq \lambda_1$ .

**M3:** There exists  $\rho : (0, \|T\|^2] \rightarrow (0, +\infty)$ , strictly increasing and of local upper type  $\beta$ , for some  $\beta \geq 0$ , such that  $\rho$  is maximal qualification of  $\{R_\alpha\}_{\alpha \in (0, \alpha_0)}$  and there exist positive constants  $a$  and  $k$  such that

$$\frac{\rho(\lambda) |r_\alpha(\lambda)|}{\rho(\alpha)} \geq a, \quad \text{for all } \alpha, \lambda \text{ such that } 0 < k \alpha \leq \lambda \leq \|T\|^2.$$

**M4:** For every  $\alpha \in (0, \alpha_0)$  the function  $\lambda \rightarrow |r_\alpha(\lambda)|^2$ ,  $\lambda \in (0, \|T\|^2]$  is convex.

Let  $\Theta(t) \doteq \sqrt{t} \rho(t)$  for  $t \in (0, \|T\|^2]$ . Then  $\psi(x, \delta) \doteq (\rho \circ \Theta^{-1})(\delta)$  for  $x \in X^\rho \doteq \mathcal{R}(\rho(T^*T)) \setminus \{0\}$  and  $\delta \in (0, \Theta(\alpha_0))$ , is saturation function of  $\{R_\alpha\}_{\alpha \in (0, \alpha_0)}$  on  $X^\rho$ .

In order to prove this theorem we will previously need two converse results that we establish in the following two Lemmas.

LEMMA 4.10. Let  $\{R_\alpha\}_{\alpha \in (0, \alpha_0)}$  be a family of spectral regularization operators for  $Tx = y$  and  $\rho : (0, \|T\|^2] \rightarrow \mathbb{R}^+$  a strictly increasing continuous function satisfying hypothesis **M3** of Theorem 4.9. If for some  $x \in X$ ,  $\|R_\alpha Tx - x\| = O(\rho(\alpha))$  for  $\alpha \rightarrow 0^+$ , then  $x \in \mathcal{R}(\rho(T^*T))$ .

*Proof.* From hypothesis **M3** it follows that

$$\|R_\alpha Tx - x\|^2 = \int_0^{\|T\|^2} r_\alpha^2(\lambda) d\|E_\lambda x\|^2 \geq a^2 \rho^2(\alpha) \int_{k\alpha}^{\|T\|^2} \rho^{-2}(\lambda) d\|E_\lambda x\|^2. \quad (4.13)$$

Since  $\|R_\alpha Tx - x\| = O(\rho(\alpha))$  for  $\alpha \rightarrow 0^+$ , it then follows that there are constants  $C > 0$  and  $\alpha^*$ ,  $0 < \alpha^* \leq \alpha_0$  such that

$$\int_{k\alpha}^{\|T\|^2} \rho^{-2}(\lambda) d\|E_\lambda x\|^2 \leq \frac{\|R_\alpha Tx - x\|^2}{a^2 \rho^2(\alpha)} \leq \frac{C^2}{a^2} \quad \text{for every } \alpha \in (0, \alpha^*).$$

Taking limit for  $\alpha \rightarrow 0^+$  we obtain that  $\int_0^{\|T\|^2} \rho^{-2}(\lambda) d\|E_\lambda x\|^2 < +\infty$ , from which it follows that  $w \doteq \int_0^{\|T\|^2} \rho^{-1}(\lambda) dE_\lambda x \in X$ . Then,

$$\rho(T^*T)w = \int_0^{\|T\|^2} \rho(\lambda)\rho^{-1}(\lambda) dE_\lambda x = x$$

and therefore  $x \in \mathcal{R}(\rho(T^*T))$ .  $\square$

LEMMA 4.11. Under the same hypotheses of Theorem 4.9, if for some  $x \in X$  we have that

$$\sup_{y^\delta \in B_\delta(Tx)} \inf_{\alpha \in (0, \alpha_0)} \|R_\alpha y^\delta - x\| = O(\rho(\Theta^{-1}(\delta))) \quad \text{when } \delta \rightarrow 0^+, \quad (4.14)$$

then  $x \in \mathcal{R}(\rho(T^*T))$ .

*Proof.* Without loss of generality we assume that  $\alpha_0 \leq \min\{\frac{\lambda_1}{c_1}, \frac{\lambda_1}{k}\}$  and that  $x \neq 0$  (if  $x = 0$  the result is trivial).

Let  $\bar{\lambda} \in \sigma_p(TT^*)$  be such that  $0 < c_1 \bar{\lambda} \leq \lambda_1$  (the compactness of  $T$  guarantees the existence of such  $\bar{\lambda}$ ), and define

$$\bar{\delta} = \bar{\delta}(\bar{\lambda}) \doteq \frac{\bar{\lambda}^{1/2}}{\gamma_2} \|R_{\bar{\lambda}} Tx - x\|.$$

Then, clearly the equation

$$\|R_\alpha Tx - x\|^2 = \frac{(\gamma_2 \bar{\delta})^2}{\alpha} \quad (4.15)$$

in the unknown  $\alpha$ , has  $\alpha = \bar{\lambda}$  as a solution. Moreover, from the hypothesis **M2 c)** and given that  $x \neq 0$ , it follows that  $\alpha = \bar{\lambda}$  is the unique solution of (4.15). Note also that  $\bar{\delta} \rightarrow 0^+$  if (and only if)  $\bar{\lambda} \rightarrow 0^+$ .

Now, for  $\delta > 0$  define

$$\bar{y}^\delta \doteq Tx - \delta G_{\bar{\lambda}} z, \quad \forall \delta > 0, \quad (4.16)$$

where  $G_{\bar{\lambda}} \doteq F_{c_1 \bar{\lambda}} - F_{\bar{\lambda}}$  and  $\{F_\lambda\}$  is the spectral family associated to  $TT^*$  and

$$z \doteq \begin{cases} \|G_{\bar{\lambda}} Tx\|^{-1} Tx, & \text{if } G_{\bar{\lambda}} Tx \neq 0, \\ \text{arbitrary with } \|G_{\bar{\lambda}} z\| = 1, & \text{in other case.} \end{cases}$$

Note that since  $\bar{\lambda} \in \sigma_p(TT^*)$  and  $c_1 > 1$  it follows that  $G_{\bar{\lambda}}$  is not the null operator and therefore the definition makes sense. Note also that  $\|\bar{y}^\delta - Tx\| = \delta$ , which implies that  $\bar{y}^\delta \in \overline{B_\delta(Tx)}$ .

Now, from (4.1), (4.16) and from the fact that  $g_\alpha(T^*T)T^* = T^*g_\alpha(TT^*)$  it follows that for every  $\alpha \in (0, \alpha_0)$  and  $\delta > 0$ ,

$$\begin{aligned}
\langle R_\alpha Tx - x, R_\alpha(\bar{y}^\delta - Tx) \rangle &= \langle g_\alpha(T^*T)T^*Tx - x, -g_\alpha(T^*T)T^* \delta G_{\bar{\lambda}}z \rangle \\
&= \delta \langle g_\alpha(T^*T)T^*Tx - x, -T^*g_\alpha(TT^*)G_{\bar{\lambda}}z \rangle \\
&= \delta \langle Tg_\alpha(T^*T)T^*Tx - Tx, -g_\alpha(TT^*)G_{\bar{\lambda}}z \rangle \\
&= \delta \langle (TT^*g_\alpha(TT^*) - I)Tx, -g_\alpha(TT^*)G_{\bar{\lambda}}z \rangle \\
&= \delta \langle -r_\alpha(TT^*)Tx, -g_\alpha(TT^*)G_{\bar{\lambda}}z \rangle \\
&= \delta \int_0^{\|T\|^2+} r_\alpha(\lambda)g_\alpha(\lambda) d \langle F_\lambda Tx, G_{\bar{\lambda}}z \rangle. \quad (4.17)
\end{aligned}$$

Now since  $c_1\bar{\lambda} \leq \lambda_1$ , it follows from hypothesis **M2 a)** that both  $g_\alpha(\lambda)$  and  $r_\alpha(\lambda)$  are nonnegative for all  $\lambda \in [0, c_1\bar{\lambda}]$ . On the other hand, from the definitions of  $G_{\bar{\lambda}}$  and  $z$  it follows immediately that the function  $h(\lambda) \doteq \langle F_\lambda Tx, G_{\bar{\lambda}}z \rangle$  for  $\lambda \in [0, c_1\bar{\lambda}]$  is real and non-decreasing and therefore

$$\int_0^{c_1\bar{\lambda}+} r_\alpha(\lambda)g_\alpha(\lambda) d \langle F_\lambda Tx, G_{\bar{\lambda}}z \rangle \geq 0. \quad (4.18)$$

On the other hand, since  $h(\lambda) = \langle Tx, F_\lambda G_{\bar{\lambda}}z \rangle$  and  $F_\lambda G_{\bar{\lambda}} = G_{\bar{\lambda}}$  for every  $\lambda \geq c_1\bar{\lambda}$ , it follows that  $h(\lambda)$  is constant for every  $\lambda \geq c_1\bar{\lambda}$  and therefore

$$\int_{c_1\bar{\lambda}+}^{\|T\|^2+} r_\alpha(\lambda)g_\alpha(\lambda) d \langle F_\lambda Tx, G_{\bar{\lambda}}z \rangle = 0. \quad (4.19)$$

From (4.18) and (4.19) we have that

$$\int_0^{\|T\|^2+} r_\alpha(\lambda)g_\alpha(\lambda) d \langle F_\lambda Tx, G_{\bar{\lambda}}z \rangle \geq 0,$$

which, by virtue of (4.17), implies that

$$\langle R_\alpha Tx - x, R_\alpha(\bar{y}^\delta - Tx) \rangle \geq 0. \quad (4.20)$$

By using once again (4.1) and (4.16) together with (4.20) it then follows that for every  $\alpha \in (0, \alpha_0)$ , for every  $\bar{\lambda} \in \sigma_p(TT^*)$  such that  $c_1\bar{\lambda} \leq \lambda_1$  and for every  $\delta > 0$ ,

$$\begin{aligned}
\|R_\alpha \bar{y}^\delta - x\|^2 &= \|R_\alpha Tx - x\|^2 + \|R_\alpha(\bar{y}^\delta - Tx)\|^2 + 2 \langle R_\alpha Tx - x, R_\alpha(\bar{y}^\delta - Tx) \rangle \\
&= \|R_\alpha Tx - x\|^2 + \delta^2 \|g_\alpha(T^*T)T^*G_{\bar{\lambda}}z\|^2 + 2 \langle R_\alpha Tx - x, R_\alpha(\bar{y}^\delta - Tx) \rangle \\
&\geq \|R_\alpha Tx - x\|^2 + \delta^2 \|g_\alpha(T^*T)T^*G_{\bar{\lambda}}z\|^2 \\
&= \|R_\alpha Tx - x\|^2 + \delta^2 \int_0^{\|T\|^2+} \lambda g_\alpha^2(\lambda) d \|F_\lambda G_{\bar{\lambda}}z\|^2 \\
&\geq \|R_\alpha Tx - x\|^2 + \delta^2 \int_{\bar{\lambda}}^{c_1\bar{\lambda}} \lambda g_\alpha^2(\lambda) d \|F_\lambda G_{\bar{\lambda}}z\|^2. \quad (4.21)
\end{aligned}$$

We now consider two different possible cases.

Case I:  $\alpha \leq \bar{\lambda}$ . Since  $c_1 \bar{\lambda} \leq \lambda_1$  and  $c_1 > 1$ , it follows from hypothesis **M2 e)** that

$$g_\alpha(\lambda) \geq g_\alpha(c_1 \bar{\lambda}) \geq g_\alpha(\lambda_1) \quad \text{for every } \lambda \in [\bar{\lambda}, c_1 \bar{\lambda}]. \quad (4.22)$$

On the other hand, from hypothesis **M2 a)** it follows that  $r_\alpha(\lambda_1) \leq 1$ , which implies that  $\lambda_1 g_\alpha(\lambda_1) \geq 0$  and therefore,  $g_\alpha(\lambda_1) \geq 0$ . It then follows from (4.22) that  $g_\alpha^2(\lambda) \geq g_\alpha^2(c_1 \bar{\lambda})$  for every  $\lambda \in [\bar{\lambda}, c_1 \bar{\lambda}]$ . Then,

$$\begin{aligned} \int_{\bar{\lambda}}^{c_1 \bar{\lambda}} \lambda g_\alpha^2(\lambda) d\|F_\lambda G_{\bar{\lambda}} z\|^2 &\geq \bar{\lambda} g_\alpha^2(c_1 \bar{\lambda}) \int_{\bar{\lambda}}^{c_1 \bar{\lambda}} d\|F_\lambda G_{\bar{\lambda}} z\|^2 \\ &= \bar{\lambda} g_\alpha^2(c_1 \bar{\lambda}), \end{aligned} \quad (4.23)$$

where the last equality follows from the fact that  $\int_{\bar{\lambda}}^{c_1 \bar{\lambda}} d\|F_\lambda G_{\bar{\lambda}} z\|^2 = 1$ , which is a consequence of the fact that  $\int_{\bar{\lambda}}^{c_1 \bar{\lambda}} d\|F_\lambda G_{\bar{\lambda}} z\|^2 = \|F_{c_1 \bar{\lambda}} G_{\bar{\lambda}} z\|^2 - \|F_{\bar{\lambda}} G_{\bar{\lambda}} z\|^2$ , from the definition of  $G_{\bar{\lambda}}$ , from the fact that  $F_\lambda F_\mu = F_{\min\{\lambda, \mu\}}$  for every  $\lambda, \mu \in \mathbb{R}$  and the fact that  $\|G_{\bar{\lambda}} z\| = 1$ .

At the same time, the hypotheses **M2 a)** and **M2 c)** imply that  $g_\alpha(\lambda)$  is monotone decreasing as a function of  $\alpha$  for each  $\lambda \in [0, \lambda_1]$ . Since  $\alpha \leq \bar{\lambda}$  and  $c_1 \bar{\lambda} \leq \lambda_1$ , we then have that

$$g_\alpha(c_1 \bar{\lambda}) \geq g_{\bar{\lambda}}(c_1 \bar{\lambda}), \quad (4.24)$$

and from hypothesis **M2 d)** we also have that

$$g_{\bar{\lambda}}(c_1 \bar{\lambda}) \geq \gamma_2 / \bar{\lambda} > 0. \quad (4.25)$$

From (4.24) and (4.25) we conclude that

$$g_\alpha^2(c_1 \bar{\lambda}) \geq (\gamma_2 / \bar{\lambda})^2. \quad (4.26)$$

Substituting (4.26) into (4.23) we obtain

$$\int_{\bar{\lambda}}^{c_1 \bar{\lambda}} \lambda g_\alpha^2(\lambda) d\|F_\lambda G_{\bar{\lambda}} z\|^2 \geq \gamma_2^2 / \bar{\lambda},$$

which, together with (4.21) imply that if  $\alpha \leq \bar{\lambda}$ , then  $\|R_\alpha \bar{y}^\delta - x\|^2 \geq (\gamma_2 \delta)^2 / \bar{\lambda}$ .

Case II:  $\alpha > \bar{\lambda}$ . In this case, it follows from hypothesis **M2 c)** that  $r_\alpha^2(\lambda) \geq r_{\bar{\lambda}}^2(\lambda)$  for every  $\lambda \in (0, \|T\|^2]$ . Then,

$$\|R_\alpha T x - x\|^2 = \int_0^{\|T\|^2} r_\alpha^2(\lambda) d\|E_\lambda x\|^2 \geq \int_0^{\|T\|^2} r_{\bar{\lambda}}^2(\lambda) d\|E_\lambda x\|^2 = \|R_{\bar{\lambda}} T x - x\|^2,$$

which, together with (4.21) imply that  $\|R_\alpha \bar{y}^\delta - x\|^2 \geq \|R_{\bar{\lambda}} T x - x\|^2$ .

Summarizing the results obtained in cases I and II, we can write:

$$\begin{aligned} \|R_\alpha \bar{y}^\delta - x\|^2 &\geq \begin{cases} \|R_{\bar{\lambda}} T x - x\|^2, & \text{if } \alpha > \bar{\lambda} \\ (\gamma_2 \delta)^2 / \bar{\lambda}, & \text{if } \alpha \leq \bar{\lambda}. \end{cases} \\ &\geq \min\{\|R_{\bar{\lambda}} T x - x\|^2, (\gamma_2 \delta)^2 / \bar{\lambda}\}, \end{aligned} \quad (4.27)$$

which is valid for every  $\alpha \in (0, \alpha_0)$ ,  $\bar{\lambda} \in \sigma_p(TT^*)$  such that  $c_1 \bar{\lambda} \leq \lambda_1$  and for every  $\delta > 0$ . Then

$$\begin{aligned} \min \left\{ \|R_{\bar{\lambda}}Tx - x\|, \gamma_2 \delta / \sqrt{\bar{\lambda}} \right\} &= \left( \min \{ \|R_{\bar{\lambda}}Tx - x\|^2, (\gamma_2 \delta)^2 / \bar{\lambda} \} \right)^{1/2} \\ &\leq \inf_{\alpha \in (0, \alpha_0)} \|R_{\alpha} \bar{y}^\delta - x\| \quad (\text{by (4.27)}) \\ &\leq \sup_{y^\delta \in \overline{B_\delta(Tx)}} \inf_{\alpha \in (0, \alpha_0)} \|R_{\alpha} y^\delta - x\| \quad (\text{since } \bar{y}^\delta \in \overline{B_\delta(Tx)}) \\ &= O(\rho(\Theta^{-1}(\delta))) \quad \text{for } \delta \rightarrow 0^+ \quad (\text{by hypothesis}). \end{aligned}$$

Now, given that  $\bar{\lambda} = \alpha(\bar{\delta})$  solves equation (4.15), from the previous inequality we have that

$$\|R_{\alpha(\bar{\delta})}Tx - x\| = \gamma_2 \bar{\delta} / \sqrt{\bar{\lambda}} = O(\rho(\Theta^{-1}(\bar{\delta}))) \quad \text{for } \bar{\delta} \rightarrow 0^+, \quad (4.28)$$

which implies that

$$\frac{\bar{\delta}}{\rho(\Theta^{-1}(\bar{\delta}))} = O\left(\sqrt{\alpha(\bar{\delta})}\right) \quad \text{for } \bar{\delta} \rightarrow 0^+. \quad (4.29)$$

Since  $\delta = \Theta(\Theta^{-1}(\delta))$  it follows from the definition of  $\Theta$  that  $\delta = \sqrt{\Theta^{-1}(\delta)} \rho(\Theta^{-1}(\delta))$ . Then, it follows from (4.29) that  $\sqrt{\Theta^{-1}(\bar{\delta})} = O(\sqrt{\alpha(\bar{\delta})})$  for  $\bar{\delta} \rightarrow 0^+$ . From this and (4.28) we then deduce that:

$$\|R_{\alpha(\bar{\delta})}Tx - x\| = O(\rho(\alpha(\bar{\delta}))) \quad \text{for } \bar{\delta} \rightarrow 0^+ \quad \forall \alpha(\bar{\delta}) \in \sigma_p(TT^*) : c_1 \alpha(\bar{\delta}) \leq \lambda_1. \quad (4.30)$$

Now, let  $\alpha \in \mathbb{R}^+$  such that  $\alpha \leq \max_{j \in \mathbb{N}} \{\tilde{\lambda}_j : \tilde{\lambda}_j \leq \frac{\lambda_1}{c_1}\}$ . Then, there exist  $n = n(\alpha) \in \mathbb{N}$  such that  $\tilde{\lambda}_{n+1} < \alpha \leq \tilde{\lambda}_n \leq \frac{\lambda_1}{c_1}$ . Note here that  $n \rightarrow \infty$  if (and only if)  $\alpha \rightarrow 0^+$ .

From hypothesis **M2 c)** and the fact that  $\tilde{\lambda}_n \in \sigma_p(TT^*)$  and  $\tilde{\lambda}_n \leq \frac{\lambda_1}{c_1}$  it follows that

$$\begin{aligned} \|R_{\alpha}Tx - x\|^2 &= \int_0^{\|T\|^2} r_{\alpha}^2(\lambda) d\|E_{\lambda}x\|^2 \\ &\leq \int_0^{\|T\|^2} r_{\tilde{\lambda}_n}^2(\lambda) d\|E_{\lambda}x\|^2 \\ &= \|R_{\tilde{\lambda}_n}Tx - x\|^2 \\ &= O(\rho^2(\tilde{\lambda}_n)), \quad (\text{by virtue of (4.30)}). \end{aligned} \quad (4.31)$$

From hypothesis **M1** we have that  $\tilde{\lambda}_n \leq c \tilde{\lambda}_{n+1}$  and since  $\rho$  is strictly increasing and positive (by hypothesis **M3**) it follows that for all  $n$  big enough, more precisely for all  $n$  such that  $c \tilde{\lambda}_{n+1} \leq \|T\|^2$ ,

$$\rho^2(\tilde{\lambda}_n) \leq \rho^2(c \tilde{\lambda}_{n+1}). \quad (4.32)$$

Now since  $c \geq 1$  and  $\rho$  is of local upper type  $\beta$  for some  $\beta \geq 0$  (hypothesis **M3**), there exists a positive constant  $d$  such that

$$\rho(c \tilde{\lambda}_{n+1}) \leq d c^\beta \rho\left(\frac{1}{c} c \tilde{\lambda}_{n+1}\right) = d c^\beta \rho(\tilde{\lambda}_{n+1}). \quad (4.33)$$

From (4.31), (4.32), (4.33) and from the fact that  $\rho(\tilde{\lambda}_{n+1}) < \rho(\alpha)$  it follows that  $\|R_\alpha T x - x\| = O(\rho(\alpha))$  for  $\alpha \rightarrow 0^+$ . Therefore, Lemma 4.10 now implies that  $x \in \mathcal{R}(\rho(T^*T))$ . This concludes the proof of the Lemma.  $\square$

REMARK 4.12. *From the definition of qualification (Definition 4.7) it follows that*

$$\|R_\alpha T x - x\|^2 \leq \gamma^2 \rho^2(\alpha) \int_0^{+\infty} \rho^{-2}(\lambda) d\|E_\lambda x\|^2.$$

Therefore, in Lemma 4.11, the hypothesis **M1** and the assumption that  $\rho$  be of local upper type  $\beta$  for some  $\beta \geq 0$  can be substituted by the requirement that  $\rho(T^*T)$  be invertible, or equivalently, that  $\rho^{-2}(\lambda)$  be integrable with respect to the measure  $d\|E_\lambda x\|^2$  for every  $x \in X$ .

*Proof of Theorem 4.9.* First we note that from hypotheses **M2 d)** and **M2 e)** it follows easily that

$$\mathbf{M5} : \quad \sup_{\lambda \in (0, \|T\|^2]} \sqrt{\lambda} |g_\alpha(\lambda)| \geq \frac{b}{\sqrt{\alpha}} \quad \text{for every } \alpha \in (0, \alpha_0),$$

where  $b = \gamma_2 \sqrt{c_1}$ . As in Lemma 4.11, without loss of generality we assume that  $\alpha_0 \leq \min\{\frac{\lambda_1}{c_1}, \frac{\lambda_1}{k}\}$ . First we will prove that  $\psi(x, \delta) \doteq (\rho \circ \Theta^{-1})(\delta)$  for  $x \in X^\rho$  and  $\delta \in (0, \Theta(\alpha_0))$ , is an upper bound of convergence for the total error of  $\{R_\alpha\}_{\alpha \in (0, \alpha_0)}$  in  $X^\rho$ , that is, we will show that  $\psi \in \mathcal{U}_{X^\rho}(\mathcal{E}_{\{R_\alpha\}}^{\text{tot}})$ . For every  $r \geq 0$  we define the source sets  $X^{\rho, r} \doteq \{x \in X : x = \rho(T^*T)\xi, \|\xi\| \leq r\}$ . Let  $x \in X^\rho$ , then there exists  $r \geq 1$  such that  $x \in X^{\rho, r}$ . Since  $\Theta$  is continuous and strictly increasing in  $(0, \alpha_0)$ , there exists a unique  $\tilde{\alpha} \in (0, \alpha_0)$  such that  $x \in X^{\rho, r}$  and  $\Theta(\tilde{\alpha}) = \frac{\delta}{r}$ . Therefore,

$$\begin{aligned} \mathcal{E}_{\{R_\alpha\}}^{\text{tot}}(x, \delta) &= \inf_{\alpha \in (0, \alpha_0)} \sup_{y^\delta \in B_\delta(Tx)} \|R_\alpha y^\delta - x\| \\ &\leq \sup_{y^\delta \in B_\delta(Tx)} \|R_{\tilde{\alpha}} y^\delta - x\| \\ &\leq \sup_{x \in X^{\rho, r}} \sup_{y^\delta \in B_\delta(Tx)} \|R_{\tilde{\alpha}} y^\delta - x\|. \end{aligned} \quad (4.34)$$

On the other hand, from hypotheses **H1-H4**, the fact that the function  $\rho$  is qualification of  $\{R_\alpha\}$ , the fact that  $\rho$  trivially covers  $\rho$  with constant equals to unity (see [5], Definition 2) and given that  $\Theta(\tilde{\alpha}) = \frac{\delta}{r}$ , it follows by virtue of Theorem 2 in [5], that there exists a positive constant  $K$ , independent of  $\delta$  such that

$$\sup_{x \in X^{\rho, r}} \sup_{y^\delta \in B_\delta(Tx)} \|R_{\tilde{\alpha}} y^\delta - x\| \leq K \rho \left( \Theta^{-1} \left( \frac{\delta}{r} \right) \right), \quad \text{for } 0 < \delta \leq r \Theta(\|T\|^2). \quad (4.35)$$

From (4.34) and (4.35) it follows that for every  $\delta \in (0, \Theta(\alpha_0))$ ,

$$\mathcal{E}_{\{R_\alpha\}}^{\text{tot}}(x, \delta) \leq K \rho \left( \Theta^{-1} \left( \frac{\delta}{r} \right) \right) \leq K \rho(\Theta^{-1}(\delta)) = K \psi(x, \delta),$$

where the last inequality follows from the fact that  $r \geq 1$  and both  $\rho$  and  $\Theta^{-1}$  are increasing functions. This proves that  $\psi \in \mathcal{U}_{X^\rho}(\mathcal{E}_{\{R_\alpha\}}^{\text{tot}})$ .

Next we will see that  $\psi$  satisfies condition **S1** of saturation on  $X^\rho$ . From hypotheses **H1-H4**, **M4** and **M5** and the fact that  $\rho$  is maximal qualification of  $\{R_\alpha\}$ ,

by virtue of Theorem 2.3 and Definition 2.2 in [3], it follows that for every  $x^* \in X$ ,  $x^* \neq 0$  and  $x \in X^\rho$  there exist positive constants  $a \doteq a(x, x^*)$  and  $d = d(x, x^*)$  such that

$$\frac{\mathcal{E}_{\{R_\alpha\}}^{\text{tot}}(x^*, \delta)}{\psi(x, \delta)} \geq a \quad \forall \delta \in (0, d).$$

Then,  $\limsup_{\delta \rightarrow 0^+} \frac{\mathcal{E}_{\{R_\alpha\}}^{\text{tot}}(x^*, \delta)}{\psi(x, \delta)} > 0$  for every  $x^* \in X$ ,  $x^* \neq 0$  and  $x \in X^\rho$ , that is,  $\psi$  satisfies condition *S1* on  $X^\rho$ .

Also, since  $\psi$  does not depend on  $x$ , it is invariant over  $X^\rho$ , i.e.,  $\psi$  satisfies condition *S2* of saturation.

It remains to prove that  $\psi$  satisfies condition *S3*. Suppose not. Then, there exist  $M \supsetneq X^\rho$  and  $\tilde{\psi} \in \mathcal{U}_M(\mathcal{E}_{\{R_\alpha\}}^{\text{tot}})$  such that  $\tilde{\psi}|_{X^\rho} = \psi$  and  $\tilde{\psi}$  satisfies *S1* and *S2* over  $M$ . Let  $x^* \in M \setminus X^\rho$ ,  $x^* \neq 0$ . Since  $\tilde{\psi} \in \mathcal{U}_M(\mathcal{E}_{\{R_\alpha\}}^{\text{tot}})$  we have that

$$\mathcal{E}_{\{R_\alpha\}}^{\text{tot}} \stackrel{\{x^*\}}{\preceq} \tilde{\psi}. \quad (4.36)$$

Since  $\tilde{\psi}$  is invariant over  $M$  and  $X^\rho \subset M$ , it follows that  $\tilde{\psi} \stackrel{\{x^*\}, X^\rho}{\preceq} \tilde{\psi}$  and since  $\tilde{\psi}$  coincides with  $\psi$  on  $X^\rho$ , it follows that  $\tilde{\psi} \stackrel{\{x^*\}, X^\rho}{\preceq} \psi$ . This together with (4.36) imply that  $\mathcal{E}_{\{R_\alpha\}}^{\text{tot}} \stackrel{\{x^*\}, X^\rho}{\preceq} \psi$  and therefore  $\mathcal{E}_{\{R_\alpha\}}^{\text{tot}}(x^*, \delta) = O(\rho(\Theta^{-1}(\delta)))$  for  $\delta \rightarrow 0^+$ . Lemma 4.6 then implies that there exists an *a-priori* admissible parameter choice rule  $\tilde{\alpha} : \mathbb{R}^+ \rightarrow (0, \alpha_0)$  such that

$$\sup_{y^\delta \in B_\delta(Tx^*)} \|R_{\tilde{\alpha}(\delta)} y^\delta - x^*\| = O(\rho(\Theta^{-1}(\delta))) \quad \text{for } \delta \rightarrow 0^+.$$

Then,

$$\sup_{y^\delta \in B_\delta(Tx^*)} \inf_{\alpha \in (0, \alpha_0)} \|R_\alpha y^\delta - x^*\| = O(\rho(\Theta^{-1}(\delta))) \quad \text{for } \delta \rightarrow 0^+.$$

Finally, Lemma 4.11 implies that  $x^* \in \mathcal{R}(\rho(T^*T))$  and since  $x^* \neq 0$ , we have that  $x^* \in X^\rho$ , which contradicts the fact that  $x^* \in M \setminus X^\rho$ . Hence,  $\psi$  satisfies condition *S3* and therefore,  $\psi$  is saturation function of  $\{R_\alpha\}$  on  $X^\rho$ .  $\square$

Note that both Lemma 4.5 and Lemma 4.6 remain true if hypotheses *iii)* and *iv)* of Theorem 4.4 are replaced by the requirement that there exists  $\rho : (0, \|T\|^2] \rightarrow (0, \infty)$  that is qualification of  $\{R_\alpha\}_{\alpha \in (0, \alpha_0)}$  and satisfies the inequality in the hypothesis *M3* of Theorem 4.9.

**5. Examples.** We close our investigation presenting a few examples of regularization methods possessing global saturation. For the sake of brevity we shall not provide much details here.

**Example 1:** The family of Tikhonov-Phillips regularization operators  $\{R_\alpha\}_{\alpha \in (0, \alpha_0)}$  is defined by (4.1) with  $g_\alpha(\lambda) = \frac{1}{\lambda + \alpha}$ . It is well known that this family of regularization operators possesses classical qualification of order  $\mu_0 = 1$ . It can be easily checked that the family  $\{g_\alpha\}_{\alpha \in (0, \alpha_0)}$  satisfies all hypotheses of the Theorem 4.4 with constants  $C \doteq 1$ ,  $\lambda_1 \doteq \|T\|^2$ ,  $\gamma_1 \doteq \frac{1}{2}$ ,  $c_1 \doteq \frac{3}{2}$ ,  $\gamma_2 \doteq \frac{2}{5}$ ,  $\gamma \doteq \frac{1}{2}$  and  $c \doteq 1$ . Therefore,

the function  $\psi(x, \delta) = \delta^{\frac{2}{3}}$  defined for  $x \in X_1 \doteq \mathcal{R}(T^*T) \setminus \{0\}$  and  $\delta > 0$  is global saturation of  $\{R_\alpha\}_{\alpha \in (0, \alpha_0)}$  on  $X_1$ .

**Example 2:** Given  $k \in \mathbb{R}^+$ , for  $\alpha, \lambda > 0$  let

$$h_\alpha^k(\lambda) \doteq \begin{cases} \frac{e^{-\frac{\lambda}{\sqrt{\alpha}}}}{\lambda}, & \text{for } 0 < \lambda < \alpha, \\ \frac{e^{-\sqrt{\frac{\lambda}{\alpha}}}}{\lambda}, & \text{for } \alpha \leq \lambda < 3\alpha, \\ \frac{e^{-\sqrt{\frac{\lambda}{\alpha}}}}{\lambda} + \frac{\alpha^k}{\lambda^{k+1}}, & \text{for } \lambda \geq 3\alpha, \end{cases}$$

and define  $g_\alpha^k(\lambda) \doteq \frac{1}{\lambda} - \alpha^k \sqrt{\lambda} - h_\alpha^k(\lambda)$  for  $\lambda > 0$ , and for  $\lambda = 0$  define  $g_\alpha^k(0) \doteq \lim_{\lambda \rightarrow 0^+} g_\alpha^k(\lambda) = \frac{1}{\sqrt{\alpha}}$ . It can be easily verified that for any  $\alpha_0 > 0$ ,  $\{g_\alpha\}_{\alpha \in (0, \alpha_0)}$  satisfies the hypotheses *H1-H3* and therefore the corresponding collection of operators  $\{R_\alpha\}_{\alpha \in (0, \alpha_0)}$  defined by (4.1) is a family of spectral regularization operators for  $Tx = y$ . Hypothesis *H2* is satisfied with  $C \doteq 1 + \|T\|^3 \alpha_0^k$ . Also, it can easily be proved that for any  $\lambda > 0$ ,  $\frac{\lambda^k |1 - \lambda g_\alpha^k(\lambda)|}{\alpha^k} = O(1)$  for  $\alpha \rightarrow 0^+$  and therefore  $\{R_\alpha\}_{\alpha \in (0, \alpha_0)}$  has classical qualification of order  $k$ .

Now, for  $k \geq 1$  and  $\alpha > 0$ , the function  $g_\alpha^k(\lambda)$  is non-increasing. Thus, hypothesis **ii.e)** of Theorem 4.4 holds and  $G_\alpha^k \doteq \|g_\alpha^k(\cdot)\|_\infty = g_\alpha^k(0) = \frac{1}{\sqrt{\alpha}}$ , which implies immediately that also hypothesis *H4* is verified. From now on we shall assume  $k \geq 1$ .

Defining

$$s_\alpha^k(\lambda) \doteq \begin{cases} e^{-\frac{\lambda}{\sqrt{\alpha}}}, & \text{for } 0 \leq \lambda < \alpha, \\ e^{-\sqrt{\frac{\lambda}{\alpha}}}, & \text{for } \alpha \leq \lambda < 3\alpha, \\ e^{-\sqrt{\frac{\lambda}{\alpha}}} + \left(\frac{\alpha}{\lambda}\right)^k, & \text{for } \lambda \geq 3\alpha, \end{cases}$$

it follows that  $r_\alpha^k(\lambda) = 1 - \lambda g_\alpha^k(\lambda) = \alpha^k \lambda^{\frac{3}{2}} + s_\alpha^k(\lambda)$ . Clearly,  $r_\alpha^k(\lambda) > 0$  for all  $\lambda \geq 0$ . Now let  $\alpha_0 \doteq \min\{\frac{1}{3}, \frac{\|T\|^2}{3}\}$  and  $\lambda_1 \doteq \min\{1, \|T\|^2\}$ . It can be shown that  $r_\alpha^k(\lambda) \leq 1$  for all  $\lambda \in [0, \lambda_1]$  and for all  $\alpha \in (0, \alpha_0)$ , i.e., hypothesis **ii.a)** of Theorem 4.4 is satisfied.

Also, for  $0 \leq \lambda < \alpha \leq \lambda_1$  we have that

$$r_\alpha^k(\lambda) = \alpha^k \lambda^{\frac{3}{2}} + e^{-\frac{\lambda}{\sqrt{\alpha}}} > e^{-1},$$

since  $\frac{\lambda}{\sqrt{\alpha}} < 1$ . Thus, hypothesis **ii.b)** of Theorem 4.4 is verified with  $\gamma_1 \doteq e^{-1}$ . Since  $|r_\alpha^k(\lambda)| = r_\alpha^k(\lambda)$  is monotone increasing with respect to  $\alpha$  for all  $\lambda \geq 0$ , hypothesis **ii.c)** of Theorem 4.4 is also satisfied.

On the other hand we have that

$$\alpha g_\alpha^k(2\alpha) = \frac{1 - e^{-\sqrt{2}}}{2} - \sqrt{2} \alpha^{\frac{3}{2}+k} \geq \frac{1 - e^{-\sqrt{2}}}{2} - \sqrt{2} 3^{-\frac{3}{2}-k},$$

since  $\alpha \leq \frac{1}{3}$ . Hence hypothesis **ii.d)** of Theorem 4.4 holds as well with constants  $c_1 \doteq 2$  and  $\gamma_2 \doteq \frac{1 - e^{-\sqrt{2}}}{2} - \sqrt{2} 3^{-\frac{3}{2}-k} > 0$  for all  $k \geq 1$ .

Finally, for  $\lambda \geq 3\alpha$ ,

$$\left(\frac{\lambda}{\alpha}\right)^k |r_\alpha^k(\lambda)| = \left(\frac{\lambda}{\alpha}\right)^k \left( e^{-\sqrt{\frac{\lambda}{\alpha}}} + \alpha^k \lambda^{\frac{3}{2}} + \left(\frac{\alpha}{\lambda}\right)^k \right) \geq 1,$$

from which it follows that hypothesis **iv**) of Theorem 4.4 is satisfied with constants  $c \doteq 3$  and  $\gamma \doteq 1$ . Hence, Theorem 4.4 allows us to conclude that the function  $\psi_k(x, \delta) = \delta^{\frac{2k}{2k+1}}$  for  $x \in X_k \doteq \mathcal{R}((T^*T)^k) \setminus \{0\}$  and  $\delta > 0$  is global saturation of  $\{R_\alpha\}_{\alpha \in (0, \alpha_0)}$  on  $X_k$ .

**Example 3:** Given  $\varepsilon > 0$ , for  $\lambda > 0$  and  $\alpha \in (0, \alpha_0)$  with  $\alpha_0 < e^{-1}$ , let

$$h^\varepsilon(\lambda) \doteq \begin{cases} \alpha, & \text{for } 0 \leq \lambda < \alpha, \\ \alpha^{1+\varepsilon}, & \text{for } \lambda \geq \alpha, \end{cases}$$

and define

$$g_\alpha^\varepsilon(\lambda) \doteq \frac{1 + \ln \alpha}{\lambda \ln \alpha - \lambda^{-\varepsilon} h^\varepsilon(\lambda)}.$$

It can be easily checked that  $\{g_\alpha^\varepsilon\}_{\alpha \in (0, \alpha_0)}$  satisfies the hypotheses  $H1-H4$ . In particular, hypothesis  $H2$  is satisfied with  $C \doteq 1$ . Therefore  $\{R_\alpha\}_{\alpha \in (0, \alpha_0)}$  with  $R_\alpha$  as in (4.1) is a family of regularization operators for  $Tx = y$ . Also it can be shown that for every  $\mu > 0$ ,

$$\frac{\lambda^\mu |1 - \lambda g_\alpha^\varepsilon(\lambda)|}{\alpha^\mu} \rightarrow +\infty \quad \text{for } \alpha \rightarrow 0^+ \quad \text{for every } \lambda > 0,$$

which implies that  $\{R_\alpha\}_{\alpha \in (0, \alpha_0)}$  has classical qualification of order  $\mu_0 = 0$ . Now, the function  $\rho(\alpha) \doteq -(\ln \alpha)^{-1}$  is strictly increasing and of local upper type  $\beta$  for  $\beta \doteq 1$  (moreover the constant  $d$  in Definition 4.8 can be taken to be  $d \doteq 1$ ) and it can also be proved that  $\rho$  is maximal qualification of  $\{R_\alpha\}_{\alpha \in (0, \alpha_0)}$  and satisfies the inequality in the hypothesis **M3** of Theorem 4.9 with constants  $a \doteq 1$  and  $k \doteq 1$ .

In this case we have that

$$r_\alpha^\varepsilon(\lambda) = \frac{h^\varepsilon(\lambda) + \lambda^{1+\varepsilon}}{h^\varepsilon(\lambda) - \lambda^{1+\varepsilon} \ln \alpha}.$$

Clearly,  $r_\alpha^\varepsilon(\lambda) > 0$  for all  $\lambda \geq 0$ . Also, it can be shown that  $r_\alpha^\varepsilon(\lambda) \leq 1$  for all  $\lambda \in [0, \lambda_1]$  and for all  $\alpha \in (0, \alpha_0)$ , where  $\lambda_1 \doteq \min\{0.6, \|T\|^2\}$ . Thus, hypothesis **M2 a)** of Theorem 4.9 is satisfied.

Now, for  $0 \leq \lambda < \alpha \leq \lambda_1$ , we have that

$$r_\alpha^\varepsilon(\lambda) = \frac{\alpha + \lambda^{1+\varepsilon}}{\alpha - \lambda^{1+\varepsilon} \ln \alpha} \geq \frac{1}{1 - \frac{\lambda^{1+\varepsilon}}{\alpha} \ln \alpha} > \frac{1}{1 - \alpha^\varepsilon \ln \alpha}, \quad (5.1)$$

since  $\frac{\lambda^{1+\varepsilon}}{\alpha} < \frac{\alpha^{1+\varepsilon}}{\alpha} = \alpha^\varepsilon$ . Since one can easily prove that

$$-\alpha^\varepsilon \ln \alpha \leq (3e)^{-1} \quad \text{for all } \alpha > 0, \quad (5.2)$$

it follows from (5.1) and (5.2) that  $r_\alpha^\varepsilon(\lambda) > (1 + \frac{1}{3e})^{-1}$  for  $0 \leq \lambda < \alpha \leq \lambda_1$ , which implies that hypothesis **M2 b)** of Theorem 4.9 holds with  $\gamma_1 \doteq (1 + \frac{1}{3e})^{-1}$ . Since  $|r_\alpha^\varepsilon(\lambda)| = r_\alpha^\varepsilon(\lambda)$  is monotone increasing with respect to  $\alpha$  for all  $\lambda \geq 0$ , hypothesis **M2 c)** of Theorem 4.9 is also satisfied.

On the other hand, for  $\varepsilon \in (0, 1)$  and  $\alpha \in (0, \alpha_0)$ , the function  $g_\alpha^\varepsilon(\lambda)$  is non-increasing for  $\lambda \in [\alpha, \lambda_1]$ , which implies that hypothesis **M2 e)** of Theorem 4.9 is also satisfied.

Assuming  $\varepsilon \in (0, 1)$ , since  $s^\varepsilon(\alpha) \doteq \frac{1+\ln \alpha}{2\ln \alpha - 2^{-\varepsilon}}$  is a non-increasing function for  $\alpha \in (0, \alpha_0)$  and  $2\alpha \leq \lambda_1 \leq 0.6$ , we have that

$$\alpha g_\alpha^\varepsilon(2\alpha) = \frac{1 + \ln \alpha}{2 \ln \alpha - 2^{-\varepsilon}} \geq \frac{1 + \ln 0.3}{2 \ln 0.3 - 2^{-\varepsilon}}.$$

Hence hypothesis **M2 d)** of Theorem 4.9 is satisfied with constants  $c_1 \doteq 2$  and  $\gamma_2 \doteq s^\varepsilon(0.3)$ .

Finally, for every  $\alpha \in (0, \alpha_0)$  the mapping  $\lambda \rightarrow |r_\alpha(\lambda)|^2$ ,  $\lambda \in (0, \|T\|^2]$  is convex and therefore hypothesis **M4** of Theorem 4.9 also holds. Hence, letting  $\Theta(t) \doteq \sqrt{t}\rho(t) = -\frac{\sqrt{t}}{\ln t}$  for  $t \in (0, \|T\|^2]$ , by Theorem 4.9 we conclude that  $\psi(x, \delta) \doteq (\rho \circ \Theta^{-1})(\delta)$  for  $x \in X^\rho \doteq \mathcal{R}(\rho(T^*T)) \setminus \{0\} = \mathcal{R}(-(\ln(T^*T))^{-1}) \setminus \{0\}$  and  $\delta \in (0, \Theta(\alpha_0)) = \left(0, -\frac{\sqrt{\alpha_0}}{\ln \alpha_0}\right)$ , is global saturation function of  $\{R_\alpha\}_{\alpha \in (0, \alpha_0)}$  on  $X^\rho$ .

**Example 4:** For  $\lambda > 0$  and  $\alpha \in (0, \alpha_0)$ , let  $g_\alpha(\lambda)$  be defined as:

$$g_\alpha(\lambda) \doteq \begin{cases} 0, & \text{for } 0 \leq \lambda < \alpha, \\ e^{\frac{\lambda}{\ln \alpha}}, & \text{for } \lambda \geq \alpha. \end{cases}$$

Thus

$$r_\alpha(\lambda) = \begin{cases} 1, & \text{for } 0 \leq \lambda < \alpha, \\ 1 - e^{-\frac{\lambda}{\ln \alpha}}, & \text{for } \lambda \geq \alpha. \end{cases}$$

It can be immediately shown that  $\{g_\alpha\}_{\alpha \in (0, \alpha_0)}$  satisfies the hypotheses **H1-H4** and therefore  $\{R_\alpha\}_{\alpha \in (0, \alpha_0)}$  with  $R_\alpha$  as in (4.1) is a family of regularization operators for  $Tx = y$ . Also it can be easily checked that  $\{R_\alpha\}_{\alpha \in (0, \alpha_0)}$  has classical qualification of order  $\mu_0 = 0$ . Furthermore, it can be proved that the function  $\rho(\alpha)$  defined by

$$\rho(\alpha) \doteq \alpha e^{\frac{\alpha}{\ln \alpha}}$$

is maximal qualification of  $\{R_\alpha\}_{\alpha \in (0, \alpha_0)}$  and all hypotheses of Theorem 4.9 are satisfied. Hence, letting  $\Theta(t) \doteq \sqrt{t}\rho(t) = t^{\frac{3}{2}}e^{\frac{t}{\ln t}}$  for  $t \in (0, \|T\|^2]$ , by Theorem 4.9 we conclude that  $\psi(x, \delta) \doteq (\rho \circ \Theta^{-1})(\delta)$  for  $x \in X^\rho \doteq \mathcal{R}(\rho(T^*T)) \setminus \{0\}$  and  $\delta \in (0, \Theta(\alpha_0)) = \left(0, \alpha_0^{\frac{3}{2}}e^{\frac{\alpha_0}{\ln(\alpha_0)}}\right)$ , is global saturation function of  $\{R_\alpha\}_{\alpha \in (0, \alpha_0)}$  on  $X^\rho$ .

**6. Conclusions.** In this article we have developed a general theory of global saturation for arbitrary regularization methods for inverse ill-posed problems. This concept of saturation formalizes the best global order of convergence that a method can achieve independently of the smoothness assumptions on the exact solution and on the selection of the parameter choice rule. Necessary and sufficient conditions for a methods to have global saturation have been provided. It was shown that for a method to have saturation the total error must be optimal in two senses, namely as optimal order of convergence over a certain set which at the same time, must be optimal with respect to the error. We have also proved two converse results and applied the theory to derive sufficient conditions for the existence of global saturation for spectral methods with classical qualification of finite positive order and for methods with maximal qualification. Finally, examples of regularization methods possessing global saturation were shown.

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