INTEGRAL AND DERIVATIVE OPERATORS OF FUNCTIONAL ORDER ON GENERALIZED BESOV AND TRIEBEL-LIZORKIN SPACES IN THE SETTING OF SPACES OF HOMOGENEOUS TYPE

SILVIA I. HARTZSTEIN AND BEATRIZ E. VIVIANI

ABSTRACT. In this work we define the Integral, I_ϕ, and Derivative, D_ϕ, operators of order ϕ, in the setting of spaces of homogeneous-type, where ϕ is a function of positive lower type and upper type lower than 1.
We show that I_ϕ and D_ϕ are bounded from Lipschitz spaces Λ^ξ to $\Lambda^{\xi\phi}$ and $\Lambda^{\xi/\phi}$ respectively, with suitable restrictions on the quasi-increasing function ξ for each case. We also prove that I_ϕ and D_ϕ are bounded from the generalized Besov $\dot{B}_p^{\psi,q}$, with $1 \le p, q < \infty$, and Triebel-Lizorkin spaces $\dot{F}_p^{\psi,q}$, with $1 < p, q < \infty$, of order ψ to those of order $\phi\psi$ and ψ/ϕ respectively, where ψ is the quotient of two quasi-increasing functions of adequate upper types.

1. Introduction

In the context of normal spaces of homogeneous-type (X, δ, μ) of order $\theta \le 1$, the fractional integral and derivative operators of order α, with $0 < \alpha < \theta$, were defined by Gatto, Segovia and Vàgi in [GSV] by linking them to quasi-distances constructed through the kernels $\{s_t(x, y)\}_{t>0}$ of a symmetric approximation to the identity. Namely, if $\delta_\alpha : X \times X \to [0, \infty)$ is defined by

$$\delta_\alpha(x, y) = \left(\int_0^\infty t^{\alpha-1} s_t(x, y)dt\right)^{1/\alpha-1} \quad \text{for } x \ne y \text{ and } \delta_\alpha(x, y) = 0 \text{ for } x = y; \tag{1.1}$$

and $\delta_{-\alpha} : X \times X \to [0, \infty)$ by

$$\delta_{-\alpha}(x, y) = \left(\int_0^\infty t^{-\alpha-1} s_t(x, y)dt\right)^{1/-\alpha-1} \quad \text{for } x \ne y \text{ and } \delta_{-\alpha}(x, y) = 0 \text{ for } x = y, \tag{1.2}$$

then the authors proved that δ_α and $\delta_{-\alpha}$ are quasi-metrics equivalent to δ. The fractional integral I_α was thus defined by

$$I_\alpha f(x) = \int_X \frac{f(y)}{\delta_\alpha^{1-\alpha}(x, y)} d\mu(y),$$

for $f \in \Lambda^\beta \cap L^1$, and the fractional derivative D_α by

$$D_\alpha f(x) = \int_X \frac{f(y) - f(x)}{\delta_{-\alpha}^{1+\alpha}(x, y)} d\mu(y)$$

for $f \in \Lambda^\beta \cap L^\infty$ and $\alpha < \beta \le \theta$.
The definition of the quasi-metrics and the resulting operators allowed the authors to prove that the composition $T_\alpha = D_\alpha I_\alpha$ is a Calderón-Zygmund operator and that it is invertible in L^2 for small positive values of α.

The purpose of this work is to show that these technics can also be used to define the integral, I_ϕ, and derivative, D_ϕ, operators whose kernels are equivalent to $\phi(\delta(x, y))/\delta(x, y)$ and $1/\phi(\delta(x, y))\delta(x, y)$ respectively and ϕ belongs to a class of quasi-increasing functions. This class of growth functions includes the potentials t^α, $0 < \alpha < 1$, but also functions as, for example, $\max(t^\alpha, t^\beta)$, $\min(t^\alpha, t^\beta)$, with $0 < \alpha < \beta < 1$, and $t^\beta(1 + \log^+ t)$, $0 \le \beta < 1$.

We then prove that those operators are bounded on Lipschitz spaces Λ^ξ defined by functions whose moduli of continuity are dominated by a function $\xi(t)$ in the class of growth function.

1991 *Mathematics Subject Classification.* 43-XX.
Key words and phrases. Fractional Integral Operator, Fractional Derivative operator, spaces of homogeneous type, Besov spaces, Triebel-Lizorkin spaces.
Supported by Universidad Nacional del Litoral and IMAL-CONICET.

We finally study boundedness of the integral and derivative operators on the Besov $\dot{B}_p^{\psi,q}$, $1 \leq p, q < \infty$, and Triebel-Lizorkin spaces $\dot{F}_p^{\psi,q}$, $1 < p, q < \infty$, of distributions of order ψ, where ψ is the quotient of two quasi-increasing functions of adequate upper types. These spaces defined in [HV] are a generalization of the spaces $\dot{B}_p^{\alpha,q}$ and $\dot{F}_p^{\alpha,q}$, $-\theta < \alpha < \theta$, given in [HS] in the setting of spaces of homogeneous type. The Calderón-type reproduction formulas proved in [HS] play a fundamental roll in their definition but also in proving boundedness of our operators on them.

This work is organized in the following way:

In section 2 the class of functions involved in the 'order' of the integral and derivative operators and in local regularity of our function and distribution spaces is defined. Also the structure of normal spaces of homogeneous type, the test function spaces, the notions of discrete and continuous (in the time variable) approximations to the identity and the definitions of the generalized Besov and Triebel-Lizorkin spaces are set there. The integral and derivative operators are defined in section 3 and the main theorems are stated in section 4. Known results on the class of quasi-increasing functions and some consequences of them, the Calderón-type reproduction formula and properties of the generalized Besov and Triebel-Lizorkin spaces are given in section 5. In section 6 new representations of the kernels of the integral and derivative operators are obtained and size and smoothness properties on them are proved. Theorems of boundedness of the operators on Lipschitz spaces are proved in section 7. Lemmas needed to prove boundedness theorems on Besov and Triebel-Lizorkin spaces are given in section 8. Finally, the proofs of those theorems are in section 9.

2. Preliminaries

Let first define the class of functions controlling local regularity of the distribution spaces concerning us and that are also related to the operators defined in this work.

A function $\phi(t)$ defined on $t > 0$ is said to be *quasi-increasing* if there is a positive constant C such that if $t_1 < t_2$ then $\phi(t_1) \leq C\phi(t_2)$.

Analogously, $\phi(t)$ is *quasi-decreasing* if there is a positive constant C such that if $t_1 < t_2$ then $\phi(t_2) \leq C\phi(t_1)$.

On the other hand, $\phi(t)$ is said to be of *lower type* i_ϕ, $0 \leq i_\phi < \infty$, if there is a constant $C_1 > 0$ such that

$$\phi(uv) \leq C_1 u^{i_\phi}\phi(v) \text{ for } u < 1 \text{ and } v > 0. \tag{2.3}$$

Similarly, $\phi(t)$ is of *upper type* s_ϕ, $0 \leq s_\phi < \infty$ if there is a constant $C_2 > 0$ such that

$$\phi(uv) \leq C_2 u^{s_\phi}\phi(v) \text{ for } u \geq 1 \text{ and } v > 0. \tag{2.4}$$

Obviously, the potential t^α, with $\alpha \geq 0$, is of lower and upper type α. The functions $\max(t^\alpha, t^\beta)$ and $\min(t^\alpha, t^\beta)$, with $\alpha < \beta$, are both of lower type α and upper type β. Also, $t^\beta(1 + \log^+ t)$, with $\beta \geq 0$, is of lower type β and of upper type $\beta + \epsilon$, for every $\epsilon > 0$. Clearly, if $\phi(t)$ is of both lower type i_ϕ and upper type s_ϕ then $i_\phi \leq s_\phi$. On the other hand, if $\phi(t)$ is quasi-increasing then $\phi(t)$ is of lower-type 0 and, reciprocally, if $\phi(t)$ is of lower type $i_\phi \geq 0$ then it is quasi-increasing.

To finish the definitions of this class of growth functions we say that two functions $\psi(t)$ and $\phi(t)$ are *equivalent*, $\psi \simeq \phi$, if there are positive constants C_1 and C_2 such that $C_1 \leq \phi/\psi \leq C_2$.

Let now define the structure of spaces of homogeneous type which is the underlying geometry for the test functions spaces defined in this work.

Given a set X a real valued function $\delta(x, y)$ defined on $X \times X$ is a quasi-distance on X if there exists a constant $A > 1$ such that for all $x, y, z \in X$ it verifies:

$$\begin{aligned}
&\delta(x, y) \geq 0 \quad \text{and} \quad \delta(x, y) = 0 \quad \text{if and only if} \quad x = y \\
&\delta(x, y) = \delta(y, x) \\
&\delta(x, y) \leq A[\delta(x, z) + \delta(z, y)].
\end{aligned}$$

In a set X endowed with a quasi-distance $\delta(x, y)$, the balls $B_\delta(x, r) = \{y : \delta(x, y) < r\}$ form a basis of neighborhoods of x for the topology induced by the uniform structure on X. Let μ be a positive measure on a σ- algebra of subsets of X which contains the open set and the balls $B_\delta(x, r)$. The triple $X := (X, \delta, \mu)$ is a *space of homogeneous type* if there exists a finite constant $A' > 0$ such that $\mu(B_\delta(x, 2r)) \leq A'\mu(B_\delta(x, r))$ for all $x \in X$ and $r > 0$. Macías and Segovia, [MS], showed that it is always possible to find a quasi-distance $d(x, y)$ equivalent to $\delta(x, y)$ and $0 < \theta \leq 1$, such that

$$|d(x, y) - d(x', y)| \leq Cr^{1-\theta}d(x, x')^\theta \tag{2.5}$$

holds whenever $d(x,y) < r$ and $d(x',y) < r$. If δ satisfies (2.5) then X is said to be *of order* θ. Furthermore, X is a *normal space* if $A_1 r \leq \mu(B_\delta(x,r)) \leq A_2 r$ for every $x \in X$ and $r > 0$ and some positive constants A_1 and A_2.

In this work $X := (X, \delta, \mu)$ means a normal space of homogeneous type of order θ and A denotes the constant of the triangular inequality associated to δ.

Let us now introduce the test function spaces which concern us in this work.

Given a quasi-increasing function $\xi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\lim_{t\to 0} \xi(t) = 0$ and $\lim_{t\to\infty} \xi(t) = \infty$, the Lipschitz space Λ^ξ is the class of all functions $f : X \to \mathbb{C}$ such that

$$|f(x) - f(y)| \leq C\xi(\delta(x,y)) \quad \text{for every} \quad x, y \in X,$$

and the number $|f|_\xi$ denoting the infimum of the constants C appearing above, defines a semi-norm on Λ^ξ, since $|f|_\xi = 0$ for all constants functions f.
Furthermore, given a ball B in X, $\Lambda^\xi(B)$ denotes the set of functions $f \in \Lambda^\xi$ with support in B. Since, a function belonging to this space is bounded, the number $\|f\|_\xi = \|f\|_\infty + |f|_\xi$, defines a norm that gives a Banach structure to $\Lambda^\xi(B)$.
We say that a function f belongs to Λ_0^ξ iff $f \in \Lambda^\xi(B)$ for some ball B. The space Λ_0^ξ is the inductive limit of the Banach spaces $\Lambda^\xi(B)$.
Finally, $(\Lambda_0^\xi)'$ will mean the space of all continuous linear functionals on Λ_0^ξ.
When $\xi(t) = t^\beta$, with $0 < \beta \leq \theta$, we have the classical Lipschitz spaces Λ^β and Λ_0^β.

Another suitable class of test functions, the set $M^{(\beta,\gamma)}$, was defined in [HS]. Indeed, setting $0 < \beta \leq 1$, $\gamma > 0$ and $x_0 \in X$ fix, a function f is called a *smooth molecule of type* (β,γ) *of width* d *centered in* x_0, if there exists a constant $C > 0$ such that

$$|f(x)| \leq C \frac{d}{(d + \delta(x,x_0))^{1+\gamma}}, \tag{2.6}$$

$$|f(x) - f(x')| \leq C\delta(x,x')^\beta \left(\frac{d}{(d + \delta(x,x_0))^{1+\gamma}} + \frac{d}{(d + \delta(x',x_0))^{1+\gamma}} \right), \tag{2.7}$$

$$\int f(x) d\mu(x) = 0, \tag{2.8}$$

hold for every $x \in X$.
If the norm $\|f\|_{(\beta,\gamma)}$, is defined by the infimum of the constants appearing in (2.6) and (2.7), the set $M^{(\beta,\gamma)}(x_0,d)$ of all smooth molecules of type (β,γ) of width d centered in x_0 is a Banach space. Fixing $x_0 \in X$ and $d = 1$, that space will be named $M^{(\beta,\gamma)}$, and the set of all linear continuous functionals on $M^{(\beta,\gamma)}$ will be called $(M^{(\beta,\gamma)})'$. Along this work $< h, f >$ denotes the natural application of $h \in (M^{(\beta,\gamma)})'$ to $f \in M^{(\beta,\gamma)}$.

In order to define the generalized Besov and Triebel–Lizorkin spaces of distributions the definition of an approximation to the identity as given in [HS], is needed.
A sequence $(S_k)_{k\in Z}$ of integral operators is called an *approximation to the identity*, if the kernels $S_k(x,y)$ associated to S_k are functions from $X \times X$ in $\mathbb{C}$ and there exist $0 < \epsilon \leq \theta$ and a finite constant C such that for all $k \in Z$ and $x, x', y, y' \in X$ they satisfy

$$S_k(x,y) = 0 \quad \text{if} \quad \delta(x,y) \geq (2A)^{-k} \quad \text{and} \ \|S_k\|_\infty \leq C(2A)^k, \tag{2.9}$$

$$|S_k(x,y) - S_k(x',y)| \leq C(2A)^{k(1+\epsilon)}\delta(x,x')^\epsilon, \tag{2.10}$$

$$|S_k(x,y) - S_k(x,y')| \leq C(2A)^{k(1+\epsilon)}\delta(y,y')^\epsilon, \tag{2.11}$$

$$\begin{aligned} &|[S_k(x,y) - S_k(x,y')] - [S_k(x',y) - S_k(x',y')]| \\ &\qquad\qquad \leq C(2A)^{k(1+2\epsilon)}\delta(x,x')^\epsilon \delta(y,y')^\epsilon, \end{aligned}$$

$$\int_X S_k(x,y) d\mu(y) = \int_X S_k(x,y) d\mu(x) = 1. \tag{2.12}$$

In all this paper the constant ϵ, $0 < \epsilon \leq \theta$, will denote that associated to an approximation to the identity satisfying (2.10), (2.11) and (2.12).
If $(S_k)_{k\in Z}$ is an approximation to the identity then the family of operators $D_k = S_k - S_{k-1}$ satisfy $\sum_{k\in Z} D_k = I$ in L^2, since $\lim_{k\to\infty} S_k f = f$ and $\lim_{k\to-\infty} S_k f = 0$ in L^2. Moreover, their

associated kernels $D_k(x,y)$ satisfy properties (2.9) to (2.12) and

$$\int_X D_k(x,y)d\mu(y) = \int_X D_k(x,y)d\mu(x) = 0. \tag{2.13}$$

Let us now define the spaces of distributions for us considered.
In the sequel we denote by ψ the function $\psi = \phi_1/\phi_2$, where $\phi_1(t)$ and $\phi_2(t)$ are quasi-increasing functions of upper types $s_1 < \epsilon$ and $s_2 < \epsilon$, respectively .
For $f \in (M^{(\beta,\gamma)})'$, with $0 < \beta, \gamma < \epsilon$, a norm is defined by

$$\|f\|_{\dot{B}_p^{\psi,q}} = \left(\sum_{k\in Z}\left(\frac{1}{\psi((2A)^{-k})}\|D_k f\|_p\right)^q\right)^{\frac{1}{q}} \quad \text{if} \quad 1 \le p \le \infty, 1 \le q \le \infty, \tag{2.14}$$

with the obvious change for the case $q = \infty$. By interchanging the order of the norms in L^p and l^q it is also defined the norm

$$\|f\|_{\dot{F}_p^{\psi,q}} = \left\|\left(\sum_{k\in Z}\left(\frac{1}{\psi((2A)^{-k})}|D_k f|\right)^q\right)^{\frac{1}{q}}\right\|_{L^p}, \text{ if } \ 1 < p, q < \infty. \tag{2.15}$$

The *Besov space* $\dot{B}_p^{\psi,q}$, $1 \le p, q \le \infty$, is the set of all $f \in \left(M^{(\beta,\gamma)}\right)'$, with $\beta > s_1$ and $\gamma > s_2$, such that

$$\|f\|_{\dot{B}_p^{\psi,q}} < \infty \text{ and } |\langle f, h\rangle| \le C\|f\|_{\dot{B}_p^{\psi,q}}\|h\|_{(\beta,\gamma)},$$

for all $h \in M^{(\beta,\gamma)}$.
Analogously, The *Triebel–Lizorkin space* $\dot{F}_p^{\psi,q}$, with $1 < p, q < \infty$, is the set of all $f \in \left(M^{(\beta,\gamma)}\right)'$, with $\beta > s_1$ and $\gamma > s_2$, such that

$$\|f\|_{\dot{F}_p^{\psi,q}} < \infty, \text{ and } |\langle f, h\rangle| \le \|f\|_{\dot{F}_p^{\psi,q}}\|h\|_{(\beta,\gamma)},$$

for all $h \in M^{(\beta,\gamma)}$.
When $\psi(t) = t^\alpha$ and $-\epsilon < \alpha < \epsilon$, the definitions of the Besov spaces $\dot{B}_p^{\alpha,q}$ and the Triebel-Lizorkin spaces $\dot{F}_p^{\alpha,q}$ given in [HS] are recovered.

Finally, to build our concerning operators we consider a symmetric approximation to the identity, $\{S_t\}_{t>0}$, as defined in [GSV]. This collection can be built in a similar fashion to that of the family $\{S_k\}_{k\in Z}$ and the kernel $s_t(x,y)$ associated to S_t satisfies the following properties:
There are positive constants, b_1, b_2, c_1, c_2 and c_3, such that for all $x, y \in X$ and $t > 0$, $s_t(x,y)$ satisfies

$$s_t(x,y) = s_t(y,x), \tag{2.16}$$

$$0 \le s_t(x,y) \le c_1/t, \tag{2.17}$$

$$s_t(x,y) = 0 \ \text{ if } \ \delta(x,y) > b_1 t \ \text{ and, } \quad c_2/t < s_t(x,y) \ \text{ if } \ \delta(x,y) < b_2 t, \tag{2.18}$$

$$|s_t(x,y) - s_t(x',y)| < c_3\delta^\theta(x,x')/t^{1+\theta}, \ \text{ for all } \ x, x', y \in X, \tag{2.19}$$

$$\int s_t(x,y)d\mu(y) = 1, \ \text{ for all } \ x \in X, \tag{2.20}$$

$$s_t(x,y) \text{ is continuously differenciable in } \ t. \tag{2.21}$$

3. Integral and Derivative operators of order ϕ

Let consider a symmetric approximation to the identity, $\{S_t\}_{t>0}$, whose kernels satisfy properties (2.16) to (2.21), and a quasi-increasing function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\lim_{t\to 0^+}\phi(t) = 0$.
We now define

$$K_\phi(x,y) = \int_0^\infty \frac{\phi(t)}{t} s_t(x,y)dt \quad \text{for} \quad x \ne y.$$

Clearly, $K_\phi(x,y) > 0$ and $K_\phi(x,y) = K_\phi(y,x)$ for every (x,y).
If ϕ is of positive lower type and upper type $s_\phi < 1$ the *integral operator of order* ϕ, I_ϕ, and its extension $\tilde{I}_\phi$ are defined in the following way:
Let ξ be any quasi-increasing function of upper type β.
If $\beta > 0$ and $f \in \Lambda^\xi \cap L^1$ then

$$I_\phi f(x) := \int_X K_\phi(x,y)f(y)d\mu(y), \tag{3.22}$$

If $\beta + s_\phi < \theta$ and $f \in \Lambda^\xi$ then

$$\tilde{I}_\phi f(x) := \int_X (K_\phi(x,y) - K_\phi(x_0,y))f(y)d\mu(y), \tag{3.23}$$

for every $x \in X$ and an arbitrary fix $x_0 \in X$.
On the other hand, for ϕ of finite upper-type we define

$$K_{1/\phi}(x,y) = \int_0^\infty \frac{1}{\phi(t)t} s_t(x,y)dt, \quad \text{for} \quad x \neq y,$$

Clearly $K_{1/\phi}$ is positive and symmetric.
If ϕ is a function of lower type $i_\phi > 0$ and upper type s_ϕ the *derivative operator of order* ϕ, D_ϕ, and its extension, $\tilde{D}_\phi$ are defined as follows:
For any function ξ of lower type α and of upper type β, such that $s_\phi < \alpha$,
if $f \in \Lambda^\xi \cap L^\infty$, then

$$D_\phi f(x) = \int_X K_{1/\phi}(x,y)(f(y) - f(x))d\mu(y) \quad \text{and} \tag{3.24}$$

if $f \in \Lambda^\xi$, then

$$\tilde{D}_\phi f(x) = \int_X (K_{1/\phi}(x,y)(f(y) - f(x)) - K_{1/\phi}(x_0,y)(f(y) - f(x_0)))d\mu(y) \tag{3.25}$$

for each $x \in X$ and an arbitrary, but fix, $x_0 \in X$.

4. Main theorems

The main theorems proved in this work are stated next:

Theorem **4.1.** *Let ϕ be of lower type $i_\phi > 0$ and upper type $s_\phi < 1$ and ξ a quasi-increasing function of upper type β.*
If $f \in \Lambda^\xi \cap L^1$ and $\beta > 0$ then $I_\phi f(x)$ converges absolutely for all x and if, also, $\beta + s_\phi < \theta$ then there is a constant $C > 0$, independent of f, such that

$$|I_\phi f|_{\Lambda^{\xi\phi}} \leq C|f|_{\Lambda^\xi}.$$

If $f \in \Lambda^\xi$ and $\beta + s_\phi < \theta$ then $\tilde{I}_\phi f(x)$ converges absolutely for all x and there is a constant $C > 0$, independent of f, such that

$$|\tilde{I}_\phi f|_{\Lambda^{\xi\phi}} \leq C|f|_{\Lambda^\xi}.$$

Moreover, If $f \in \Lambda^\xi \cap L^1$, then $\tilde{I}_\phi f$ coincides with $I_\phi f$ as an element of $\Lambda^{\xi\phi}$ (since $\tilde{I}_\phi f(x) = I_\phi f(x) - I_\phi f(x_0)$.)

Theorem **4.2.** *Let ϕ be a function of lower type $i_\phi > 0$ and upper type s_ϕ.*
Let also ξ be a quasi-increasing function of lower type α and upper type β.
If $f \in \Lambda^\xi \cap L^\infty$ and $s_\phi < \alpha$ then $D_\phi f(x)$ is absolutely convergent for every $x \in X$ and if, also, $\beta < \theta + i_\phi$ then

$$\|D_\phi f\|_{\xi/\phi} \leq C\|f\|_\xi.$$

If $f \in \Lambda^\xi$, $s_\phi < \alpha$ and $\beta < \theta + i_\phi$ then $\tilde{D}_\phi f(x)$ is absolutely convergent for every $x \in X$ and

$$|\tilde{D}_\phi f|_{\xi/\phi} \leq C|f|_\xi.$$

Moreover, if $f \in \Lambda^\xi \cap L^\infty$, then $\tilde{D}_\phi f$ coincides with $D_\phi f$ as an element of Λ^ξ, (since $\tilde{D}_\phi f(x) = D_\phi f(x) - D_\phi f(x_0)$.)

Theorem **4.3.** *Let ϕ be a function of lower type $i_\phi > 0$ and upper type $s_\phi < \epsilon$.*
Let also denote $\psi = \psi_1/\psi_2$, where ψ_1 and ψ_2 are quasi-increasing functions of upper types s_1 and s_2 respectively.
If $s_1 + s_\phi < \epsilon$ and $s_2 + s_\phi - i_\phi < \epsilon$, then I_ϕ is a linear continuous operator from $\dot{F}_p^{\psi,q}$ to $\dot{F}_p^{\phi\psi,q}$, for $1 < p, q < \infty$.

Theorem **4.4.** *Suppose that ϕ and ψ satisfy the same hypothesis as in the previous theorem.*
If $s_1 + s_\phi < \epsilon$ and $s_2 + s_\phi - i_\phi < \epsilon$ then I_ϕ is a linear continuous operator from $\dot{B}_p^{\psi,q}$ to $\dot{B}_p^{\phi\psi,q}$, for $1 \leq p, q < \infty$.

Theorem **4.5.** *Let ϕ be a function of positive lower type and of upper type $s_\phi < \epsilon$. Also denote $\psi = \psi_1/\psi_2$, where ψ_1 and ψ_2 are quasi-increasing functions of upper type s_1 and s_2, respectively. If $s_1 < \epsilon$ and $s_\phi + s_2 < \epsilon$ then, D_ϕ is a linear continuous operator from $\dot{F}_p^{\psi,q}$ to $\dot{F}_p^{\psi/\phi,q}$, for $1 < p, q < \infty$.*

Theorem **4.6.** *Let ϕ be of positive lower type and of upper type $s_\phi < \epsilon$. Consider $\psi = \psi_1/\psi_2$, where ψ_1 and ψ_2 are quasi-increasing functions of upper type s_1 and s_2, respectively. If $s_1 < \epsilon$ and $s_\phi + s_2 < \epsilon$ then D_ϕ is a linear continuous operator from $\dot{B}_p^{\psi,q}$ to $\dot{B}_p^{\psi/\phi,q}$, for $1 \leq p, q < \infty$.*

5. Previous results

A straightforward proof shows that if $\phi(t)$ is of upper type s_ϕ then there is a constant $C > 0$ such that

$$\phi(uv) \geq \frac{1}{C} u^{s_\phi} \phi(v), \text{ for } u < 1, v > 0. \tag{5.26}$$

Similarly, if $\phi(t)$ is of lower type i_ϕ then there is a constant $C > 0$ such that

$$\phi(uv) \geq \frac{1}{C} u^{i_\phi} \phi(v), \text{ for } u \geq 1, v > 0. \tag{5.27}$$

Also, it is easy to check that

Proposition **5.1.** *If $\phi(t)$ is of lower type i_ϕ and $\xi(t)$ is of upper type $\lambda \leq i_\phi$ then $\phi(t)/\xi(t)$ is quasi-increasing. On the other hand, if $\phi(t)$ is of upper type s_ϕ and $\xi(t)$ is of lower type $\lambda \geq s_\phi$ then $\phi(t)/\xi(t)$ is quasi-decreasing.*

After some manipulation it comes out that

Proposition **5.2.** *If $\phi(t)$ is of lower type $\alpha > 0$ and upper type $\beta \in \mathbb{R}$ and $0 < \gamma < \alpha$ then the function*

$$\psi(t) = t^\gamma \int_0^t \frac{\phi(u)}{u^{\gamma+1}} du$$

is equivalent to ϕ, continuous, increasing and invertible. Moreover, its inverse ψ^{-1} is of lower type β^{-1} and of upper type α^{-1}.

Corollary **5.1.** *If ϕ is a quasi-increasing function of upper type $s_\phi < 1$ then there is an equivalent function $\tilde{\phi}$ such that $\tilde{\phi}(t)/t$ is decreasing, continuous and invertible on $t > 0$.*

Indeed, since $t/\phi(t)$ is of lower type $1 - s_\phi > 0$ and of upper type 1, Proposition 5.2 claims that there exists an increasing and invertible function, $\psi(t)$, equivalent to $t/\phi(t)$ and having the same lower and upper types. It is an exercise to prove that $\tilde{\phi} = t/\psi(t)$ satisfies the statements of Corollary 5.1.

Corollary **5.2.** *If $\phi(t)$ is a quasi-increasing function of finite upper type then there exists a function $\hat{\phi}(t)$ equivalent to $\phi(t)$, such that $t\hat{\phi}(t)$ is increasing, continuous and invertible in $\mathbb{R}^+$.*

This comes out by defining $\hat{\phi}(t) = \hat{\psi}(t)/t$, where $\hat{\psi}(t)$ is the function equivalent to $t\phi(t)$, of lower type 1 and upper type $1 + s_\phi$, given by Proposition 5.2.

The following properties will be useful when studying local regularity and integrability of the functions and distributions that concern us. Their proof is based on dyadic partition.

Proposition **5.3.** *Let $\phi_i(t)$ be a function of lower type α_i and of upper type β_i, i= 1,2. The following inequalities hold for $x \in X$ and $r > 0$:*

$$\text{If } \alpha_1 > \beta_2 \text{ then } \int_{\delta(x,y)\leq r} \frac{\phi_1(\delta(x,y))}{\phi_2(\delta(x,y))\delta(x,y)} d\mu(y) \leq C \frac{\phi_1(r)}{\phi_2(r)}. \tag{5.28}$$

$$\text{If } \beta_1 < \alpha_2 \text{ then } \int_{\delta(x,y)\geq r} \frac{\phi_1(\delta(x,y))}{\phi_2(\delta(x,y))\delta(x,y)} d\mu(y) \leq C \frac{\phi_1(r)}{\phi_2(r)}. \tag{5.29}$$

The Calderón-type reproduction formulas, proved in [HS] in the context of spaces of homogeneous type, are needed to define the Besov and Triebel-Lizorkin spaces and to prove boundedness theorems on them. They are stated next.

THEOREM **5.1.** *Let $(S_k)_{k\in Z}$ be an approximation to the identity and set $D_k = S_k - S_{k-1}$. Then, there exist families of operators $(\tilde{D}_k)_{k\in Z}$ and $(\hat{D}_k)_{k\in Z}$ such that for all $f \in M^{(\beta,\gamma)}$*

$$f = \sum_{k=-\infty}^{\infty} \tilde{D}_k D_k f = \sum_{k=-\infty}^{\infty} D_k \hat{D}_k f,$$

where the series converges in $M^{(\beta',\gamma')}$, for $\beta' < \beta$ and $\gamma' < \gamma$.

If $(\tilde{D}_k)_{k\in Z}$ y $(\hat{D}_k)_{k\in Z}$ are like in Theorem (5.1) then their associated kernels $\tilde{D}_k(x,y)$ and $\hat{D}_k(x,y)$ are (ϵ',ϵ')-smooth molecules of width $(2A)^{-k}$, as functions of the first and second variable respectively, for each $0 < \epsilon' < \epsilon$. Then $\tilde{D}_k^* f$ and $\hat{D}_k^* f \in M^{(\beta,\gamma)}$, whenever $f \in M^{(\beta,\gamma)}$, $0 < \beta, \gamma < \epsilon$.
Thus, for $h \in \left(M^{(\beta,\gamma)}\right)'$ $\tilde{D}_k h$ and $\hat{D}_k h$ are defined as elements of $\left(M^{(\beta,\gamma)}\right)'$ by $< \tilde{D}_k h, f >=< h, \tilde{D}_k^* f >$ and $< \hat{D}_k h, f >=< h, \hat{D}_k^* f >$. Therefore, the formulas in Theorem (5.1) will also hold true in the sense of distributions. More precisely,

THEOREM **5.2.** *Let $(D_k)_{k\in Z}$, $(\tilde{D}_k)_{k\in Z}$ and $(\hat{D}_k)_{k\in Z}$ be like in Theorem (5.1). Then for all $f \in (M^{(\beta,\gamma)})'$,*

$$f = \sum_{k=-\infty}^{\infty} \tilde{D}_k D_k f = \sum_{k=-\infty}^{\infty} D_k \hat{D}_k f,$$

in the sense of

$$\langle f, g\rangle = \lim_{M\to\infty} \langle \sum_{|k|\le M} \tilde{D}_k D_k f, g\rangle = \lim_{M\to\infty} \langle \sum_{|k|\le M} D_k \hat{D}_k f, g\rangle$$

for all $g \in M^{(\beta',\gamma')}$, with $\beta' > \beta$ and $\gamma' > \gamma$.

Using the Calderón–type reproduction formula it can be proved that if the operators D_k in the definitions of the norms are replaced by $E_k = P_k - P_{k-1}$, with $(P_k)_{k\in Z}$ another approximation to the identity of order $\epsilon \le \theta$, the resulting norms are equivalent to those defined in (2.14) and (2.15), (see [H]). The same result is true if the operators D_k are replaced by $\tilde{D}_k^*$ or $\hat{D}_k$.

In the following two lemmas the main properties of the generalized Besov and Triebel-Lizorkin spaces are stated without proof, for the sake of briefness.

LEMMA **5.3.** *The classes $\dot{B}_p^{\psi,q}$, $1 \le p, q < \infty$ and $\dot{F}_p^{\psi,q}$, $1 < p, q < \infty$ are Banach spaces and their dual spaces are $\dot{B}_{p'}^{1/\psi,q'}$ and $\dot{F}_{p'}^{1/\psi,q'}$ respectively, with $1/p + 1/p' = 1$ and $1/q + 1/q' = 1$.*

LEMMA **5.4.** *The molecular space $M^{(\beta,\gamma)}$ is embedded in $\dot{B}_p^{\psi,q}$, $1 \le p, q < \infty$ and $\dot{F}_p^{\psi,q}$, $1 < p, q < \infty$, when $s_1 < \beta$ and $s_2 < \gamma$. Moreover, $M^{(\epsilon',\epsilon')}$ is dense in $\dot{B}_p^{\psi,q}$, $1 \le p, q < \infty$ and $\dot{F}_p^{\psi,q}$,$1 < p, q < \infty$, for all ϵ', such that $\max(s_1, s_2) < \epsilon' < \epsilon$.*

In the setting of $\mathbb{R}^n$ and for $q = \infty$ unified approaches between Besov spaces of order ξ related to a Banach space E of functions (in our definitions $E = L^p$) and Lipschitz classes of distributions whose moduli of continuity in E is dominated by ξ are treated in [J] and [B]. Also see [I] for the inhomogeneous case. The identification between the Sobolev space of fractional order $\dot{L}^{p,\alpha}$ and $\dot{F}_p^{\alpha,q}$ in the setting of spaces of homogeneous type is treated in [GV].

6. Main lemmas

Let now define two quasi-metrics associated to ϕ and equivalent to δ and obtain new representations of the kernels of I_ϕ and D_ϕ in terms of each quasi-metric.
Consider a quasi-increasing function ϕ of upper-type $s_\phi < 1$ and a fix function $\tilde{\phi}$, as given in Corollary 5.1.
We define $\delta_\phi : X \times X \to \mathbb{R}$ in the following way: for every pair $(x,y) \in X$, $\delta_\phi(x,y)$ is the unique solution of

$$\begin{aligned} \frac{\tilde{\phi}(\delta_\phi(x,y))}{\delta_\phi(x,y)} &= \int_0^\infty \frac{\phi(t)}{t} s_t(x,y)dt \quad \text{if} \quad x \ne y, \\ \delta_\phi(x,y) &= 0 \quad \text{if} \quad x = y. \end{aligned} \tag{6.30}$$

We then have that

$$K_\phi(x,y) = \frac{\tilde{\phi}(\delta_\phi(x,y))}{\delta_\phi(x,y)} \quad \text{for} \quad x \neq y.$$

When $\phi(t) = t^\alpha$, $0 < \alpha < 1$, we can choose $\tilde{\phi} = \phi$ and then $\delta_\alpha := \delta_\phi$ is the quasi-metric associated to I_α defined in (1.1).

The next lemma proves that $K_\phi(x,y)$ is equivalent to $\phi(\delta(x,y))/\delta(x,y)$.

LEMMA **6.1.** *If ϕ is of upper type $s_\phi < 1$ then there are positive constants C_1 and C_2 such that for $\delta(x,y) > 0$,*

$$C_2 \frac{\phi(\delta(x,y))}{\delta(x,y)} \leq \frac{\tilde{\phi}(\delta_\phi(x,y))}{\delta_\phi(x,y)} \leq C_1 \frac{\phi(\delta(x,y))}{\delta(x,y)} \tag{6.31}$$

PROOF: By (2.17) and (2.18), it holds that

$$\int_0^\infty \frac{\phi(t)}{t} s_t(x,y)dt \leq c_1 \int_{\delta(x,y)/b_1}^\infty \frac{\phi(t)}{t^2} dt.$$

The substitution $t = u\delta(x,y)/b_1$ and inequality (2.4) yield to

$$\int_0^\infty \frac{\phi(t)}{t} s_t(x,y)dt \leq \frac{c_1 b_1}{\delta(x,y)} \phi(\frac{\delta(x,y)}{b_1}) \int_1^\infty \frac{1}{u^{2-s_\phi}} du \leq C_1 \frac{\phi(\delta(x,y))}{\delta(x,y)} \tag{6.32}$$

since $s_\phi < 1$ and $\phi(s/b_1) \leq C \max(1, 1/b_1^{s_\phi})\phi(s)$ for all $s > 0$.

On the other hand, by (2.18) and the fact that ϕ is quasi-increasing, it follows that

$$\begin{aligned} \int_0^\infty \frac{\phi(t)}{t} s_t(x,y)dt &\geq c_2 \int_{\delta(x,y)/b_2}^\infty \frac{\phi(t)}{t^2} dt \\ &\geq C c_2 \frac{\phi(\delta(x,y)/b_2)}{\delta(x,y)/b_2} \int_1^\infty \frac{1}{u^2} du = C_2 \frac{\phi(\delta(x,y))}{\delta(x,y)}, \end{aligned} \tag{6.33}$$

since $\phi(s/b_2) \geq C \min(1, 1/b_2^{s_\phi})\phi(s)$ for all $s > 0$. From definition (6.30) and the above inequalities then (6.31) follows. ♢

An immediate consequence of the previous lemma is that

$$0 < K_\phi(x,y) \leq C \frac{\phi(\delta(x,y))}{\delta(x,y)}. \tag{6.34}$$

LEMMA **6.2.** *If $\phi(t)$ is of upper type $s_\phi < 1$ then δ_ϕ is a quasi-metric equivalent to δ.*

PROOF: Since $s_\phi < 1$, from (6.31) and $\phi \simeq \tilde{\phi}$ it follows that

$$C_2' \frac{\tilde{\phi}(\delta(x,y))}{\delta(x,y)} \leq \frac{\tilde{\phi}(\delta_\phi(x,y))}{\delta_\phi(x,y)} \leq C_1' \frac{\tilde{\phi}(\delta(x,y))}{\delta(x,y)}. \tag{6.35}$$

Nevertheless, since $\psi(t) = t/\tilde{\phi}(t)$ is increasing and invertible and its inverse function is of finite upper type, it follows that

$$C_1'' \delta(x,y) \leq \delta_\phi(x,y) \leq C_2'' \delta(x,y).$$

Clearly, from the above equivalence turns out that δ_ϕ is a quasi-metric.♢

The next two lemmas state smoothness and cancellation properties of K_ϕ.

LEMMA **6.3.** *Let ϕ be of upper type $s_\phi < 1$. Then*

$$|K_\phi(x,y) - K_\phi(x',y)| \leq C \left(\frac{\delta(x,x')}{\delta(x,y)}\right)^\theta \frac{\phi(\delta(x,y))}{\delta(x,y)} \tag{6.36}$$

whenever $\delta(x,y) \geq 2A\delta(x,x')$.

PROOF: Let $a = b_1^{-1} \min\{\delta(x,y), \delta(x',y)\}$. where b_1 is that defined in (2.18). From (6.30), it follows that

$$|K_\phi(x,y) - K_\phi(x',y)| \leq \int_a^\infty \frac{\phi(t)}{t} |s_t(x,y) - s_t(x',y)| dt,$$

From the smoothness property (2.19) of s_t it follows that

$$|K_\phi(x,y) - K_\phi(x',y)| \leq \int_a^\infty \frac{\phi(t)}{t} \frac{\delta(x,x')^\theta}{t^{1+\theta}} dt \tag{6.37}$$

Since $s_\phi < 1$, Proposition 5.1 says that $\phi(t)/t$ is quasi-decreasing and then,

$$|K_\phi(x,y) - K_\phi(x',y)| \leq C\delta(x,x')^\theta \frac{\phi(a)}{a^{1+\theta}} \tag{6.38}$$

Since $\delta(x,y) \geq 2A\delta(x,x')$ then $\delta(x,y) \leq 2A\delta(x',y)$ and thus, $\delta(x,y) \leq 2Ab_1 a$. But, since $\phi(t)/t^l$ is quasi-decreasing whenever $l > s_\phi$, it follows that

$$|K_\phi(x,y) - K_\phi(x',y)| \leq C\delta(x,x')^\theta \frac{\phi(\delta(x,y))}{\delta(x,y)^{1+\theta}}$$

which is our statement.◊

Lemma **6.4.** *Let ϕ be of upper type $s_\phi < \theta$. Then*

$$\int_X [K_\phi(x,y) - K_\phi(x',y)]d\mu(y) = 0, \tag{6.39}$$

for every x and $x' \in X$.

Proof: First notice that the integral in (6.39) is absolutely convergent. Indeed, (6.30), (2.17) and (2.18) yield to

$$\begin{aligned} &\int_X \int_0^1 \frac{\phi(t)}{t}|s_t(x,y) - s_t(x',y)|dt d\mu(y) \\ \leq\; & C\int_0^1 \frac{\phi(t)}{t}\int_X (|s_t(x,y)| + |s_t(x',y)|)d\mu(y)dt \leq C\int_0^1 \frac{\phi(t)}{t}dt < \infty. \end{aligned} \tag{6.40}$$

Moreover, from (2.19) it follows that

$$\begin{aligned} &\int_X \int_1^\infty \frac{\phi(t)}{t}|s_t(x,y) - s_t(x',y)|dt d\mu(y) \\ \leq\; & C(\delta(x,x'))^\theta \int_1^\infty \frac{\phi(t)}{t^{2+\theta}} \int_{\delta(x,y)<b_1 t \text{ o } \delta(x',y)<b_1 t} d\mu(y)dt \\ \leq\; & C(\delta(x,x'))^\theta \int_1^\infty \frac{\phi(t)}{t^{1+\theta}}dt \leq C(\delta(x,x'))^\theta \int_1^\infty \frac{1}{t^{1+\theta-s_\phi}}dt < \infty. \end{aligned} \tag{6.41}$$

Therefore, (6.39) is obtained by Fubini's theorem and (2.20).◊

Let now consider a quasi-increasing function ϕ of finite upper type and the function $\hat{\phi}$, as given by Corollary 5.2.

We then define $\delta_{1/\phi} : X \times X \to \mathbb{R}$ such that $\delta_{1/\phi}(x,y)$ is the unique solution of the equation

$$\begin{aligned} \frac{1}{\hat{\phi}(\delta_{1/\phi}(x,y))\delta_{1/\phi}(x,y)} &= \int_0^\infty \frac{1}{\phi(t)t}s_t(x,y)dt \quad \text{if} \quad x \neq y, \quad \text{and} \\ \delta_{1/\phi}(x,y) &= 0 \quad \text{if} \quad x = y. \end{aligned} \tag{6.42}$$

Hence we have that

$$K_{1/\phi}(x,y) = \frac{1}{\hat{\phi}(\delta_{1/\phi}(x,y))\delta_{1/\phi}(x,y)} \quad \text{for} \quad x \neq y.$$

When $\phi(t) = t^\alpha$, $0 < \alpha < 1$, choosing $\hat{\phi} = \phi$ it turns out that $\delta_{-\alpha} := \delta_{t^{-\alpha}}$ is the quasi-metric associated to D_α defined in (1.2).

The following results are obtained in analogous way to the case of I_ϕ and their proof is ommited for the sake of briefness. The first one sets that $K_{1/\phi}(x,y)$ is equivalent to $1/(\phi(\delta(x,y))\delta(x,y))$.

Lemma **6.5.** *If ϕ is a quasi-increasing function of finite upper type then there are positive constants C_1 and C_2 such that*

$$C_1 \frac{1}{\phi(\delta(x,y))\delta(x,y)} \leq \frac{1}{\hat{\phi}(\delta_{1/\phi}(x,y))\delta_{1/\phi}(x,y)} \leq C_2 \frac{1}{\phi(\delta(x,y))\delta(x,y)}. \tag{6.43}$$

Moreover, $\delta_{1/\phi}$ is a quasi-metric equivalent to δ.

Immediately follows from the above lemma ths size estimate

$$0 < K_{1/\phi}(x,y) < C\frac{1}{\phi(\delta(x,y))\delta(x,y)} \tag{6.44}$$

LEMMA **6.6.** *If ϕ is a quasi-increasing function of finite upper type then*

$$|K_{1/\phi}(x,y) - K_{1/\phi}(x',y)| + |K_{1/\phi}(y,x) - K_{1/\phi}(y,x')| \leq C\left(\frac{\delta(x,x')}{\delta(x,y)}\right)^{\theta}\frac{1}{\phi(\delta(x,y))\delta(x,y)} \tag{6.45}$$

for $\delta(x,y) \geq 2A\delta(x,x')$.

7. PROOF OF THEOREMS 4.1 AND 4.2

PROOF OF THEOREM 4.1
Let us first see that $I_\phi f(x)$ is absolutely convergent for every $x \in X$. From (6.30), it follows that

$$\begin{aligned}\int_X |K_\phi(x,y)||f(y)|d\mu(y) &\leq C\int_X \frac{\phi(\delta(x,y))}{\delta(x,y)}|f(y)|d\mu(y),\\ &= C\left(\int_{\delta(x,y)\leq 1} + \int_{\delta(x,y)>1}\right)\frac{\phi(\delta(x,y))}{\delta(x,y)}|f(y)|d\mu(y) = I_1 + I_2.\end{aligned} \tag{7.46}$$

Applying (5.28) -since $i_\phi > 0$-, and the fact that ξ is quasi-increasing it follows that

$$\begin{aligned}I_1 &\leq C\int_{\delta(x,y)\leq 1}\frac{\phi(\delta(x,y))}{\delta(x,y)}(|f(y)-f(x)| + |f(x)|)d\mu(y)\\ &\leq C|f|_\xi\int_{\delta(x,y)\leq 1}\frac{(\phi\xi)(\delta(x,y))}{\delta(x,y)}d\mu(y) + C|f(x)|\int_{\delta(x,y)\leq 1}\frac{\phi(\delta(x,y))}{\delta(x,y)}d\mu(y)\\ &\leq C(\xi(1)|f|_\xi + |f(x)|)\\ &\leq C(|f|_\xi + |f(x)|)\end{aligned} \tag{7.47}$$

Furthermore, since $s_\phi < 1$ then $\phi(t)/t$ is quasi-decreasing and

$$I_2 \leq C\int_{\delta(x,y)>1}\frac{\phi(\delta(x,y))}{\delta(x,y)}|f(y)|d\mu(y) \leq C\|f\|_1 \tag{7.48}$$

Inequalities (7.47) and (7.48) lead to the bound

$$|I_\phi f(x)| \leq \int_X |K_\phi(x,y)||f(y)|d\mu(y) < C(|f|_\xi + |f(x)| + \|f\|_1), \tag{7.49}$$

for every $x \in X$.

In order to prove that $\tilde{I}_\phi$ is well defined we follow the idea of the above proof. In fact, using (5.28) for $\delta(x,y) \leq 2A\delta(x,x_0)$ and, on the other hand, (6.36), the fact that $s_\phi + \beta < \theta$, and (5.29) for $\delta(x,y) \geq 2A\delta(x,x_0)$, it is not hard to prove that

$$\int_X |K_\phi(x,y) - K_\phi(x_0,y)||f(y)|d\mu(y) < C\phi(\delta(x,x_0))(\xi(\delta(x,x_0))|f|_\xi + |f(x)|),$$

for every $x, x_0 \in X$.

To prove that $|I_\phi f|_{\phi\xi} \leq C|f|_\xi$ and $|\tilde{I}_\phi f|_{\phi\xi} \leq C|f|_\xi$ it is enough to consider $x_1, x_2 \in X$, $x_1 \neq x_2$, set $r = \delta(x_1,x_2)$ and show that there is a constant $C > 0$ such that

$$|\tilde{I}_\phi f(x_2) - \tilde{I}_\phi f(x_1)| = |I_\phi f(x_2) - I_\phi f(x_1)| \leq C|f|_\xi \xi(r)\phi(r), \tag{7.50}$$

where it must be understood that $f \in \Lambda^\xi$ for $\tilde{I}_\phi$ and $f \in \Lambda^\xi \cap L^1$ for I_ϕ.
By Lema 6.4 we can write

$$\tilde{I}_\phi f(x_2) - \tilde{I}_\phi f(x_1) = I_\phi f(x_2) - I_\phi f(x_1) = \int_X (f(y) - f(x_2))(K_\phi(x_2,y) - K_\phi(x_1,y))d\mu(y).$$

and the right member in the above equalities is bounded by

$$\begin{aligned}
&\int_{\delta(y,x_2)\le 2Ar} \frac{\phi(\delta(y,x_2))}{\delta(y,x_2)}|f(y)-f(x_2)|d\mu(y)\\
&\quad+\int_{\delta(y,x_2)\le 2Ar} \frac{\phi(\delta(y,x_1))}{\delta(y,x_1)}|f(y)-f(x_2)|d\mu(y)\\
&\quad+\int_{\delta(y,x_2)>2Ar} |K_\phi(x_2,y)-K_\phi(x_1,y)||f(y)-f(x_2)|d\mu(y)\\
&\quad= J_1+J_2+J_3. \qquad (7.51)
\end{aligned}$$

Let denote $B=B(x_2,2Ar)$ and B^c its complement. From the smoothness condition of f and since $\xi\phi$ is of positive lower type it holds that

$$J_1 \le C|f|_\xi \int_B \frac{\phi(\delta(y,x_2))}{\delta(y,x_2)}\xi(\delta(y,x_2))d\mu(y) \le C|f|_{\xi,}\xi(r)\phi(r) \qquad (7.52)$$

On the other, since $B\subset B(x_1,A(2A+1)r)$, ξ is quasi-increasing and ϕ is of positive lower type, it holds that

$$J_2 \le C|f|_\xi \int_B \frac{\phi(\delta(y,x_1))}{\delta(y,x_1)}\xi(\delta(y,x_2))d\mu(y) \le C|f|_\xi\xi(r)\phi(r) \qquad (7.53)$$

Finally, the smoothness conditions on the kernel and on f, the condition $\beta+s_\phi<\theta$ and Proposition 5.3 are used to get

$$J_3 \le C|f|_\xi r^\theta \int_{B^c} \frac{\phi(\delta(y,x_2))\xi(\delta(y,x_2))}{\delta(y,x_2)^{1+\theta}}d\mu(y) \le C|f|_\xi\xi(r)\phi(r).\diamondsuit \qquad (7.54)$$

Remarks **7.1.** *From inequality (7.49) it also follows that I_ϕ is a linear continuous operator from $M^{(\beta_1,\gamma_1)}$ to $\left(M^{(\beta_2,\gamma_2)}\right)'$, for every β_1,γ_1,β_2 and $\gamma_2>0$. More precisely, there is a finite constant C such that*
$|<I_\phi f,g>| \le C\|f\|_{M^{(\beta_1,\gamma_1)}}\|g\|_{M^{(\beta_2,\gamma_2)}}$, *for every pair $f\in M^{(\beta_1,\gamma_1)}$ and $g\in M^{(\beta_2,\gamma_2)}$ and, moreover, it holds that*

$$<I_\phi f,g>=<f,I_\phi g>=\int\int K_\phi(x,y)f(y)g(x)d\mu(y)d\mu(x). \qquad (7.55)$$

Proof of Theorem 4.2
By (6.43) we have

$$\begin{aligned}
&\int_X |K_{1/\phi}(x,y)||f(y)-f(x)|d\mu(y) \le C\int_{\delta(x,y)\le 1}\frac{|f(y)-f(x)|}{\phi(\delta(x,y))\delta(x,y)}d\mu(y)\\
&+C\int_{\delta(x,y)>1}\frac{|f(y)-f(x)|}{\phi(\delta(x,y))\delta(x,y)}d\mu(y)=I_1+I_2, \qquad (7.56)
\end{aligned}$$

Since $s_\phi<\alpha$, from (5.28) it follows that

$$I_1 \le C|f|_\xi \int_{\delta(x,y)\le 1}\frac{\xi(\delta(x,y))}{\phi(\delta(x,y))\delta(x,y)}d\mu(y) \le C|f|_\xi. \qquad (7.57)$$

Furthermore, since $i_\phi>0$ and $f\in L^\infty$, (5.29) leads to

$$I_2 \le 2C\|f\|_\infty \int_{\delta(x,y)>1}\frac{1}{\phi(\delta(x,y))\delta(x,y)}d\mu(y) \le C\|f\|_\infty, \qquad (7.58)$$

and thus, from (7.57) and (7.58),

$$\int_X |K_{1/\phi}(x,y)||f(y)-f(x)|d\mu(y) \le C\|f\|_\xi \;\text{ for every }\; x\in X, \qquad (7.59)$$

which implies that

$$\|D_\phi f\|_\infty \le C\|f\|_\xi \;\text{ for }\; s_\phi<\alpha \qquad (7.60)$$

To show that $\tilde{D}_\phi f(x)$ is absolutely convergent for $f \in \Lambda_\xi$ and $|\tilde{D}_\phi f|_{\xi/\phi} = |D_\phi f|_{\xi/\phi} \leq C|f|_\xi$ is enough to prove that

$$\int_X |K_{1/\phi}(x,y)(f(y)-f(x)) - K_{1/\phi}(x_0,y)(f(y)-f(x_0))|d\mu(y)$$
$$\leq C|f|_\xi \frac{\xi(\delta(x,x_0))}{\phi(\delta(x,x_0))}, \quad \text{for every } \ x, x_0 \in X. \tag{7.61}$$

Firstly, if $y \in B = B(x, 2A\delta(x,x_0))$ then $\delta(y,x_0) \leq A(2A+1)\delta(x,x_0)$, and proceeding as in (7.57), since $s_\phi < \alpha$, we have

$$\int_B |K_{1/\phi}(x,y)(f(y)-f(x)) - K_{1/\phi}(x_0,y)(f(y)-f(x_0))|d\mu(y) \leq C|f|_\xi \frac{\xi(\delta(x,x_0))}{\phi(\delta(x,x_0))}. \tag{7.62}$$

Moreover, by reordering the integrand, it follows that

$$\int_{B^c} K_{1/\phi}(x,y)(f(y)-f(x)) - K_{1/\phi}(x_0,y)(f(y)-f(x_0))|d\mu(y)$$
$$\leq \int_{B^c} K_{1/\phi}(x,y)|f(x_0)-f(x)|d\mu(y)$$
$$+ \int_{B^c} |f(y)-f(x_0)||K_{1/\phi}(x,y) - K_{1/\phi}(x_0,y)|d\mu(y) = J_1 + J_2. \tag{7.63}$$

From (5.29) and $i_\phi > 0$ it follows that

$$J_1 \quad \leq \quad C|f|_\xi \frac{\xi(\delta(x,x_0))}{\phi(\delta(x,x_0))}.$$

On the other hand, Proposition 6.45, the facts that if $y \in B^c$ then $\delta(y,x_0) \leq C\delta(x,y)$, ξ is quasi-increasing and finally, inequality (5.29), since $\beta < \theta + i_\phi$, lead to the bound

$$J_2 \leq C|f|_\xi \delta(x,x_0)^\theta \int_{B^c} \frac{\xi(\delta(x,y))}{\delta(x,y)^{1+\theta}\phi(\delta(x,y))} d\mu(y) \leq C|f|_\xi \frac{\xi(\delta(x,x_0))}{\phi(\delta(x,x_0))}.$$

We then arrived to inequality (7.61).$\diamondsuit$

REMARKS **7.2.** *Let ξ_i be a function of lower type α_i and upper type β_i for $i = 1,2$ and let $s_\phi < \alpha_1$ then*

$$< D_\phi f, g >= \iint K_{1/\phi}(x,y)(f(y)-f(x))g(x)d\mu(x)d\mu(y), \tag{7.64}$$

for any $f \in \Lambda^{\xi_1} \cap L^\infty$ and $g \in L^1$.
Furthermore, if $f \in \Lambda^{\xi_1} \cap L^\infty \cap L^1$, $g \in \Lambda^{\xi_2} \cap L^\infty \cap L^1$, and $s_\phi < \alpha_2$ then

$$< D_\phi f, g >=< D_\phi g, f > . \tag{7.65}$$

Indeed, by (7.60), if $f \in \Lambda^{\xi_1} \cap L^\infty$, with $s_\phi < \alpha_1$, then $D_\phi f \in L^\infty$ and $< D_\phi f, g >$ is well defined for $g \in L^1$ and the left side of (7.64) is absolutely convergent. The identity then follows from Fubini's theorem.
Moreover, we have that $| < D_\phi f, g > | \leq C\|f\|_\xi \|g\|_{L^1}$
Furthermore, if $f \in \Lambda^{\xi_1} \cap L^\infty \cap L^1$, $g \in \Lambda^{\xi_2} \cap L^\infty \cap L^1$, and $s_\phi < \alpha_2$, the previous argument also leads to the identity

$$< D_\phi g, f >= \iint K_{1/\phi}(x,y)(g(y)-g(x))f(x)d\mu(x)d\mu(y). \tag{7.66}$$

Therefore,

$$< D_\phi f, g > - < D_\phi g, f >= \iint K_{1/\phi}(x,y)(f(y)g(x) - f(x)g(y))d\mu(x)d\mu(y) = 0 \tag{7.67}$$

since the integrand $h(x,y)$ satisfies the condition $h(x,y) = -h(y,x)$ and
$\int\int h(x,y)d\mu(x)d\mu(y)$ is absolutely convergent.

REMARKS **7.3.** *Since $M^{(\beta,\gamma)} \subset \Lambda^\beta \cap L^\infty \cap L^1$, for any β and $\gamma > 0$, from Remark (7.2) follows that D_ϕ is a linear continuous operator from $M^{(\beta_1,\gamma_1)}$ in $(M^{(\beta_2,\gamma_2)})'$, for $s_\phi < \beta_1$, $\gamma_1, \gamma_2 > 0$ and $\beta_2 > 0$. Moreover, if also $s_\phi < \beta_2$ then $< D_\phi f, g >=< D_\phi g, f >$.*

8. Lemmas needed to prove Theorems 4.3, 4.4, 4.5 and 4.6

Seeking for the continuity of the operator I_ϕ on the generalized Besov and Triebel-Lizorkin spaces, a representation of the operator I_ϕ in terms of the Calderón-type reproduction formulas is needed. Let consider an approximation to the identity $\{S_k\}_{k\in\mathbb{Z}}$ of order $\epsilon \leq \theta$ and the family $\{D_k = S_k - S_{k-1}\}_{k\in\mathbb{Z}}$. Given $f \in M^{(\beta_1,\gamma_1)}$, with $0 < \beta_1 \leq 1$ and $\gamma_1 > 0$, by Theorem (5.1) it follows that

$$f = \lim_{M\to\infty} \sum_{|j|\leq M} D_j \hat{D}_j f, \tag{8.68}$$

where the series converges in $M^{(\beta',\gamma')}$ for every $\beta' < \beta_1$ and $\gamma' < \gamma_1$. Moreover, by Remark (7.1) I_ϕ is a linear continuous operator from $M^{(\beta',\gamma')}$
into $\left(M^{(\beta'',\gamma'')}\right)'$, for every $\beta'' > 0$ and $\gamma'' > 0$. Then, for $g \in M^{(\beta_2,\gamma_2)}$, $0 < \beta_2 \leq 1$ and $\gamma_2 > 0$, it holds that

$$< I_\phi f, g >= \lim_{M\to\infty} \sum_{|j|\leq M} < I_\phi D_j \hat{D}_j f, g > . \tag{8.69}$$

Choosing $\beta'' < \beta_2$ and $\gamma'' < \gamma_2$ and now applying the Theorem (5.2) it follows that

$$\begin{aligned} < I_\phi f, g > &= \lim_{M\to\infty} \lim_{N\to\infty} \sum_{|k|\leq N} \sum_{|j|\leq M} < \tilde{D}_k D_k I_\phi D_j \hat{D}_j f, g > \\ &= \lim_{M\to\infty} \lim_{N\to\infty} \sum_{|k|\leq N} \sum_{|j|\leq M} < D_k I_\phi D_j (\hat{D}_j f), \tilde{D}_k^* g >, \end{aligned}$$

It is easy to check that the kernel associated to the operator $I_{\phi,kj} = D_k I_\phi D_j$ is defined by

$$K_{\phi,kj}(x,y) =< D_k(x,.), I_\phi D_j(.,y) >= \int\int D_k(x,z) K_\phi(z,u) D_j(u,y) d\mu(u) d\mu(z), \tag{8.70}$$

and it satisfies

$$K_{\phi,kj}(x,y) = K_{\phi,jk}(y,x), \text{ for every } x,y \in X \text{ and } k,j \in \mathbb{Z}. \tag{8.71}$$

In analogous way, Remark (7.3) yields a representation of D_ϕ in terms of the Calderón-type reproduction formulas. Indeed, we get that

$$\begin{aligned} < D_\phi f, g > &= \lim_{M\to\infty} \lim_{N\to\infty} \sum_{|k|\leq N} \sum_{|j|\leq M} < \tilde{D}_k D_k D_\phi D_j \hat{D}_j f, g > \\ &= \lim_{M\to\infty} \lim_{N\to\infty} \sum_{|k|\leq N} \sum_{|j|\leq M} < D_k D_\phi D_j (\hat{D}_j f), \tilde{D}_k^* g >, \end{aligned}$$

for every pair of functions $f \in M^{(\beta_1,\gamma_1)}$ and $g \in M^{(\beta_2,\gamma_2)}$, with $s_\phi < \beta_1$ and $\beta_2, \gamma_1, \gamma_2 > 0$. The kernel associated to the operator $D_{\phi,kj} = D_k D_\phi D_j$ is given by

$$\begin{aligned} K_{1/\phi,kj}(x,y) &= < D_\phi D_j(.,y), D_k(x,.) > \\ &= \iint D_k(x,z) K_{1/\phi}(z,u)(D_j(u,y) - D_j(z,y)) d\mu(u) d\mu(z), \end{aligned} \tag{8.72}$$

which is well defined by Remark (7.2). Moreover, since the kernels $D_k(x,z)$ and $D_j(u,y)$ are symmetric, from (7.65) it follows that

$$K_{1/\phi,kj}(x,y) = K_{1/\phi,jk}(y,x). \tag{8.73}$$

A sharp bound for $K_{\phi,kj}(x,y)$ will be obtained in the following lemma.

Lemma **8.1.** *If ϕ is of positive lower type and of upper type $s_\phi < \epsilon \leq \theta$ then the kernel $K_{\phi,kj}$ satisfies the inequality*

$$|K_{\phi,kj}(x,y)| \leq C\phi((2A)^{-(k\vee j)}) \frac{(2A)^{-(k\vee j)(\epsilon - s_\phi)}}{((2A)^{-(k\wedge j)} + \delta(x,y))^{1+(\epsilon-s_\phi)}}. \tag{8.74}$$

where $a \vee b = \max(a,b)$ and $a \wedge b = \min(a,b)$.

PROOF: It is enough to consider the case $k \geq j$ since the other immediately follows from this by (8.71).
From (8.70) and as D_k has null mean in each variable, the kernel can be rewritten in the form

$$K_{\phi,kj}(x,y) = \int\int D_k(x,z)[K_\phi(z,u) - K_\phi(x,u)]D_j(u,y)d\mu(u)d\mu(z). \tag{8.75}$$

Let first consider the case $\delta(x,y) \leq 4A^2C(2A)^{-j}$.
Defining $\eta(t) \in \Lambda^\epsilon$, with $\eta(t) = 1$ if $|t| \leq A$ and $\eta(t) = 0$ if $|t| > 4A^2$, by Lemma (6.4) it holds that

$$\begin{aligned}
&K_{\phi,kj}(x,y) \\
&= \int\int D_k(x,z)\left(K_\phi(z,u) - K_\phi(x,u)\right)\left(D_j(u,y) - D_j(x,y)\right)\eta(\frac{\delta(x,u)}{(2A)^{-k}})d\mu(u)d\mu(z) \\
&+ \int\int D_k(x,z)\left(K_\phi(z,u) - K_\phi(x,u)\right)\left(D_j(u,y) - D_j(x,y)\right) \\
&\times \left(1 - \eta(\frac{\delta(x,u)}{(2A)^{-k}})\right) d\mu(u)d\mu(z) = D + B.
\end{aligned}$$

The first term D satisfies

$$\begin{aligned}
&|D| \leq \\
&C\int |D_k(x,z)| \int_{\delta(x,u)\leq(2A)^{-k+1}} |K_\phi(z,u) - K_\phi(x,u)||D_j(u,y) - D_j(x,y)|d\mu(u)d\mu(z) \\
&\leq C(2A)^{j(1+\epsilon)} \int |D_k(x,z)| \int_{\delta(x,u)\leq C(2A)^{-k}} (K_\phi(z,u) + K_\phi(x,u))\delta(x,u)^\epsilon d\mu(u)d\mu(z),
\end{aligned}$$

but if $\delta(x,z) \leq C(2A)^{-k}$ and $\delta(x,u) \leq C(2A)^{-k}$, then $\delta(z,u) \leq C(2A)^{-k}$. Moreover, since ϕ is of positive lower type, inequality (5.28) and the size condition (6.34) can be used to get

$$\begin{aligned}
&|D| \leq \\
&C(2A)^j(2A)^{-(k-j)\epsilon}\left(\int_{\delta(x,u)\leq C(2A)^{-k}} K_\phi(x,u)d\mu(u) + \int_{\delta(z,u)\leq C(2A)^{-k}} K_\phi(z,u)d\mu(u)\right) \\
&\leq C(2A)^j(2A)^{-(k-j)\epsilon}\phi((2A)^{-k}).
\end{aligned} \tag{8.76}$$

On the other hand,

$$\begin{aligned}
&|B| \leq \\
&\int\int_{\delta(x,u)\geq C(2A)^{-k+1}} |D_k(x,z)||K_\phi(z,u) - K_\phi(x,u)||D_j(u,y) - D_j(x,y)|d\mu(u)d\mu(z) \\
&= \left(\int\int_{C(2A)^{-k+1}\leq\delta(x,u)\leq C(2A)^{-j+1}} + \int\int_{\delta(x,u)\geq C(2A)^{-j+1}}\right) \\
&\qquad |D_k(x,z)||K_\phi(z,u) - K_\phi(x,u)||D_j(u,y) - D_j(x,y)|d\mu(u)d\mu(z) \\
&= B_1 + B_2.
\end{aligned}$$

As $\delta(x,u) \geq 2A\delta(x,z)$ for u in the domain of B_1 and B_2, Lemma 6.3 can be applied. Moreover, denoting $C_i = \{C(2A)^{-k+i} \leq \delta(x,u) \leq C(2A)^{-k+i+1}\}$, $i = 1, 2, \ldots$, since $s_\phi > 0$ and $\phi(t)/t$ is quasi-decreasing and $\epsilon \le \theta$ then

$$\begin{aligned} B_1 &\le C(2A)^{j(1+\epsilon)} \int |D_k(x,z)| \sum_{i=1}^{k-j} \int_{C_i} |K_\phi(z,u) - K_\phi(x,u)| \delta(x,u)^\epsilon d\mu(u) d\mu(z) \\ &\le C(2A)^{j(1+\epsilon)} \int |D_k(x,z)| \sum_{i=1}^{k-j} \int_{C_i} \delta(x,z)^\epsilon \frac{\phi(\delta(x,u))}{\delta(x,u)} d\mu(u) d\mu(z) \\ &\le C(2A)^{j(1+\epsilon)} (2A)^{-k\epsilon} \sum_{i=1}^{k-j} \phi((2A)^{-k+i}) \\ &\le C(2A)^j \phi((2A)^{-k}) (2A)^{-(k-j)\epsilon} \sum_{i=1}^{k-j} (2A)^{is_\phi} \\ &\le C(2A)^j \phi((2A)^{-k}) (2A)^{-(k-j)(\epsilon - s_\phi)}. \end{aligned}$$

On the other side, since $s_\phi < \epsilon \le \theta$, from (5.29) it follows that

$$\begin{aligned} B_2 &\le C(2A)^j \int |D_k(x,z)| \int_{\delta(x,u) \ge C(2A)^{-j+1}} |K_\phi(z,u) - K_\phi(x,u)| d\mu(u) d\mu(z) \\ &\le C(2A)^j \int |D_k(x,z)| \int_{\delta(x,u) \ge C(2A)^{-j+1}} \frac{\delta(x,z)^\epsilon}{\delta(x,u)^\epsilon} \frac{\phi(\delta(x,u))}{\delta(x,u)} d\mu(u) d\mu(z) \\ &\le C(2A)^j (2A)^{-(k-j)\epsilon} \phi((2A)^{-j}) \\ &\le C(2A)^j \phi((2A)^{-k}) (2A)^{-(k-j)(\epsilon - s_\phi)}. \end{aligned} \tag{8.77}$$

Nevertheless, since $t^\epsilon < t^{\epsilon - s_\phi}$ for $t < 1$ and $(2A)^{j(1+\epsilon-s_\phi)} \le C1/((2A)^{-j} + \delta(x,y))^{1+\epsilon-s_\phi}$ for $\delta(x,y) \le 4A^2(2A)^{-j}$, inequality (8.74) arises from (8.76) to (8.77).
Let now consider the case $\delta(x,y) \ge 4A^2C(2A)^{-j}$.
If $D_j(u,y) \ne 0$ then $\delta(u,y) < C(2A)^{-j}$ and thus, $\delta(x,u) \ge 2AC(2A)^{-j} > 2A\delta(x,z)$, moreover, the equivalence $\delta(x,u) \simeq (2A)^{-j} + \delta(x,y)$ holds. Therefore, using Lemma 6.3 and (5.26), from (8.75) it follows that

$$\begin{aligned} &|K_{\phi,kj}(x,y)| \\ &\quad \le C \int |D_k(x,z)| \int_{\delta(u,y) < C(2A)^{-j}} \frac{\delta(x,z)^\epsilon}{\delta(x,u)^{1+\epsilon}} \phi(\delta(x,u)) |D_j(u,y)| d\mu(u) d\mu(z) \\ &\quad \le C \int |D_k(x,z)| \int_{\delta(u,y) < C(2A)^{-j}} \frac{\delta(x,z)^{\epsilon - s_\phi}}{\delta(x,u)^{1+\epsilon - s_\phi}} \phi(\delta(x,z)) |D_j(u,y)| d\mu(u) d\mu(z) \\ &\quad \le C\phi\left((2A)^{-k}\right) \frac{(2A)^{-k(\epsilon - s_\phi)}}{((2A)^{-j} + \delta(x,y))^{1+\epsilon - s_\phi}}. \diamondsuit \end{aligned}$$

The next lemma follows easily from Lemma 8.1.

LEMMA **8.2.** *If ϕ is of positive lower type and of upper type $s_\phi < \epsilon$ then*

$$\int |K_{\phi,kj}(x,y)| d\mu(x) + \int |K_{\phi,kj}(x,y)| d\mu(y) \le C\phi((2A)^{-(k \vee j)}) (2A)^{-|k-j|(\epsilon - s_\phi)}.$$

An estimate of $I_{\phi,kj}$ in terms of the Hardy-Littlewood maximal operator derives from Lemma (8.1).

LEMMA **8.3.** *If ϕ is of positive lower type and of upper type $s_\phi < \epsilon$ then*

$$|I_{\phi,kj} h(x)| \le C\phi((2A)^{-(k \vee j)}) (2A)^{-|k-j|(\epsilon - s_\phi)} M|h|(x), \tag{8.78}$$

where M denotes the Hardy-Littlewood maximal operator.

PROOF: As in the proof of Lemma (8.1), it is enough to consider the case $k \geq j$. From that lemma it follows that

$$\begin{aligned}
&\int |K_{\phi,kj}(x,y)||h(y)|d\mu(y) \\
&\quad\leq C\phi((2A)^{-k}) \Bigg(((2A)^{-k(\epsilon-s_\phi)})(2A)^{j(\epsilon-s_\phi)+1} \int_{\delta(x,y)\leq 4A^2C(2A)^{-j}} |h(y)|d\mu(y) \\
&\quad+ \int_{\delta(x,y)>4A^2C(2A)^{-j}} \frac{(2A)^{-k(\epsilon-s_\phi)}}{\delta(x,y)^{(\epsilon-s_\phi)+1}}|h(y)|d\mu(y) \Bigg) = I_1 + I_2.
\end{aligned}$$

Clearly,

$$I_1 \leq C\phi((2A)^{-k})((2A)^{-(k-j)(\epsilon-s_\phi)})M(|h|)(x).$$

Finally, defining the sets $Q_i = \{y : C(2A)^{i-j} \leq \delta(x,y) \leq C(2A)^{i+1-j}\}$, $i = 2, 3, \dots$, since $s_\phi < \epsilon$ then

$$\begin{aligned}
&I_2 \leq C\phi((2A)^{-k}) \sum_{i=2}^{\infty} \int_{Q_i} \frac{(2A)^{-k(\epsilon-s_\phi)}}{\delta(x,y)^{1+(\epsilon-s_\phi)}}|h(y)|d\mu(y) \\
&\leq C\phi((2A)^{-k})(2A)^{-(k-j)(\epsilon-s_\phi)} \sum_{i=2}^{\infty} (2A)^{-i(\epsilon-s_\phi)}(2A)^{i-j} \int_{\delta(x,y)\leq C(2A)^{i+1-j}} |h(y)|d\mu(y) \\
&\leq C\phi((2A)^{-k})(2A)^{-(k-j)(\epsilon-s_\phi)} M|h|(x). \diamondsuit
\end{aligned}$$

Corresponding results are obtained for the kernel $K_{1/\phi,kj}$ and the operator $D_{\phi,kj}$ in the following lemmas

LEMMA **8.4.** *Let ϕ be of lower type $i_\phi > 0$ and upper type $s_\phi < \epsilon$. Then, there is a constant $C > 0$ such that*

$$|K_{1/\phi,kj}(x,y)| \leq C \frac{1}{\phi((2A)^{-(k\vee j)})} \frac{(2A)^{-(k\vee j)\epsilon}}{((2A)^{-(k\wedge j)} + \delta(x,y))^{1+\epsilon}},$$

where $a \vee b = \max(a,b)$ and $a \wedge b = \min(a,b)$.

PROOF: It is enough to consider the case $k \geq j$ since the other one immediately follows from this by (8.73). Let first consider the case $\delta(x,y) \leq 4A^2(2A)^{-j}$.
Since D_k has null mean in the z variable, the kernel defined in (8.72) can be rewritten as

$$\begin{aligned}
&K_{1/\phi,kj}(x,y) = \\
&\quad\iint D_k(x,z)[K_{1/\phi}(z,u)(D_j(u,y) - D_j(z,y)) \\
&\qquad -K_{1/\phi}(x,u)(D_j(u,y) - D_j(x,y))]d\mu(u)d\mu(z).
\end{aligned}$$

Fix $\eta(t) \in \Lambda_0^\epsilon(\mathbb{R})$, such that $\eta(t) = 1$ for $|t| \leq A$ and $\eta = 0$, for $|t| \geq 2A$. Then,

$$\begin{aligned}
&K_{1/\phi,kj}(x,y) \\
&= \iint D_k(x,z) \left(K_{1/\phi}(z,u)(D_j(u,y) - D_j(z,y)) - K_{1/\phi}(x,u)(D_j(u,y) - D_j(x,y))\right) \\
&\quad \times\eta(\frac{\delta(x,u)}{(2A)^{-k}})d\mu(u)d\mu(z) \\
&+ \iint D_k(x,z) \left(K_{1/\phi}(z,u)(D_j(u,y) - D_j(z,y)) - K_{1/\phi}(x,u)(D_j(u,y) - D_j(x,y))\right) \\
&\quad \times(1 - \eta(\frac{\delta(x,u)}{(2A)^{-k}}))d\mu(u)d\mu(z) = D + B.
\end{aligned}$$

First notice that if $\delta(x,z) \leq C(2A)^{-k}$ and $\delta(x,u) \leq C(2A)^{-k}$, then $\delta(z,u) \leq CA(2A)^{-k}$. Therefore, from (2.10), applied to D_j, and (5.28) it follows that

$$\begin{aligned} |D| &\leq \int |D_k(x,z)|(\int_{\delta(z,u)\leq CA(2A)^{-k}} |K_{1/\phi}(z,u)|(2A)^{j(1+\epsilon)}\delta(z,u)^\epsilon d\mu(u)d\mu(z) \\ &\quad + \int |D_k(x,z)|(\int_{\delta(x,u)\leq c(2A)^{-k}} |K_{1/\phi}(x,u)|(2A)^{j(1+\epsilon)}\delta(x,u)^\epsilon d\mu(u)d\mu(z) \\ &\leq C(2A)^{j(1+\epsilon)}\frac{(2A)^{-k\epsilon}}{\phi((2A)^{-k})}. \end{aligned} \tag{8.79}$$

On the other hand, it holds that

$$\begin{aligned} |B| \leq &\iint_{\delta(x,u)\geq(2A)^{-k+1}} |D_k(x,z)| \\ &\big(|K_{1/\phi}(z,u) - K_{1/\phi}(x,u)||D_j(u,y) - D_j(x,y)| \\ &+K_{1/\phi}(z,u)|D_j(x,y) - D_j(z,y)|\big)\, d\mu(u)d\mu(z) = B_1 + B_2. \end{aligned}$$

But, if $D_k(x,z) \neq 0$ and $\delta(x,u) \geq (2A)^{-k+1}$ then $\delta(z,u) \geq C(2A)^{-k}$. Moreover, since $i_\phi > 0$, from (6.44), (2.10) and (5.29), we deduce that

$$\begin{aligned} B_2 &\leq C(2A)^{j(1+\epsilon)}\int |D_k(x,z)| \int_{\delta(z,u)\geq c(2A)^{-k}} \frac{\delta(z,x)^\epsilon}{\phi(\delta(z,u))\delta(z,u)} d\mu(u)d\mu(z) \\ &\leq C(2A)^j(2A)^{-(k-j)\epsilon}\int |D_k(x,z)| \int_{\delta(z,u)\geq C(2A)^{-k}} \frac{1}{\phi(\delta(z,u))\delta(z,u)} d\mu(u)d\mu(z) \\ &\leq C\frac{(2A)^{j(1+\epsilon)}(2A)^{-k\epsilon}}{\phi((2A)^{-k})}. \end{aligned} \tag{8.80}$$

We now split B_1 in the form

$$\begin{aligned} B_1 &\leq \left(\iint_{(2A)^{-k+1}\leq\delta(x,u)\leq(2A)^{-j+1}} + \iint_{\delta(x,u)\geq(2A)^{-j+1}}\right) \\ &\quad |D_k(x,z)||K_{1/\phi}(z,u) - K_{1/\phi}(x,u)||D_j(u,y) - D_j(x,y)|d\mu(u)d\mu(z) \\ &= B_{1,1} + B_{1,2}. \end{aligned}$$

Since $\delta(x,z) \leq C(2A)^{-k}$ and $\delta(x,u) \geq 2A\delta(x,z)$, smoothness conditions (6.45) and (2.10) and, also, (2.3) lead to the bound

$$\begin{aligned} B_{1,1} &\leq C(2A)^{j(1+\epsilon)} \times \\ &\quad \int |D_k(x,z)| \int_{(2A)^{-k+1}\leq\delta(x,u)\leq(2A)^{-j+1}} \delta(x,u)^\epsilon \frac{\delta(x,z)^\epsilon}{\delta(x,u)^{1+\epsilon}}\frac{1}{\phi(\delta(x,u))} d\mu(u)d\mu(z) \\ &\leq C(2A)^j(2A)^{-(k-j)\epsilon}\sum_{i=1}^{k-j}\int_{(2A)^{-k+i}\leq\delta(x,u)\leq(2A)^{-k+i+1}} \frac{1}{\phi(\delta(x,u))\delta(x,u)} d\mu(u) \\ &\leq C(2A)^j(2A)^{-(k-j)\epsilon}\frac{1}{\phi((2A)^{-k})}\sum_{i=1}^{k-j}(2A)^{-ii_\phi} \\ &\leq C\frac{(2A)^j(2A)^{-(k-j)\epsilon}}{\phi((2A)^{-k})}\sum_{i=1}^{\infty}(2A)^{-ii_\phi} \leq C\frac{(2A)^j(2A)^{-(k-j)\epsilon}}{\phi((2A)^{-k})}. \end{aligned} \tag{8.81}$$

On the other hand, using (6.45), (2.9), (5.29) and the fact that ϕ is quasi-increasing, we obtain

$$\begin{aligned} B_{1,2} &\leq C(2A)^j\int |D_k(x,z)| \int_{\delta(x,u)\geq(2A)^{-j+1}} \frac{\delta(x,z)^\epsilon}{\delta(x,u)^{1+\epsilon}}\frac{1}{\phi(\delta(x,u))} d\mu(u)d\mu(z) \\ &\leq C(2A)^j(2A)^{-k\epsilon}\frac{1}{(2A)^{-j\epsilon}\phi((2A)^{-j})} \leq C(2A)^j(2A)^{-(k-j)\epsilon}\frac{1}{\phi((2A)^{-k})}. \end{aligned} \tag{8.82}$$

From inequalities (8.79), (8.80), (8.81) and (8.82), we conclude that
if $\delta(x,y) \le 4A^2(2A)^{-j}$ then

$$|K_{1/\phi,kj}(x,y)| \le C(2A)^{j(1+\epsilon)}(2A)^{-k\epsilon}\frac{1}{\phi((2A)^{-k})} \le C\frac{(2A)^{-k\epsilon}}{((2A)^{-j}+\delta(x,y))^{1+\epsilon}}\frac{1}{\phi((2A)^{-k})}.$$

To finish the proof, we consider the case $\delta(x,y) \ge C4A^2(2A)^{-j}$.
Notice that if $\delta(x,z) \le C(2A)^{-k}$ then $\delta(z,y) \ge CA(2A)^{-j}$ and therefore $D_j(z,y) = 0$. Moreover, the condition $\int D_k(x,z)d\mu(z) = 0$ enables us to rewrite the kernel in (8.72) in the form

$$K_{1/\phi,kj}(x,y) = \int D_k(x,z)\int (K_{1/\phi}(z,u) - K_{1/\phi}(x,u))D_j(u,y)d\mu(u)d\mu(z).$$

But, also, since $\delta(u,y) \le C(2A)^{-j}$ then $\delta(x,u) \ge C(2A)^{-j} \ge C(2A)^{-k} \ge 2A\delta(x,z)$ and $\delta(x,u) \ge C(\delta(x,y) + (2A)^{-j})$. Therefore, from (6.45) and the fact that $\phi(t)$ is quasi-increasing, we deduce that

$$\begin{aligned}
&|K_{1/\phi,kj}(x,y)|\\
\le\ & \int |D_k(x,z)| \int_{\delta(u,y)<(2A)^{-j}} \frac{\delta(x,z)^\epsilon |D_j(u,y)|}{\delta(x,u)^{1+\epsilon}\phi(\delta(x,u))}d\mu(u)d\mu(z)\\
\le\ & C\frac{(2A)^{-k\epsilon}}{((2A)^{-j}+\delta(x,y))^{1+\epsilon}}\frac{1}{\phi((2A)^{-k})}\int |D_k(x,z)|\int |D_j(u,y)|d\mu(u)d\mu(z)\\
\le\ & C\frac{(2A)^{-k\epsilon}}{((2A)^{-j}+\delta(x,y))^{1+\epsilon}}\frac{1}{\phi((2A)^{-k})}.\diamondsuit
\end{aligned}$$

The proof of the following two lemmas are similar to those given for the integral of order ϕ and so they will be omitted.

Lemma 8.5. *If ϕ is of lower type $i_\phi > 0$ and upper type $s_\phi < \epsilon$, then there is a constant $C > 0$ such that*

$$\int |K_{1/\phi,kj}(x,y)|d\mu(x) + \int |K_{1/\phi,kj}(x,y)|d\mu(y) \le C\frac{(2A)^{-|k-j|\epsilon}}{\phi((2A)^{-(k\vee j)})}. \tag{8.83}$$

Lemma 8.6. *If ϕ is of lower type $i_\phi > 0$ and upper type $s_\phi < \epsilon$, then there is a constant $C > 0$ such that*

$$|D_{\phi,kj}h(x)| \le C\frac{(2A)^{-|k-j|\epsilon}}{\phi((2A)^{-(k\vee j)})}M|h|(x), \tag{8.84}$$

where M denotes the Hardy-Littlewood maximal operator.

9. Proof of Theorems 4.3, 4.4, 4.5, and 4.6

If $\max(s_1, s_2) < \epsilon$ then the space $M^{(\epsilon,\epsilon)}$ is dense in $\dot{F}_p^{\psi,q}$ and $\dot{B}_p^{\psi,q}$ and hence, in all the theorems, it is enough to prove the boundedness of the operators on such molecules. Proof of Theorem 4.3
For $f \in M^{(\epsilon,\epsilon)}$, by using (8.69) we obtain

$$\begin{aligned}
\|I_\phi f\|_{\dot{F}_p^{\phi\psi,q}} &= \| \left(\sum_{k\in Z}\left(\frac{1}{\phi((2A)^{-k})\psi((2A)^{-k})}|D_k I_\phi f)|\right)^q\right)^{1/q} \|_p\\
&\le \| \left(\sum_{k\in Z}\left(\frac{1}{\phi((2A)^{-k})\psi((2A)^{-k})}\sum_{j\in Z}|D_k I_\phi D_j(\hat{D}_j f)|\right)^q\right)^{1/q} \|_p\\
&\le \| \left(\sum_{k\in Z}\left(\frac{1}{\phi((2A)^{-k})\psi((2A)^{-k})}\sum_{j\le k}|I_{\phi,kj}(\hat{D}_j f)|\right)^q\right)^{1/q}\\
&+ \left(\sum_{k\in Z}\left(\frac{1}{\phi((2A)^{-k})\psi((2A)^{-k})}\sum_{j>k}|I_{\phi,k,j}(\hat{D}_j f)|\right)^q\right)^{1/q} \|_p = \|S_1 + S_2\|_p.
\end{aligned}$$

First notice that as ψ_2 is quasi-increasing and ψ_1 is of upper-type s_1, for $k \geq j$ it holds that

$$\frac{1}{\psi((2A)^{-k})} = \frac{\psi_2((2A)^{-k})}{\psi_1((2A)^{-k})} \leq C(2A)^{(k-j)s_1}\frac{\psi_2((2A)^{-j})}{\psi_1((2A)^{-j})} = C\frac{(2A)^{(k-j)s_1}}{\psi((2A)^{-j})}, \tag{9.85}$$

Also, since ψ_1 is quasi-increasing and ψ_2 is of upper-type s_2 then, for $k < j$,

$$\frac{1}{\psi((2A)^{-k})} = \frac{\psi_2((2A)^{-k})}{\psi_1((2A)^{-k})} \leq C(2A)^{(j-k)s_2}\frac{\psi_2((2A)^{-j})}{\psi_1((2A)^{-j})} = C\frac{(2A)^{(j-k)s_2}}{\psi((2A)^{-j})}, \tag{9.86}$$

Therefore, applying (8.78) and then (9.85) it follows that

$$\begin{aligned} S_1(x) &\leq \left(\sum_{k\in Z}\left(\sum_{j\leq k}\frac{(2A)^{-(k-j)(\epsilon-s_\phi-s_1)}}{\psi((2A)^{-j})}M|\hat{D}_j f|(x)\right)^q\right)^{1/q} \\ &= \left(\sum_{k\in Z}\left(\sum_{j\geq 0}\frac{(2A)^{-j(\epsilon-s_\phi-s_1)}}{\psi((2A)^{-(k-j)})}M|\hat{D}_{k-j} f|(x)\right)^q\right)^{1/q}. \end{aligned} \tag{9.87}$$

On the other hand, using (8.78), (9.86) and inequality

$$\phi((2A)^{-j}) \leq C(2A)^{-(j-k)i_\phi}\phi((2A)^{-k}), \quad \text{for } j > k, \tag{9.88}$$

it follows that

$$S_2(x) \leq \left(\sum_{k\in Z}\left(\sum_{j>k}\frac{(2A)^{-(j-k)(\epsilon-s_\phi+i_\phi-s_2)}}{\psi((2A)^{-j})}M|\hat{D}_j f|(x)\right)^q\right)^{1/q} \tag{9.89}$$

From Minkowski's inequality and the hypothesis $s_\phi + s_1 < \epsilon$ for (9.87), and $s_\phi - i_\phi + s_2 < \epsilon$ for (9.89), it follows that

$$S_1(x) + S_2(x) \leq C\left(\sum_{k\in Z}\left(\frac{M|\hat{D}_k f|(x)}{\psi((2A)^{-k})}\right)^q\right)^{1/q} \tag{9.90}$$

for every $x \in X$. Since $1 < p, q < \infty$, we are able to apply the Fefferman-Stein vector valued maximal inequality to get that

$$\|S_1 + S_2\|_p \leq C\|\left(\sum_{k\in Z}\left(\frac{|\hat{D}_k f|}{\psi((2A)^{-k})}\right)^q\right)^{1/q}\|_p \leq C\|f\|_{\dot{F}_p^{\psi,q}}.\diamondsuit$$

Proof of Theorem 4.4

For $f \in M^{(\epsilon,\epsilon)}$, by (8.69), it holds that

$$\begin{aligned} \|I_\phi f\|_{\dot{B}_p^{\phi\psi,q}} &\leq \left(\sum_{k\in Z}\left(\frac{1}{\phi((2A)^{-k})\psi((2A)^{-k})}\sum_{j\in Z}\|D_k I_\phi D_j(\hat{D}_j f)\|_p\right)^q\right)^{1/q} \\ &\leq \left(\sum_{k\in Z}\left(\frac{1}{\phi((2A)^{-k})\psi((2A)^{-k})}\sum_{j\leq k}\|I_{\phi,k,j}\|_{p,p}\|(\hat{D}_j f)\|_p\right)^q\right)^{1/q} \\ &+ \left(\sum_{k\in Z}\left(\frac{1}{\phi((2A)^{-k})\psi((2A)^{-k})}\sum_{j>k}\|I_{\phi,k,j}\|_{p,p}\|(\hat{D}_j f)\|_p\right)^q\right)^{1/q} = S_1 + S_2. \end{aligned}$$

Nevertheless, from Lemma 8.2, it follows that

$$\|I_{\phi,k,j}\|_{p,p} \leq C\phi((2A)^{-(k\vee j)})(2A)^{-|k-j|(\epsilon-s_\phi)}. \tag{9.91}$$

In fact, for $1<p<\infty$, it holds that

$$\begin{aligned}\|I_{\phi,kj}h\|_p &\leq \left(\int\left(\int |K_{\phi,kj}(x,y)||h(y)|d\mu(y)\right)^p d\mu(x)\right)^{1/p}\\ &\leq \left(\int\left(\int |K_{\phi,kj}(x,y)|d\mu(y)\right)^{p/p'}\left(\int |K_{\phi,kj}(x,y)||h(y)|^p d\mu(y)\right)d\mu(x)\right)^{1/p};\end{aligned} \tag{9.92}$$

and, for $p=1$,

$$\|I_{\phi,kj}h\|_1 \leq \int\int |K_{\phi,kj}(x,y)||h(y)|d\mu(y)d\mu(x). \tag{9.93}$$

Then applying Lemma (8.2) in (9.92) and (9.93), it follows that

$$\begin{aligned}&\|I_{\phi,kj}h\|_p\\ &\quad\leq C\left(\phi((2A)^{-(k\vee j)}(2A)^{-|k-j|(\epsilon-s_\phi)})\right)^{1/p'}\left(\int\int |K_{\phi,kj}(x,y)||h(y)|^p d\mu(y)d\mu(x)\right)^{1/p}\\ &\quad\leq C\phi((2A)^{-(k\vee j)})(2A)^{-|k-j|(\epsilon-s_\phi)}\|h\|_p,\end{aligned}$$

for $1<p<\infty$, and

$$\begin{aligned}\|I_{\phi,kj}h\|_1 &\leq C\phi((2A)^{-(k\vee j)})(2A)^{-|k-j|(\epsilon-s_\phi)}\int |h(y)|d\mu(y)\\ &= C\phi((2A)^{-(k\vee j)})(2A)^{-|k-j|(\epsilon-s_\phi)}\|h\|_1.\end{aligned}$$

for $p=1$. Thus inequality (9.91) results. Substituting it in S_1 and using (9.85), it follows that

$$S_1 \leq C\left(\sum_{k\in Z}\left(\sum_{j\leq k}(2A)^{-(k-j)(\epsilon-s_\phi-s_1)}\frac{\|(\hat{D}_j f)\|_p}{\psi((2A)^{-j})}\right)^q\right)^{1/q} \tag{9.94}$$

On the other hand, using (9.91), (9.85) and (9.88) it follows that

$$S_2 \leq C\left(\sum_{k\in Z}\left(\sum_{j>k}(2A)^{-(j-k)(\epsilon-s_\phi+i_\phi-s_2)}\frac{\|(\hat{D}_j f)\|_p}{\psi((2A)^{-j})}\right)^q\right)^{1/q}. \tag{9.95}$$

For $1\leq q<\infty$, Minkowski's inequality and conditions $s_\phi+s_1<\epsilon$ for (9.94) and $s_\phi-i_\phi+s_2<\epsilon$ for (9.95) lead to the bound

$$S_1+S_2\leq C\|f\|_{\dot{B}_p^{\psi,q}}.$$

Proof of Theorem 4.5

For $f\in M^{(\epsilon,\epsilon)}$, proceeding as in the above proofs and applying (8.72) it follows that

$$\begin{aligned}\|D_\phi f\|_{\dot{F}_p^{\psi/\phi,q}} &\leq \Bigg\| \left(\sum_{k\in Z}\left(\frac{\phi((2A)^{-k})}{\psi((2A)^{-k})}\sum_{j\leq k}|D_{\phi,kj}(\hat{D}_j f)|\right)^q\right)^{1/q}\\ &+\left(\sum_{k\in Z}\left(\frac{\phi((2A)^{-k})}{\psi((2A)^{-k})}\sum_{j>k}|D_{\phi,kj}(\hat{D}_j f)|\right)^q\right)^{1/q}\Bigg\|_p = \|S_1(x)+S_2(x)\|_p.\end{aligned}$$

Using (8.84) and (9.85) it follows that

$$S_1(x)\leq C\left(\sum_{k\in Z}\left(\sum_{j\leq k}(2A)^{-(k-j)(\epsilon-s_1)}\frac{M|\hat{D}_j f|(x)}{\psi((2A)^{-j})}\right)^q\right)^{1/q} \tag{9.96}$$

On the other side, again using (8.84) and inequalities (9.86) and

$$\phi((2A)^{-k})\leq C(2A)^{(j-k)s_\phi}\phi((2A)^{-j}),\quad\text{for}\quad k<j,$$

then

$$S_2(x) \leq C \left(\sum_{k \in Z} \left(\sum_{j>k} (2A)^{-(j-k)(\epsilon - s_\phi - s_2)} \frac{M|\hat{D}_j f|(x)}{\psi((2A)^{-j})} \right)^q \right)^{1/q} \tag{9.97}$$

From Minkowski's inequality and the hipothesis $s_1 < \epsilon$ for (9.96) and $s_\phi + s_2 < \epsilon$ for (9.97) it follows that

$$S_1(x) + S_2(x) \leq C \left(\sum_{k \in Z} \left(\frac{M|\hat{D}_k f|(x)}{\psi((2A)^{-k})} \right)^q \right)^{1/q}.$$

From the Fefferman–Stein vector valued maximal inequality, for $1 < p, q < \infty$, it follows that

$$\|S_1 + S_2\|_p \leq C \| \left(\sum_{k \in Z} \left(\frac{|\hat{D}_k f|(x)}{\psi((2A)^{-k})} \right)^q \right)^{1/q} \|_p \leq C \|f\|_{\dot{F}_p^{\psi,q}}. \diamondsuit$$

Since the proof of Theorem 4.6 is similar to the previous ones it is ommited.

References

[B] O. Blasco *Weighted Lipschitz Spaces Defined by a Banach Space*, García-Cuerva, J. et al. Fourier Analysis and Partial Differential Equations. CRC, 1995, Ch.7.

[FJW] M.Frazier, B. Jawerth y G. Weiss, *Littlewood–Paley theory and the study of function spaces*, CBMS, Regional Conference Series in Math., No. 79, 1991.

[GSV] A.E. Gatto, C. Segovia y S.Vági, *On fractional differentiation and integration on spaces of homogeneous type* Rev. Mat. Iberoamericana, Vol.12, n.2, 1996, 111–145.

[GV] A.E. Gatto y S.Vági, *On Sobolev Spaces of Fractional Order and ϵ–Families of Operators on Spaces of Homogeneous Type*, Studia Math 133 (1), 1999, 19–27.

[H] S.I.Hartzstein, *Acotación de operadores de Calderón–Zygmund en espacios de Triebel–Lizorkin y de Besov generalizados sobre espacios de tipo homogéneo.* Thesis, 2000, UNL, Santa Fe, Argentina.

[HV] S.I.Hartzstein y B.E.Viviani, $T1$ *theorems on generalized Besov and Triebel-Lizorkin spaces over spaces of homogeneous type*, (to appear in Revista de la Unión Matemática Argentina 2001)

[HS] Y.-S. Han, E.T.Sawyer, *Littlewood-Paley theory on spaces of homogeneous type and the classical function spaces*, Memoirs of the American Mathematical Society, Vol.110, N.530, 1994.

[I] B. Iaffei. *Espacios Lipschitz generalizados y operadores invariantes por traslaciones.* Thesis, UNL, 1996.

[J] S. Janson, *Generalization on Lipschitz spaces and applications to Hardy spaces and bounded mean oscillation*, Duke Math. J. 47, 1980, 959-982.

[MS] R.A. Macías y C. Segovia, *Lipschitz functions on spaces of homogeneous type*, Adv. in Math. 33, 1979, 257–270.

Departamento de Matemática, Facultad de Ingeniería Química, Universidad Nacional del Litoral and IMAL, CONICET, Santiago del Estero 2829, 3000 Santa Fe, Argentina

E-mail address: shrtzstn@fiqus.unl.edu.ar, viviani@ceride.gov.ar