Mecánica Computacional Vol. XXI, pp. 2712–2731
S. R. Idelsohn, V. E. Sonzogni and A. Cardona (Eds.)
Santa Fe–Paraná, Argentina, October 2002

AN EFFICIENT FINITE DIFFERENCE SENSITIVITY ANALYSIS METHOD ALLOWING REMESHING IN LARGE DEFORMATION PROBLEMS

Pablo A. Muñoz-Rojas[*,♰], Jun S. O. Fonseca[♰] and Guillermo J. Creus[♰,✧]

[*] Departamento de Engenharia Mecânica, UDESC, Joinville, SC
e-mail: dem2pmr@joinville.udesc.br

[♰] PROMEC – PROMEC-UFRGS,Porto Alegre, RS
e-mail: jun@ufrgs.br

[✧] CEMACOM – UFRGS, Porto Alegre, RS
e-mail: creus@ufrgs.br

Key words: Sensitivity Analysis, Non-linear Finite Element Analysis, Remeshing.

Abstract: *The use of finite difference sensitivity analysis in large scale industrial problems is frequently avoided due to its high computational cost and approximation errors when compared to the analytical approach. Although in iteratively solved problems the high cost may be greatly reduced, a major problem of the usual finite difference approach is that it fails when remeshing is required. This happens because the errors caused by parametric inversion and interpolation in variables transfer from the old to the new mesh may have the same order of magnitude than the sought gradient components. This paper presents an efficient finite difference approach that allows remeshing, not being affected by the mentioned errors. The low cost and the accuracy of the sensitivity fields obtained after remeshing are shown in two examples, with shape and constitutive design variables respectively. Although it is presented in the frame of large strain elastoplasticity with linear isotropic hardening, the method opens the possibility of dealing in a simple manner with more complex material and contact laws including thermo-mechanical coupling, damage, etc.*

1. INTRODUCTION

Sensitivity analysis is an important tool for decision making in engineering practice and a necessary step in optimization procedures and reliability analyses. Efficient methodologies for sensitivity determination in linear problems are well established at least since the 80's (Haug, Choi and Komkov[1]). In nonlinear analysis different additional issues must be taken into consideration, depending on the type of the nonlinearity (Kleiber et al.[2]). Both references are centered in analytical sensitivity methods.

In nonlinear problems such as those involving large strain elastoplasticity with frictional contact, the computation of each displacement increment depends on the constitutive model and the stress integration algorithm employed. This dependence affects directly the displacements, stresses and internal variables history. Hence, in order to perform the displacements sensitivity analysis at the *n-th* load increment, it is necessary to differentiate displacements, stresses and internal variables at load increment *n-1*.

The analytical sensitivity analysis may clearly be difficult when non-differentiable functions are involved and when complex constitutive laws are considered, especially in large strain situations. The difficulty increases when the design variables are the shape parameters that define the integration domain and when boundary conditions change along the process.

There are three main approaches for sensitivity analysis in nonlinear problems. One is the analytical method, efficient and accurate but frequently limited to particular cases. Another one is the finite difference method which is very simple and general but usually has a much higher computational cost. Moreover, in large strain situations which require remeshing, the finite difference method has been shown not to give acceptable results[3,4], whereas the analytical method still provides very accurate ones. The third is the semi-analytical aproach which is cheaper but still suffers the remeshing limitation.

This work presents a very simple strategy that reduces considerably the time spent by the finite difference method and solves its limitation in dealing with remeshing situations, resulting in very accurate sensitivity fields. In this way, the proposed procedure is very efficient and comprehensive and thus adequate for any kind of sensitivity determination either in path-dependent or path-independent problems.

The implementation used the METAFOR® finite element code[5], originally developed at the University of Liège and available at CEMACOM – UFRGS through a cooperation agreement. It uses a Lagrangian – or Arbitrary Lagrangian Eulerian - description elastic-(visco)plastic incremental approach and applies frictional contact by penalization. Different material and contact models are available. The stress integration algorithm uses the combination of the method of instantaneous rotation[6] and radial return.

2. FINITE DIFFERENCE SENSITIVITY METHOD IN NONLINEAR ANALYSES WITHOUT REMESHING

A path-dependent nonlinear structural problem may be posed by a set of incremental equations of the type

$$\boldsymbol{f}^{int}\left(\boldsymbol{u}^{n},\boldsymbol{U}^{n-1},\boldsymbol{b}\right)=\mu^{n}\boldsymbol{f}^{ext}\left(\boldsymbol{b}\right) \quad (1)$$

$$\boldsymbol{U}^n = \sum_{i=1}^{n} \boldsymbol{u}^i \tag{2}$$

established for the *n-th* load increment. In Eq. (1), $\boldsymbol{f}^{int}$ is the vector of internal forces, which is function of the displacement increment vector $\boldsymbol{u}^n$, the displacement vector $\boldsymbol{U}^{n-1}$ and the design variables vector $\boldsymbol{b}$, $\boldsymbol{f}^{ext}$ is the vector of external loads and μ^n is an amplitude parameter. Due to the dependence of Eq. (1) on $\boldsymbol{u}^n$, it must be solved iteratively. Using the Newton-Raphson method one has

$$\boldsymbol{f}^{int}\left(\boldsymbol{u}_1^n, \boldsymbol{U}^{n-1}, \boldsymbol{b}\right) \approx \boldsymbol{f}^{int}\left(\boldsymbol{u}_0^n, \boldsymbol{U}^{n-1}, \boldsymbol{b}\right) + \boldsymbol{K}_T\left(\boldsymbol{u}_1^n - \boldsymbol{u}_0^n\right) = \mu^n \boldsymbol{f}^{ext}\left(\boldsymbol{b}\right) \tag{3}$$

where $\boldsymbol{u}_0^n$ is the initial displacement increment estimate and

$$\boldsymbol{K_T} = \frac{\partial \boldsymbol{f}^{int}\left(\boldsymbol{u}_0^n, \boldsymbol{b}\right)}{\partial \boldsymbol{u}^n} \tag{4}$$

is the tangential stiffness matrix.

Hence,

$$\boldsymbol{u}_{k+1}^n = \boldsymbol{u}_k^n + \boldsymbol{K}_T^{-1}\left[\mu^n \boldsymbol{f}^{ext}\left(\boldsymbol{b}\right) - \boldsymbol{f}^{int}\left(\boldsymbol{u}_k^n, \boldsymbol{U}^{n-1}, \boldsymbol{b}\right)\right] = \boldsymbol{u}_k^n - \boldsymbol{K}_T^{-1}\boldsymbol{R}_k^n \tag{5}$$

where $\boldsymbol{R}_k^n$ is the residual of the equilibrium equations at the *n-th* step and *k-th* iteration. The solution is obtained when the residual $\boldsymbol{R}_k^n$ reaches a prescribed convergence tolerance.

In order to obtain the desired sensitivity fields, one possibility is to employ overall finite differences. In this case the equilibrium equations (1) are perturbed with respect to the *j-th* design variable, resulting

$$\boldsymbol{f}^{int}\left(\boldsymbol{u}_\Delta^n, \boldsymbol{U}_\Delta^{n-1}, \boldsymbol{b} + \delta\boldsymbol{b}_j\right) = \mu^n \boldsymbol{f}^{ext}\left(\boldsymbol{b} + \delta\boldsymbol{b}_j\right) \tag{6}$$

$$\boldsymbol{U}_\Delta^n = \sum_{i=1}^{n} \boldsymbol{u}_\Delta^i \tag{7}$$

where

$$\delta\boldsymbol{b}_j = \begin{bmatrix} b_1 & \cdots & \delta b_j & \ldots & b_{nvp} \end{bmatrix}^T \tag{8}$$

δb_j is the perturbation and *nvp* is the number of design variables. The gradients are then evaluated by the approximation

$$\frac{d\boldsymbol{U}^n}{db_j} \approx \frac{\boldsymbol{U}_\Delta^n - \boldsymbol{U}^n}{\delta b_j}. \tag{9}$$

The numerical perturbations must not be too small nor too large in order to give acceptable results. Too small perturbations have their effect polluted by the computer finite precision residuals. Too large perturbations provide a secant rather than a tangent approximation[7].

In nonlinear problems there exist additional inaccuracy sources because of the iterative nature of the solution procedures and the associated residuals. Fig. 1 depicts the behavior of a hypothetical displacement U_j as a function of a design variable b_k. The red lines limit the maximum deviation allowed, as a consequence of the prescribed residual tolerance. Note that at the unperturbed situation, either U_j^1 or U_j^2 are acceptable solutions, situated inside the tolerance range. The same occurs with $U_{j\Delta}^1$ and $U_{j\Delta}^2$ relative to the perturbed situation. Nevertheless, it is possible to obtain completely different numerically computed gradients, some of them completely wrong. This situation is worse when the perturbation is smaller and the tolerance is larger[8,9].

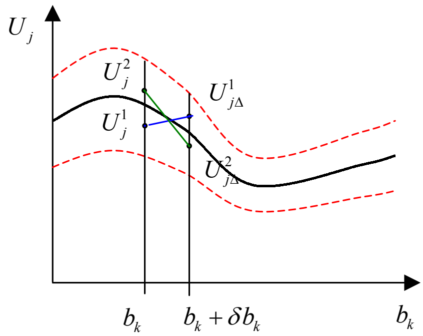
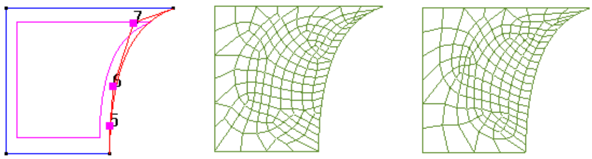


Figure 1. Finite difference sensitivity inaccuracy in nonlinear problems.

Numerical perturbations may be applied directly according to Eq. (6) in the case of constitutive parameters or any design variable that does not affect the unperturbed mesh. However, in the case of shape sensitivity, when using unstructured mesh generators, special care must be taken in order not to change the mesh topology, as will be discussed next.

2.1.Shape parameters finite difference sensitivities using unstructured mesh generators

The satisfaction of the mesh topology invariance requirement is usually not met when unstructured mesh generators are employed for the sensitivity task. An example is given in Fig. 2, in which a slight boundary perturbation causes a complete mesh topology change. Fig. 2(a) presents the geometrical model which is given by 3 straight lines and 1 spline. Points 5, 6 and 7 are control points of the spline. Point 7 is perturbed upwards by 10^{-3} times the spline's length, causing a modification in the generated mesh. Fig. 2(b) and Fig. 2(c) show the meshes generated for the unperturbed and perturbed geometries respectively. The problem was noticed early by researchers as Braibant & Morelle[10] who suggested to adopt a Laplacian nodal smoothing as a possible solution. In this kind of procedure, the sensitivity of the boundary nodal coordinates with respect to shape parameters is obtained analytically and introduced as a non-homogeneous boundary condition of a properly defined Laplacian problem, leading to the determination of the sensitivity in the domain. The obtained field is called "mesh sensitivity" or "velocity field" because the shape parameter may be seen as a pseudo-time variable. Duysinx et al.[11] studied the application of different Laplacian schemes in linear elasticity problems. In the present work, this approach is employed and based on a comparative study applied to metal forming optimization applications[12], the power Laplacian scheme[13] was selected.



(a) (b) (c)

Figure 2. Effect of perturbation in mesh topology using an unstructured generator.

The invariance of the mesh topology is necessary because otherwise interpolated values would bring interpolation errors polluting the sensitivity calculation. Consider

$$\frac{\Delta \tilde{U}_j}{\Delta b_k} = \frac{\tilde{U}_{j\Delta} - U_j}{\delta b_k} \tag{10}$$

in which $\tilde{U}_{j\Delta}$ is an interpolated value of $U_{j\Delta}$.

Then,

$$\tilde{U}_{j\Delta} = U_{j\Delta} + err \tag{11}$$

where *err* is the interpolation error.

Replacing Eq. (11) in Eq. (10),

$$\frac{\Delta \tilde{U}_j}{\Delta b_k} = \frac{U_{j\Delta} - U_j}{\delta b_k} + \frac{err}{\delta b_k} \tag{12}$$

that is

$$\frac{\Delta \tilde{U}_j}{\Delta b_k} = \frac{\delta U_j}{\delta b_k} + \frac{err}{\delta b_k} \tag{13}$$

Notice that although the interpolation error is negligible if compared to U_j, for small perturbations it will be "large" if compared to δU_j, hence, destroying the gradient approximation. Some examples of this nature are given in Srikanth et al.[3].

Although the traditional finite difference method is very general and simple, it is extremely expensive because it demands one complete additional nonlinear analysis for each design variable. Hence, cheaper alternatives need to be considered.

3. SEMI-ANALYTICAL SENSITIVITY METHOD WITHOUT REMESHING

One alternative to the traditional finite difference sensitivity method which avoids the analytical differentiation of the constitutive equations and provides approximate results at a fraction of the cost of the former is the semi-analytical approach. In this method, the derivative of the nonlinear equations residual is approximated using finite differences.

Hisada[14] pioneered the topic for path-dependent problems. He developed a semi-analytical method for large strain elastoplasticity which was later extended by Chen to include frictional contact[15,16].

In 1993, Haftka established a relationship between the semi-analytical and an efficient finite difference method that he developed for iteratively solved path-independent problems[8]. This relationship shows that the former is equal to the efficient finite difference method based in only one iteration. The extension to path-dependent problems is straightforward if the effect of the perturbation in the whole loading history is taken into consideration.

3.1.Hisada and Chen's semi-analytical sensitivity approach

In elastic-plastic problems, the discretized equilibrium equations may be put in the infinitesimal form

$$\boldsymbol{K}_T d\boldsymbol{U} = d\boldsymbol{F} \tag{14}$$

From (14) and due to the path-dependent nature of the problem, it is correct to affirm that the integral below is valid provided the integration limits cover the whole loading history,

$$\int_0^{\boldsymbol{U}} \boldsymbol{K}_T d\boldsymbol{U} = \int_0^{\boldsymbol{F}} d\boldsymbol{F} \tag{15}$$

In practice, the loading $\boldsymbol{f}^{ext} = \boldsymbol{F}$ is divided in *ninc* sufficiently small finite increments, such that material behavior can be considered path-independent inside each interval. Hence,

$$\boldsymbol{F} = \boldsymbol{F}^1 + \boldsymbol{F}^2 + \ldots\ldots + \boldsymbol{F}^{ninc} . \tag{16}$$

Therefore, the integral

$$\int_{\boldsymbol{U}^{n-1}}^{\boldsymbol{U}^n} \boldsymbol{K}_T d\boldsymbol{U} = \boldsymbol{F}^n \tag{17}$$

is path-independent and may be solved using an iterative procedure. Hisada's semi-analytical approach is based in this equation.

Consider that an arbitrary parameter b_j suffers a variation δb_j $(=\alpha b_j; |\alpha| << 1)$ and the tangent stiffness matrix is given by $\boldsymbol{K}_T^a$. In such a case, from (17), one has

$$\int_{\boldsymbol{U}^n}^{\boldsymbol{U}^n_\Delta} \boldsymbol{K}_T^\alpha d\boldsymbol{U} = -\left(\int_{\boldsymbol{U}^{n-1}_\Delta}^{\boldsymbol{U}^n} \boldsymbol{K}_T^\alpha d\boldsymbol{U} - \boldsymbol{F}^n \right) \tag{18}$$

The expression (18) may be approximated by

$$\boldsymbol{K}_{\mathrm{T}}\left(\boldsymbol{U}^n\right)\delta\boldsymbol{U}^n = -\delta\boldsymbol{R}^n \tag{19}$$

with

$$\delta\boldsymbol{R}^n = \int_{\boldsymbol{U}_\Delta^{n-1}}^{\boldsymbol{U}^n} \boldsymbol{K}_T^\alpha d\boldsymbol{U} - \boldsymbol{F}^n \tag{20}$$

or, from (17)

$$\delta\boldsymbol{R}^n = \int_{\boldsymbol{U}_\Delta^{n-1}}^{\boldsymbol{U}^n} \boldsymbol{K}_T^\alpha d\boldsymbol{U} - \int_{\boldsymbol{U}^{n-1}}^{\boldsymbol{U}^n} \boldsymbol{K}_T d\boldsymbol{U} \tag{21}$$

The expression for $\delta\boldsymbol{R}^n$ can still be put as

$$\begin{aligned}\delta\boldsymbol{R}^n = &\left(\int_0^{\boldsymbol{U}_\Delta^1} \boldsymbol{K}_T^\alpha d\boldsymbol{U} + \int_{\boldsymbol{U}_\Delta^1}^{\boldsymbol{U}_\Delta^2} \boldsymbol{K}_T^\alpha d\boldsymbol{U} + \cdots + \int_{\boldsymbol{U}_\Delta^{n-1}}^{\boldsymbol{U}^n} \boldsymbol{K}_T^\alpha d\boldsymbol{U} \right) - \\ &- \left(\int_0^{\boldsymbol{U}^1} \boldsymbol{K}_T d\boldsymbol{U} + \int_{\boldsymbol{U}^1}^{\boldsymbol{U}^2} \boldsymbol{K}_T d\boldsymbol{U} + \cdots + \int_{\boldsymbol{U}^{n-1}}^{\boldsymbol{U}^n} \boldsymbol{K}_T d\boldsymbol{U} \right)\end{aligned} \tag{22}$$

Therefore,

$$\frac{d\boldsymbol{U}^n}{db_j} \simeq \frac{\delta\boldsymbol{U}^n}{\delta b_j} = -\boldsymbol{K}_T\left(\boldsymbol{U}^n\right)^{-1}\frac{\delta\boldsymbol{R}^n}{\delta b_j} \tag{23}$$

The terms inside the second parenthesis in (22) are determined by the standard integration procedure adopted for the unperturbed problem. However, in order to evaluate the terms in the first parenthesis, it is important to note that the upper integration limit of the last integral is configuration $\boldsymbol{U}^n$. As a consequence, the internal variables and stresses found are relative to this configuration. In the next increment, however, it will be necessary to start with the lower integration limit relative to configuration $\boldsymbol{U}_\Delta^n$. These displacement values are obtained by linear approximation, i.e.,

$$\boldsymbol{U}_\Delta^n = \boldsymbol{U}^n + \frac{d\boldsymbol{U}^n}{db_j}\delta b_j \tag{24}$$

Next, a new integration of the constitutive equations is performed, time in the interval $\left[\boldsymbol{U}_\Delta^{n-1}, \boldsymbol{U}_\Delta^n\right]$, obtaining the stress and internal variables state in configuration $\boldsymbol{U}_\Delta^n$.

For a generic degree of freedom, the described procedure has the geometrical interpretation depicted in Fig. 3. The displacement sensitivity obtained may be visualized in the $U \times b$ plane, given by the tangent of the angle between the line defined by $U_{\Delta *}^n$ and the line defined by U^n (parallel to the b axis). This value has two main inaccuracy sources: the first one is related to the use of matrix K to estimate the value of U_Δ^n, which leads to point 1 instead of point 2. This fact is very important since the problem is path-dependent and the error at each incremental solution will accumulate. This will clearly happen since after finding the value of $U^n_{\Delta *}$, according to point 1, the integration of the constitutive equations for this configuration will lead to point 1'. Hence, the internal variables and stresses found will be relative to point 1', not point 1, resulting in a cumulative error in the incremental determination of sensitivity. The second inaccuracy source is relative to the magnitude of the perturbation δb_k, which should be small in order to compute a tangent rather than a secant. If δb_k tends to zero, these errors become negligible but conditioning errors appear due to the computer finite precision.

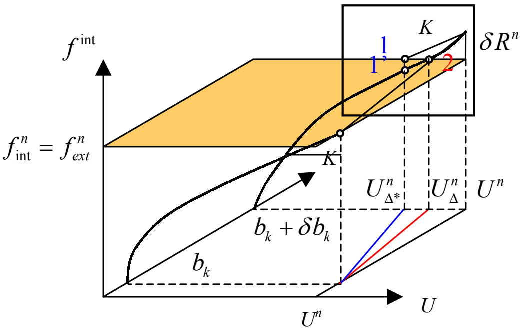


Figure 3. Structural response with respect to *b*.

4. EFFICIENT FINITE DIFFERENCE SENSITIVITY METHOD WITHOUT REMESHING

Kleinermann[17,18], studied inverse optimization problems including preforms, die-shapes and parameters identification in metal forming. In his developments, he employed a variation of Hisada's and Haftka's methods, establishing an efficient finite difference method for path-dependent problems. In view of the incremental nature of the nonlinear solution method adopted [Eq. (1)-(2) and (6)-(7)], the sensitivity field is given by

$$\frac{d\boldsymbol{U}^n}{db_k} = \sum_{i=1}^{n} \frac{d\boldsymbol{u}^i}{db_j} \simeq \sum_{i=1}^{n} \frac{\delta \boldsymbol{u}^i}{\delta b_j} \tag{25}$$

with
$$\delta \boldsymbol{u}^i = \boldsymbol{u}^i_\Delta - \boldsymbol{u}^i . \tag{26}$$

Based in Kleinermann's developments this work presents a slightly modified efficient finite difference sensitivity approach. As in Hisada's method, the analysis begins with a separate set of data for each problem, the unperturbed and the perturbed ones. In the case of shape design variables, the mesh topology must be maintained. The load history is divided in *ninc* increments, and the procedure starts with the search of equilibrium for the first load increment of the unperturbed problem. Once this condition is achieved within the tolerance imposed for the iterative solution method, all the nodal and Gauss points information is stored in a binary file. Then, the data of the first perturbed problem is loaded and a new analysis takes place, but this time the converged configuration of the unperturbed problem is taken as first the solution estimate of the perturbed problem. Moreover, the tangent matrix of the unperturbed problem is used in a modified Newton-Raphson scheme, which converges rapidly and at low cost. The same tolerance of the unperturbed problem is imposed to the perturbed ones. After convergence, the information is stored in a binary file. This procedure is repeated for every design variable. In each load step, after all the perturbed problems have converged, the sensitivity obtained in the increment is determined by finite differences and the total sensitivity up to the step is given by Eqs. (25) and (26). At the end of the cycle, the data of the unperturbed problem is reloaded, the next load increment is considered and the procedure continues until the last load step.

The perturbed incremental displacement is given by the iterative solution of

$$\boldsymbol{u}^n_{\Delta,k+1} = \boldsymbol{u}^n_{\Delta,k} \pm \boldsymbol{K}_T^{-1} \frac{d\boldsymbol{R}^n_k}{db_j}\delta b_j \quad ; \quad \boldsymbol{u}^n_{\Delta,0} = \boldsymbol{u}^n \tag{27}$$

where the signal depends if the finite difference is forward (negative signal) or backward (positive signal). The relation of the method to Hisada's approach is clear by comparing (27) to (24) and (23).

Following Kleinermann, who employed central finite differences for the sensitivity evaluation, the residual derivative is approximated using forward and backward finite differences respectively, that is

$$\left(\frac{d\boldsymbol{R}^n_k}{db}\right)_{ij} = \frac{\left[\boldsymbol{R}^n_k\left(\boldsymbol{u}^n;\boldsymbol{U}^{n-1}_{\Delta_{j+}};\boldsymbol{b}+\delta\boldsymbol{b}_j\right)\right]_i - \left[\boldsymbol{R}^n\left(\boldsymbol{u}^n;\boldsymbol{U}^{n-1};\boldsymbol{b}\right)\right]_i}{\delta b_j} \tag{28}$$

$$\left(\frac{d\boldsymbol{R}^n_k}{db}\right)_{ij} = \frac{\left[\boldsymbol{R}^n_k\left(\boldsymbol{u}^n;\boldsymbol{U}^{n-1};\boldsymbol{b}\right)\right]_i - \left[\boldsymbol{R}^n_k\left(\boldsymbol{u}^n;\boldsymbol{U}^{n-1}_{\Delta_{j-}};\boldsymbol{b}-\delta\boldsymbol{b}_j\right)\right]_i}{\delta b_j} \tag{29}$$

Replacing (28) and (29) in (27) gives, respectively

$$\boldsymbol{u}^n_{\Delta_{j+}} = \boldsymbol{u}^n - \boldsymbol{K}_T^{-1}\left[\boldsymbol{R}^n\left(\boldsymbol{u}^n;\boldsymbol{U}^{n-1}_{\Delta_{j+}};\boldsymbol{b}+\delta\boldsymbol{b}_j\right) - \boldsymbol{R}^n\left(\boldsymbol{u}^n;\boldsymbol{U}^{n-1};\boldsymbol{b}\right)\right] \tag{30}$$

$$\boldsymbol{U}^n_{\Delta_{j+}} = \sum_{i=1}^{n}\boldsymbol{u}^i_{\Delta_{j+}} \tag{31}$$

and

$$\boldsymbol{u}^n_{\Delta_{j-}} = \boldsymbol{u}^n - \boldsymbol{K}_T^{-1}\left[\boldsymbol{R}^n\left(\boldsymbol{u}^n;\boldsymbol{u}^{n-1}_{\Delta_{j-}};\boldsymbol{b}-\delta\boldsymbol{b}_j\right) - \boldsymbol{R}^n\left(\boldsymbol{u}^n;\boldsymbol{u}^{n-1};\boldsymbol{b}\right)\right]. \tag{32}$$

$$\boldsymbol{U}^n_{\Delta_{j-}} = \sum_{i=1}^{n}\boldsymbol{u}^i_{\Delta_{j-}} \tag{33}$$

The final sensitivity matrix is obtained simply from

$$\frac{d\boldsymbol{U}^n}{d\boldsymbol{b}} = \frac{\left(\boldsymbol{U}^n_{\Delta_{j+}} - \boldsymbol{U}^n_{\Delta_{j-}}\right)_i}{2\delta b_j} \tag{34}$$

In his efficient finite difference method, Haftka[8] showed that the use of unperturbed solution as an initial estimate for the perturbed problem may lead to completely erroneous results if special care is not taken. The solution that he proposed is equivalent to Kleinermann's approach and is revised here.

For unilateral finite differences, two finite element problems are established which are solved using the Newton-Raphson method. The first one is the unperturbed problem, given by

$$\boldsymbol{u}^n_{k+1} = \boldsymbol{u}^n_k - \boldsymbol{K}_T^{-1}\boldsymbol{R}^n_k\left(\boldsymbol{u}^n_k;\boldsymbol{U}^{n-1};\boldsymbol{b}\right) \tag{35}$$

with
$$\boldsymbol{K}_T = \frac{\partial \boldsymbol{R}^n_k}{\partial \boldsymbol{u}^n}. \tag{36}$$

The second problem is the perturbed one whose equation is given by (30) or (32). Here, forward finite differences will be adopted, yielding

$$\boldsymbol{u}^n_{\Delta,k+1} = \boldsymbol{u}^n_{\Delta,k} - \boldsymbol{K}_T^{-1}\boldsymbol{R}^n_k\left(\boldsymbol{u}^n_{\Delta,k};\boldsymbol{U}^{n-1}_\Delta;\boldsymbol{b}+\delta\boldsymbol{b}_j\right) \quad ; \quad \boldsymbol{u}^n_{\Delta,0} = \boldsymbol{u}^n \tag{37}$$

The solution of the sensitivity problem in the *n-th* load increment is found by the iterative solution of (35) and (37) until the residuals reach a value below the tolerance imposed, that is

$$\boldsymbol{R}^n\left(\tilde{\boldsymbol{u}}^n,\boldsymbol{U}^{n-1},\boldsymbol{b}\right) \cong \boldsymbol{0} \text{ and } \boldsymbol{R}^n\left(\tilde{\boldsymbol{u}}^n_\Delta,\boldsymbol{U}^{n-1}_\Delta,\boldsymbol{b}+\delta\boldsymbol{b}_j\right) \cong \boldsymbol{0} \tag{38)-(39}$$

Haftka showed that in order to make the sensitivity results nearly independent of the initial estimate, it suffices to modify the original and perturbed problems to

$$\boldsymbol{R}^n\left(\boldsymbol{u}^n,\boldsymbol{U}^{n-1},\boldsymbol{b}\right) - \boldsymbol{R}^n\left(\tilde{\boldsymbol{u}}^n,\boldsymbol{U}^{n-1},\boldsymbol{b}\right) = \boldsymbol{0} \tag{40}$$

and
$$\boldsymbol{R}^n\left(\boldsymbol{u}^n_\Delta,\boldsymbol{U}^{n-1}_\Delta,\boldsymbol{b}+\delta\boldsymbol{b}_j\right) - \boldsymbol{R}^n\left(\tilde{\boldsymbol{u}}^n,\boldsymbol{U}^{n-1},\boldsymbol{b}\right) = \boldsymbol{0}, \tag{41}$$

where the tilde indicates the configuration at which the iterative search was stopped because the residual reached a value below the imposed tolerance.

The problem (40) is negligibly different from the original problem since after convergence, $\boldsymbol{R}^n\left(\tilde{\boldsymbol{u}}^n,\boldsymbol{U}^{n-1},\boldsymbol{b}\right)$ is almost zero. Since for $\delta\boldsymbol{b}_j = \boldsymbol{0}$ the exact solution of Eq. (41) is $\tilde{\boldsymbol{u}}^n$, the iterative process will only reflect the perturbation effect. Therefore, adding the residual of the original problem in the perturbed one provides

$$\boldsymbol{u}^n_{\Delta,k+1} = \boldsymbol{u}^n_{\Delta,k} - \boldsymbol{K}_T^{-1}\left[\boldsymbol{R}^n_k\left(\boldsymbol{u}^n_\Delta,\boldsymbol{U}^{n-1}_\Delta,\boldsymbol{b}+\delta\boldsymbol{b}_j\right) - \boldsymbol{R}^n\left(\tilde{\boldsymbol{u}}^n,\boldsymbol{U}^{n-1},\boldsymbol{b}\right)\right] \quad ; \quad \boldsymbol{u}^n_{\Delta,0} = \boldsymbol{u}^n \tag{42}$$

The expression is iterated until the term in brackets satisfies a predefined tolerance. In this work the same tolerance that was imposed for the unperturbed residual was applied.

Thus, the displacement sensitivity (9) may be obtained more accurately by

$$\frac{\Delta \boldsymbol{U}^n}{\Delta b_j} = \frac{\boldsymbol{U}_\Delta^n - \tilde{\boldsymbol{U}}^n}{\delta b_j} \tag{43}$$

The iterative procedure on equation (42) is interpreted graphically in Fig. 4, which is a zoom of the box in Fig.3. It displays a modified Newton-Raphson method applied to the curve $f^{\text{int}}(\boldsymbol{b}+\delta \boldsymbol{b}_k)$ using the unperturbed stiffness matrix K and initial estimate U^n, approaching point 2 at each load increment. From Figure 4 it is clear that if only one iteration is performed, Hisada's semi-analytical method is obtained. This is in accordance with Haftka's results for path-independent problems[19].

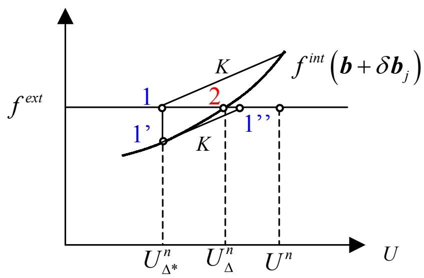


Figure 4. Iterative procedure for U_Δ^n determination.

The code METAFOR® allows to employ an automatic load incrementation procedure aiming to accelerate the analysis convergence rate and to avoid the termination of the process due to non convergence caused by a large increment. In the automatic incrementation method it is possible to give an initial configuration estimate through extrapolation based in previous iterations. For the efficient finite difference method implementation one can choose to adopt automatic or imposed increments in the unperturbed problem. However, in perturbed problems imposed incrementation is necessary in order to use the converged unperturbed solution as initial estimate. This situation is shown in Fig. 5, in which the blue and the red colors represent the unperturbed and the perturbed problems respectively. The point t^* is automatically found in the unperturbed problem but it is imposed in the perturbed ones.

5. EFFECTS OF REMESHING ON FINITE DIFFERENCE SENSITIVITY FIELDS: PROBLEMS AND SOLUTIONS

Frequently in large strain problems it becomes necessary to remesh for avoiding numerical errors associated to mesh distortion. This is the case in non-steady metal forming simulations such as forging. In sensitivity analysis it has been shown that while the gradients obtained using the analytical approach are unaffected by remeshing procedures, the gradients obtained via finite differences might be severely affected producing unacceptable results. This happens because small interpolation errors related to parametric inversion and variables transfer may contaminate the gradients approximations in the same way that was indicated by Eq. (13)[3].

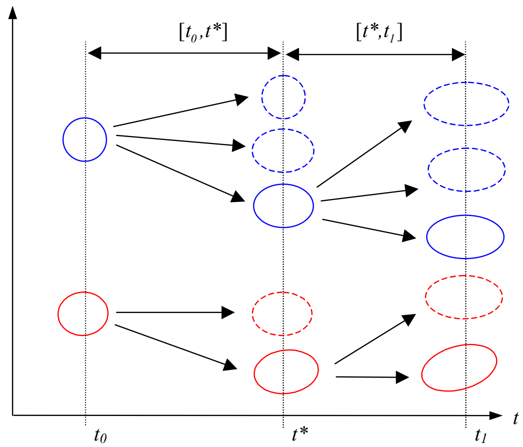


Figure 5. Search for equilibrium implemented in METAFOR using the "efficient finite difference method".

However, a simple solution for this problem is possible. If the efficient finite difference method presented so far is employed, at each load step the unperturbed and the perturbed problems are solved sequentially. In this case, the program must perform a remeshing based in the distortion criterion *only* in the unperturbed problem. After the new mesh is defined and the parametric coordinates (ξ_A,η_A) of every new mesh node A are found – this is a necessary step in any variables transfer procedure – these parametric coordinates are stored as depicted in Fig. 6 and Table 1.

The new mesh of the perturbed problems is then defined by

$$\tilde{x}_A = \sum_{i=1}^{nnel} N_i\left(\xi_A,\eta_A\right) x_i \;, \tag{44}$$

where $\tilde{x}_A$ represents a generalized coordinate of point A in the new mesh, x_i represents the generalized coordinate of the *i-th* node of the old mesh element that contains node A and N_i are the shape functions. The unperturbed and perturbed mesh connectivities are identical. In this way, the interpolation errors in the unperturbed and perturbed problems will be almost identical and will tend to cancel out providing a "clean" sensitivity field.

In the present work the remeshing strategy presented by Olmi[20] was adopted. This strategy defines a global remeshing estimator given by the sum of all the elements that exceed a limiting distortion value. In other words, in each converged load increment the distortion level of every element is verified. When the user defined allowable percentage of elements is reached, a remeshing is performed. The new mesh is generated using the routine proposed by

Alquati & Groehs[21] yielding triangular and/or quadrilateral elements by a frontal scheme starting at the boundary. Afterwards, Laplacian smoothing is applied for mesh regularization.

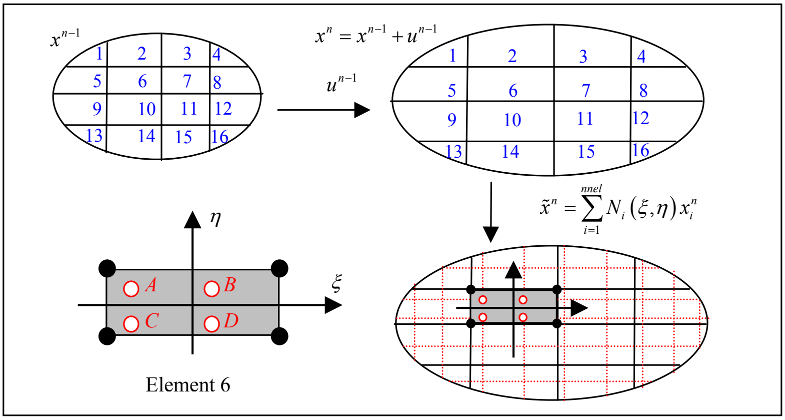


Figure 6. Parametric inversion for variables transfer and remeshing of perturbed problems.

Table 1. Mesh topology and local coordinates of the new mesh nodes with respect to the old mesh.

New mesh node	Old mesh element	ξ	η
A	6	ξ_A	η_A
⋮	⋮	⋮	⋮

The variables transfer strategy implemented by Olmi consists in transferring the deviatoric and volumetric stress tensor components and the effective plastic strain at the Gauss points. The approach follows the Gauss to node, node to node, node to Gauss procedure. Although this strategy has the inconvenient of not assuring equilibrium and flow laws after remeshing, the good results obtained by Olmi make believe that these are not critical issues. Hence, the same procedure is used in this development. It must be clear, however, that the idea for dealing with remeshing in sensitivity analysis proposed in this work independs on the remeshing and variables transfer strategies adopted. The use of variables transfer schemes that satisfy both equilibrium equations and constitutive laws, including the plastic increment incompressibility condition, has been object of many discussions in the scientific literature[3,22,23,24,25,26].

6. NUMERICAL EXAMPLES

In order to demonstrate the potential of the method, the upsetting of a cylinder of 200mm height × 115mm radius is considered, as depicted in Fig. 7. Due to symmetry conditions, only the right upper quadrant is discretized. The material parameters employed are: E=2.1×10^5

MPa, $\nu = 0.3$, H=2.1×10^3 MPa and $\sigma_Y = 270$ MPa. Sticking contact is imposed with normal penalization factor given by $P_N=10^8$.

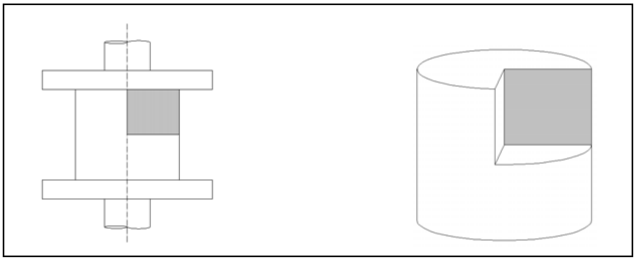
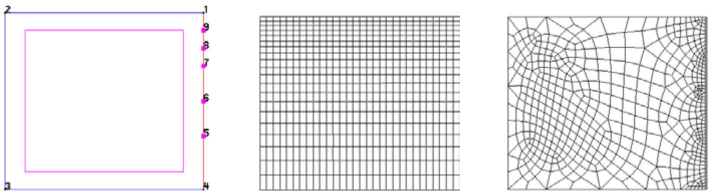


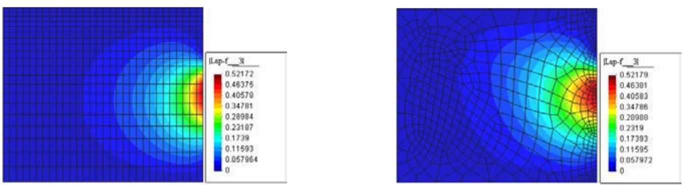
Figure 7. Cylinder upsetting problem and discretized quadrant.

First, a geometrical model is needed to parameterize the boundary. For this purpose, three straight lines (1-2, 2-3 and 3-4) and one spline (4-1) are used, according to Fig. 8(a). The red dots on the spline are control points. Then, the domain must be meshed. A structured mesh and an unstructured one are displayed in Figs. 8(b) and 8(c). Their velocity fields obtained by the power Laplacian method (with power p=1.5) are shown in Figs. 9(a) and 9(b). In these Figures the influence of the mesh is clear but in both cases meaningful results are obtained.



(a) (b) (c)

Fig.8. (a) Geometric model, (b) structured mesh and (c) unstructured mesh.



(a) (b)

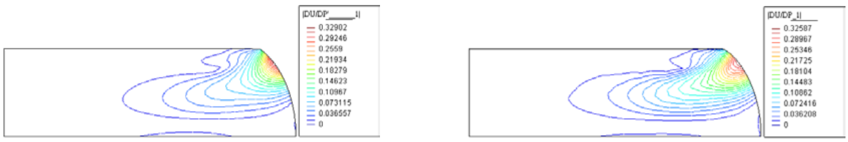
Fig.9. Power Laplacian velocity fields: (a) Structured mesh and (b) unstructured mesh.

The final deformed configuration of the meshes are depicted in Figs. 10(a) and 10(b).

(a) (b)

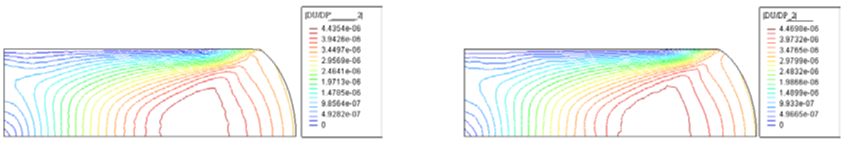
Fig.10. (a) Deformed structured mesh and (b) deformed unstructured F.E. mesh.

For exemplification purposes, two design variables are considered. The first one is the horizontal position of point 6 which is a spline control point. The second is a material parameter, the elasticity modulus (E). In the case of the shape parameter the perturbation is set as $\Delta b_1 = 10^{-4} L_S$, where L_S is the spline length L_S=100mm. In the case of the material parameter, the perturbation is set as $\Delta b_2 = 10^{-3} E$. The displacement sensitivity fields evaluated by the traditional finite difference method are displayed in Figs. 11 and 12 for the structured and unstructured meshes. The same results obtained via the efficient finite difference method are presented in Figs. 13 and 14 showing excellent agreement. The Newton-Raphson tolerance is set to 10^{-6}. An automatic load incrementation strategy with 10 compulsory matching points is applied.



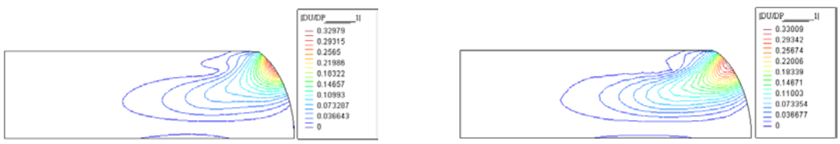
(a) (b)

Fig.11. Traditional finite difference displacement sensitivity with respect to shape parameter at the end of deformation (no remeshing). (a) structured mesh and (b) unstructured mesh.



(a) (b)

Fig.12. Traditional finite difference displacement sensitivity with respect to material parameter at the end of deformation (no remeshing). (a) structured mesh and (b) unstructured mesh.



(a) (b)

Fig.13. Efficient finite difference displacement sensitivity with respect to shape parameter at the end of deformation (no remeshing). (a) structured mesh and (b) unstructured mesh.

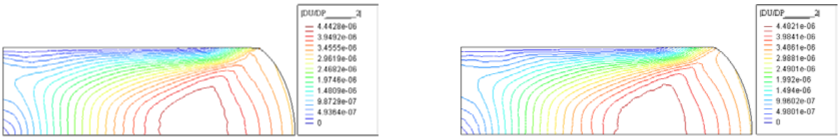


Fig.14. Efficient finite difference displacement sensitivity with respect to material parameter at the end of deformation (no remeshing). (a) structured mesh and (b) unstructured mesh.

In order to make clear the efficiency of the proposed method compared to the traditional finite difference approach, both methods are employed to obtain the sensitivity of an increasing number of parameters. To simulate this situation, the sensitivity with respect to the same shape design variable considered up to now is calculated repeatedly. Fig. 15, below presents the time required for the calculations in the efficient finite difference method, compared to the minimum time that would be required by using the traditional finite difference method. In the analysis, the structured mesh of Fig. 8(b) with 600 nodes is used.

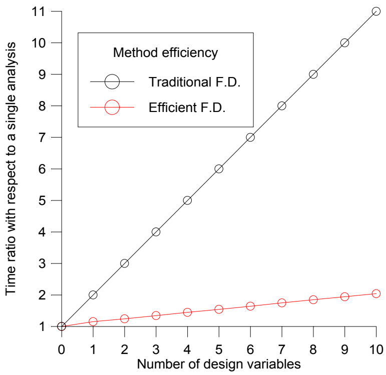
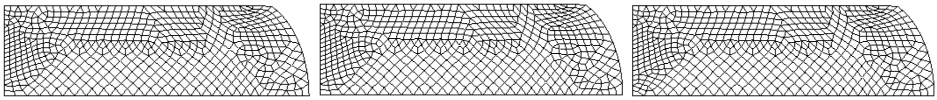


Fig 15. Efficiency comparison between traditional and efficient finite difference methods.

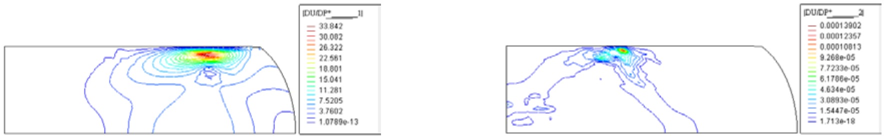
Note that, for this mesh, when using the efficient finite difference approach, the cost of two analyses is reached only after the consideration of 10 design variables. This efficiency difference tends to get even more pronounced when the number of nodes increases because the cost of the tangent matrix calculation increases in the third power of its order.

In order to analyze a remeshing case, the same problem depicted in Fig. 8(b) is considered, this time allowing a limit of 2% of elements over the distortion threshold. Firstly the traditional finite difference method is applied. Figs. 16(a), 16(b) and 16(c) display the final deformed meshes of the unperturbed problem and the ones perturbed by the shape and material parameters, respectively. In this case, the meshes are slightly different but as Figs. 17(a) and 17(b) illustrate, the interpolation errors are sufficiently large to pollute the sensitivity fields obtained by the small perturbations applied, in accordance to the results reported by Srikanth at al.[3].



(a) (b) (c)

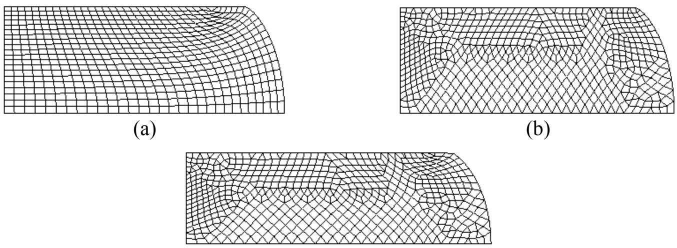
Fig 16. Final configuration of the deformed remeshed meshes. Meshes relative to: (a) unperturbed problem, (b) perturbation on the shape design variable and (c) perturbation on the material design variable.



(a) (b)

Fig 17. Displacement sensitivity fields using the traditional finite difference method for (a) shape parameter and (b) material parameter.

On the other hand, the application of the method proposed in this work is displayed in Figs. 18 to 20. Figs. 18(a) and 18(b) display the unperturbed meshes before and after remeshing respectively and Fig. 18(c) shows the final deformed configuration of the new mesh.



(a) (b) (c)

Fig. 18. Meshes of the unperturbed problem (a) at the remeshing moment before remeshing, (b) at the remeshing moment after remeshing and (c) at the end of the analysis.

Figs. 19(a) and 19(b) illustrate the shape sensitivity field transfer from the old to the new mesh and Fig. 19(c) presents the final shape finite difference sensitivity field. Figs. 20(a), 20(b) and 20(c) play the same role for the sensitivity with respect to the material parameter.

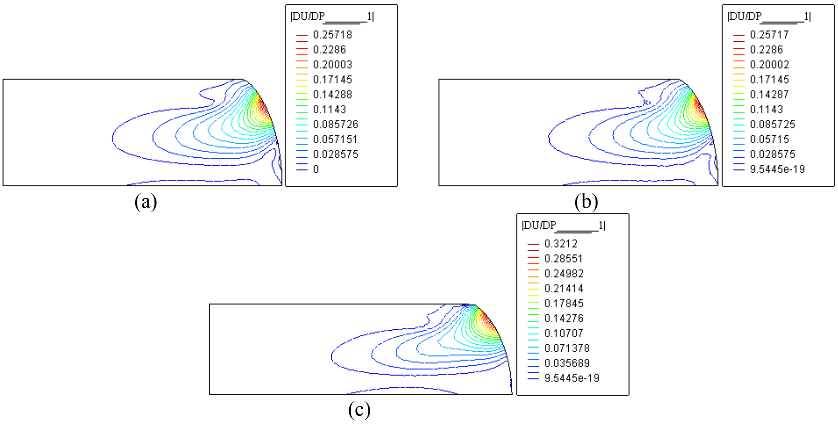


Fig. 19. Displacement sensitivity field with respect to the shape parameter. (a) in the old mesh at the remeshing moment, (b) in the new mesh at the remeshing moment (transferred field) and (c) at the final configuration.

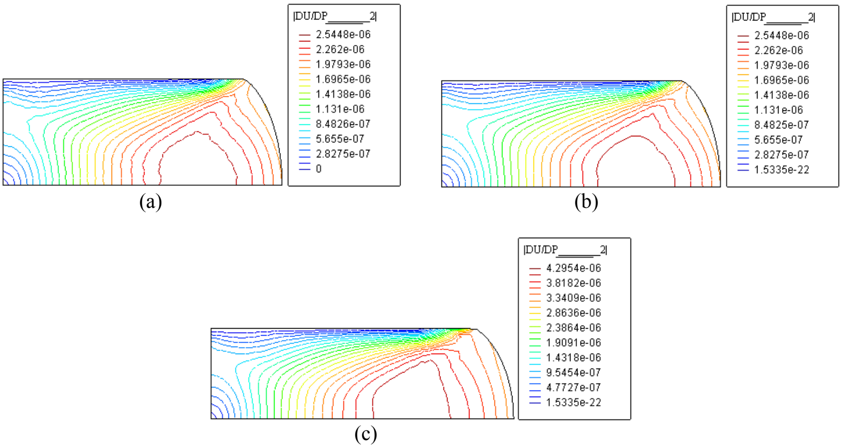


Fig. 20. Displacement sensitivity field with respect to the material parameter. (a) in the old mesh at the remeshing moment, (b) in the new mesh at the remeshing moment (transferred field) and (c) at the final configuration.

7. CONCLUSIONS

An efficient procedure for finite difference sensitivity evaluation which allows remeshing has been developed and its formulation and computational details are presented.

With this procedure the finite difference method becomes competitive, particularly in highly non linear history dependent problems that include remeshing.

As shown in the examples with this new procedure considerable amount of computational time was saved as compared with the traditional finite difference sensitivity method, while clean and precise displacement sensitivity fields were obtained after remeshing for the shape and material parameters considered even for very small perturbations. The extension for other sensitivity fields determination is straightforward.

8. ACKNOWLEDGEMENTS

The authors would greatly like to acknowledge the contribution of Prof. M. Hogge and Prof. J.P. Ponthot, who allowed the use of the METAFOR® code and gave valuable theoretical support relative to its implementation. Prof. Hisada, Prof. Chen, Prof. M. Kleiber and Dr. J.P. Kleinermann also contributed through personal communications about their semi-analytical methods. Prof. P. Duysinx provided valuable research material relative to Laplacian smoothing applied to finite difference sensitivity calculations. Finally, M.Sc. F. Olmi and Prof. L.A.B. da Cunda provided their implementation of remeshing and variables transfer in METAFOR® code, which was used as the basis of this work.

The authors would also like to acknowledge the financial support given by CNPq and CAPES to this research. Finally, the first author wishes to express his gratitude to UDESC for the waiving of his lecturing duties during his Ph.D. studies.

9. REFERENCES

[1] Haug, E., Choi, K.K. & Komkov, V., "Design Sensitivity Analysis of Structural Systems", Academic Press, 1986.

[2] Kleiber,M., Antúnez,H., Hien, T.D. & Kowalczyk, P., "Parameter Sensitivity in Nonlinear Mechanics: Theory and Finite Element Computations", John Wiley & Sons, 1997.

[3] Srikanth, A., Zabaras, N. & Frazier, W.G. "An Updated Lagrangian Finite Element Sensitivity Analysis of Large Deformations using Quadrilateral Elements", *Int. J. Numer. Meth. Engng.*, 52: (10), pp. 1131-1163, 2001.

[4] Forestier, R., Chastel, Y. & Massoni, E., "3D Inverse Analysis Using Semi-Analytical Differentiation for Mechanical Parameter Estimation", *4th International Conference on Inverse Problems in Engineering,* Rio de Janeiro, Brazil, 2002.

[5] Ponthot, J.P. and Hogge, M., "The Use of the Euler Lagrange Finite Element Method in Metal Forming Including Contact and Adaptive Mesh", in: *Proceedings of the ASME Winter Annual Meeting*, Atlanta, USA, 1991.

[6] Nagtegaal, J.C., "On the Implementation of Inelastic Constitutive Equations with Special Reference to Large Deformation Problems", *CMAME*, 33, pp 469-484, 1982.

[7] Haftka R.T. & Gürdal, Z., "Elements of Structural Optimization", 3rd edition, Kluwer Academic Publishers, 1996.

[8] Haftka, R.T., "Sensitivity Calculations for Iteratively Solved Problems", *Int. J. Numer. Meth. Engng*, 21, pp. 1535-1546, 1985.

[9]Tortorelli, D.A., Non-Linear and Time Dependent Structural Systems: Sensitivity Analysis and Optimization, Lecture Notes, DCAMM, Lingby, Denmark, 1997.

[10]Braibant, V. & Morelle, P., Shape Optimal Design and Free Mesh Generation, *Structural Optimization*, vol. 2, pp. 223-231, 1990.
[11]Duysinx, P., Zhang, W.H. & Fleury, C., "Sensitivity Analysis with Unstructured Free Mesh Generators in 2-D Shape Optimization", Report AO-31, Aerospace Laboratory, LTAS, University of Liège, Belgium, 1993/ *World Congress on Optimal Design of Structural Systems, "Structural Optimization 93"*, Rio de Janeiro, Brasil, 1993.
[12]Muñoz-Rojas, P.A, Fonseca, J.S.O. & Creus, G.J., "A New Approach for 2D Preform Optimization Using an Elastic-Plastic Material Model and Unstructured Mesh Generators", *CILAMCE*, 2001.
[13]Robinson, B., Batina, J. & Yang, H., Aeroelastic analysis of wings using Euler equations with deforming mesh, *Journal of Aircraft*, vol 28, n^o 11, 1991.
[14]Hisada, T., "Sensitivity Analysis of Nonlinear FEM", *Proc. ASCE EMD/GTD/STD Specialty Conference on Probabilistic Methods,* p. 164, 1988.
[15]Chen, X., "Nonlinear Finite Element Sensitivity Analysis for Large Deformation Elastoplastic and Contact Problems", Ph.D. Thesis, University of Tokyo, Japan, 1994.
[16]Chen, X., Hisada, T., Nakamura, K. & Mori, M. "Sensitivity Analysis of Inelastic Structures", *Theoretical and Applied Mechanics*, vol. 48, pp. 39-47, 1999.
[17]Kleinermann, J.P. & Ponthot, J.P., "Optimization Methods for Inverse Problems in Large Strain Plasticity", *CanCNSM*, 1999.
[18]Kleinermann, J.P., "Identification Parametrique et Optimisation des Procedes de Mise a Forme par Problemes Inverses", Ph.D. Thesis, University of Liège, Belgium, 2000.
[19]Haftka, R.T., "Semi-Analytical Static Nonlinear Structural Sensitivity Analysis", *AIAA Journal*, vol. 31, no 7, 1993.
[20]Olmi, F., Bittencourt, E. & Creus, G.J., "An Interactive Remeshing Technique Applied to Two Dimensional Problems Involving Large Elasto-Plastic Deformations", *Proceedings of the 5th International Conference on Computational Plasticity - COMPLAS*, Owen, D.R.J, Oñate, E. & Hinton, E. (eds.), pp. 619-625, Barcelona, Spain, 1997.
[21]Alquati, E.L.G. & Groehs, A.G., "Transformação Triângulos/Quadriláteros em Malhas Não Estruturadas para Elementos Finitos", *Proc. XVI CILAMCE*, v. 1, pp. 478-487, 1995.
[22]Boroomand,B. & Zienkiewicz, O.C., "Recovery Procedures in Error Estimation and Adaptivity. Part II: Adaptivity in Nonlinear Problems of Elastoplasticity Behaviour", *CMAME*, 176, pp. 127-146, 1999.
[23]Camacho, G.T. & Ortiz, M., "Adaptive Lagrangian Modelling of Ballistic Penetration of Metallic Targets", *CMAME*, 142, pp. 269-301, 1997.
[24]Peric, D., Hochard, Ch, Dutko, M. & Owen, D.R.J., "Transfer Operators for Evolving Meshes in Small Strain Elasto-Plasticity", *CMAME*, 137, pp. 331-344, 1996.
[25]Lee, N.S. & Bathe, K.J., "Error Indicators and Adaptive Remeshing in Large Deformation Finite Element Analysis", *Finite Elements in Analysis and Design*, 16, pp. 99-139, 1994.
[26]Ortiz,M. & Quigley, J.J., "Adaptive Mesh Refinement in Strain Localization Problems", *CMAME*, 90, pp. 781-804, 1991.