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ON SOME CLOSED-FORM SOLUTIONS OF DUHAMEL'S INTEGRAL

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Abstract. In engineering analyses, the dynamic behavior of mechanical oscillators, or the electrical performance of RLC circuits, is often modeled by linear, second order, inhomogeneous, ordinary differential equations, with constant coefficients. Their general solution can be expressed in terms of Duhamel's convolution integral, which involves the forcing function contained in the inhomogeneous term of those equations. Depending on the complexity of this function, the integration may, or may not, yield a closed-form solution. This article presents a general, and relatively compact, expression for the N^{th} recursive integration by parts of Duhamel's integral. The obtained solution is based on the use of second order recursive coefficients, which can be written in closed form. For forcing functions with zero N^{th} derivatives, the proposed expression is an exact and closed-form solution of the original integral. This is the case for polynomial forcing functions of $(N-1)^{th}$ degree. In this article, due to space limitations, only final expressions are included, but their derivation process is summarized. The summation format of the presented expressions allows for the proper identification of all components contributing to the response. They are indicated as force-derivative and as initial-force-derivative components. An example shows the use of the proposed exact, closed-form, solution. It employs a forcing function defined as a quartic polynomial pulse. All different terms contributing to either the displacement or velocity response are identified and analyzed. The proposed expressions constitute ready tools for the solution of linear, second order differential equations subjected to polynomial forcing functions.

1 INTRODUCTION

The behavior of linear mechanical oscillators, or linear RLC circuits, is governed by the same well-known linear, second order, differential equation. This work presents exact solutions of those equations for certain type of forcing functions. They are based on the Duhamel's integral solving procedure. For continuous and continuously differentiable forcing functions, with zero N^{th} derivatives, the solutions are exact. The proposed expressions are obtained by successive integrations by parts of Duhamel's integral. The final expression is exact and is written in terms of recursive coefficients. Closed forms of those coefficients are also presented. Hence, the proposed solution is not only exact, but also a closed-form expression.

Unfortunately, the complete derivation process is excessively long and cannot be included in this article. Most derivation steps are presented in a Journal article that is being simultaneously prepared and submitted for possible publication. Even though the length of the derivation process is large, the final expressions are relatively compact. Their compactness is based on the use of recursive coefficients derived via a mathematical inductive process. This work presents a summary of the steps, and intermediate auxiliary expressions, leading to the final proposed solution.

Particular characteristics of the attained solution allow for the proper identification of contributions to the response from different sources. They include contributions from the forcing function and its time derivatives as well as from all initial conditions of the forcing action at time zero.

An example is fully developed to show the implementation of the proposed expressions. It uses a quartic polynomial pulse as the forcing function. This pulse is similar in shape to the haversine pulse commonly used to represent a frontal barrier collision in vehicle safety research (Varat, 2003).

2 FORMULATION

2.1 Equation of Motion of a Single-Degree-of-Freedom (SDF) Oscillator

The motion of a SDF, mechanical, linear oscillator subjected to an arbitrary, external, time-varying forcing function, $f(t)$, and to initial displacement and velocity conditions, is described by the following second-order, linear ordinary differential equation:

$$m \overset{(2)}{u}(t) + c \overset{(1)}{u}(t) + k \overset{(0)}{u}(t) = f(t) \quad \text{with} \quad u(0) = u_0 \quad \text{and} \quad \overset{(1)}{u}(0) = \overset{(1)}{u_0} . \tag{1}$$

In this article, the notation (j), on top of a variable, indicates its j^{th} derivative with respect to time. For $j = 0$, the superscript (0) indicates the original function. Thus, $\overset{(0)}{u}(t) = u(t)$ is the relative displacement of the oscillating mass with respect to the point of static equilibrium; $\overset{(1)}{u}(t)$ is its relative velocity; and $\overset{(2)}{u}(t)$ is its relative acceleration. Quantities m, c, and k are constant values representing, respectively, the mass, damping, and stiffness coefficients of the oscillator. Its natural undamped and damped circular frequencies are $\omega = \sqrt{k/m}$ and $\omega_d = \omega\sqrt{1-\zeta^2}$, respectively. ζ is the oscillator damping ratio, $\zeta = c / c_{cr} = c / 2m\omega$, and $c_{cr} = 2m\omega = \sqrt{4km}$ is its critical damping value. In this work, we consider subcritically damped oscillators with $0 \le \zeta < 1$. In particular, the undamped case, $\zeta = 0$, is also included.

The stiffness and damping coefficients can be written in terms of the oscillating mass, the natural circular frequency, and the damping ratio: $k = \omega^2 m$; $c = 2\zeta\,\omega m$. Substitution of these expressions into Eq. (1), and division by the mass, results in the following equation of motion normalized with respect to the mass:

$$\overset{(2)}{u}(t) + 2\zeta\,\omega\,\overset{(1)}{u}(t) + \omega^2\,u(t) = \frac{f(t)}{m}. \tag{2}$$

The right-hand side of this equation represents the induced external acceleration acting on the oscillator.

Alternatively, all terms of Eq. (2) can be expressed in units of displacement. For this, the mass is substituted by k/ω^2 and the equation is divided by ω^2:

$$\frac{\overset{(2)}{u}(t)}{\omega^2} + \frac{2\zeta}{\omega}\overset{(1)}{u}(t) + u(t) = \frac{f(t)}{k}. \tag{3}$$

The right-hand side of Eq. (3) is equivalent to a static displacement $u_{st}(t) = f(t)/k$. This displacement would occur if the forcing function acts in a slow enough fashion to produce negligible velocity and acceleration (no dynamic effects). Details on the mathematical modeling of linear oscillators are found in most textbooks on vibrations and dynamics (Chopra, 2007).

2.2 General Solution of the Equation of Motion

The complete displacement response, or general solution, of the above mass-spring-damper system, consists of two distinctive parts: the complementary, or homogeneous, component $u_H(t)$, and the particular component $u_P(t)$:

$$u(t) = u_H(t) + u_P(t). \tag{4}$$

To obtain this solution, we use Duhamel's integral approach. It considers that $u_H(t)$ is the response of the unforced oscillator subjected only to the initial conditions of the motion (u_0 and $\overset{(1)}{u}_0$), and that $u_P(t)$ is the response of the forced oscillator with zero initial displacement and velocity. The corresponding equations for both components are:

$$\overset{(2)}{u}_H(t) + 2\zeta\,\omega\,\overset{(1)}{u}_H(t) + \omega^2\,u_H(t) = 0 \quad \text{with} \quad u_H(0) = u(0) = u_0 \quad \text{and} \quad \overset{(1)}{u}_H(0) = \overset{(1)}{u}(0) = \overset{(1)}{u}_0 \tag{5}$$

$$\overset{(2)}{u}_P(t) + 2\zeta\,\omega\,\overset{(1)}{u}_P(t) + \omega^2\,u_P(t) = \frac{f(t)}{m} \quad \text{with} \quad u_P(0) = 0 \quad \text{and} \quad \overset{(1)}{u}_P(0) = 0 \tag{6}$$

For subcritically damped oscillators ($0 \le \zeta < 1$), $u_H(t)$ has the following expression:

$$u_H(t) = u_0\,h_C(t) + \left(u_0\,\zeta + \frac{\overset{(1)}{u}_0}{\omega} \right) h_S(t), \tag{7}$$

where $h_S(t)$ and $h_C(t)$ are non-dimensional, exponentially decaying, sinusoidal functions:

$$h_S(t) = \left[e^{-\zeta\omega t} \sin\left(\omega_d\, t\right) \right] \Big/ \sqrt{1-\zeta^2}\ , \tag{8}$$

$$h_C(t) = e^{-\zeta\omega t} \cos\left(\omega_d\, t\right). \tag{9}$$

The sub-indices *S* and *C*, in $h_S(t)$ and $h_C(t)$, are used to indicate their respective sine and cosine factors.

Component $u_P(t)$ can be obtained in terms of Duhamel's convolution integral:

$$u_P(t) = \int_0^t h(\tau)\ f(t-\tau)\,d\tau = \int_0^t h(t-\tau)\ f(\tau)\,d\tau\,, \tag{10}$$

where $0 \le \tau \le t$. The time function $h(t)$ is the unit impulse response function. For subcritically damped oscillators, $0 < \zeta < 1$. Its expression is:

$$h(t) = \frac{e^{-\zeta\omega t} \sin(\omega_d\, t)}{m\ \ \omega_d}. \tag{11}$$

Since $h(t)$ and $h_S(t)$ are related as $h(t) = h_S(t)\big/ m\,\omega$, Duhamel's integral can be expressed in terms of $h_S(t)$:

$$u_P(t) = \int_0^t h(t-\tau)\ f(\tau)\,d\tau = \frac{\omega}{k}\int_0^t h_S(t-\tau)\ \ f(\tau)\ d\tau\,. \tag{12}$$

A compact form of the general solution of the equation of motion, $u(t) = u_H(t) + u_P(t)$, is obtained by adding Eqs. (7) and (12):

$$u(t) = u_0\, h_C(t) + \left(u_0\,\zeta + \frac{\overset{(1)}{u}_0}{\omega} \right) h_S(t) + \frac{\omega}{k}\int_0^t h_S(t-\tau)\ \ f(\tau)\,d\tau \tag{13}$$

2.3 N^{th} Integration by Parts of Duhamel's Integral - Summary

This section presents a summary of the process leading to a compact expression for the N^{th} integration by parts of Duhamel's integral. The right-hand side of Eq. (12) is the selected form of the original integral to be successively integrated by parts. Herein, it is denoted as $D_0(t)$:

$$D_0(t) = \int_0^t h(t-\tau)\ f(\tau)\,d\tau = \frac{\omega}{k}\int_0^t h_S(t-\tau)\ f(\tau)\,d\tau\,. \tag{14}$$

The zero subscript in $D_0(t)$ indicates the original expression without having been integrated by parts yet. After *j* successive integrations, the resulting expression is denoted $D_j(t)$. Each successive integration by parts expresses the same entity in a different mathematical form. That is:

$$D_0(t) = D_1(t) = D_2(t) = D_3(t) = \cdots = D_j(t) = \cdots = D_N(t)\,. \tag{15}$$

For a continuous, and *N*-times differentiable forcing function $f(\tau)$, Duhamel's integral can be successively integrated *N*-times by parts. In order to assist this integration process, four auxiliary expressions were obtained. The first two are the antiderivatives of $h_S(t-\tau)$ and $h_C(t-\tau)$ with respect to τ. Their expressions are:

$$\int h_S(t-\tau)\,d\tau = \frac{1}{\omega}\left[\zeta\, h_S\left(t-\tau\right) + h_C\left(t-\tau\right)\right] + C_S\,, \tag{16}$$

$$\int h_C(t-\tau)\,d\tau = \frac{1}{\omega}\left[\zeta\, h_C(t-\tau) + \left(\zeta^2 - 1\right) h_S(t-\tau)\right] + C_C\,, \tag{17}$$

where C_S and C_C are integration constants. These antiderivatives were used to obtain two additional expressions. They are for the integration by parts of $I_S(t)_j = \int_0^t h_S(t-\tau) \overset{(j-1)}{f}(\tau)\,d\tau$ and $I_C(t)_j = \int_0^t h_C(t-\tau) \overset{(j-1)}{f}(\tau)\,d\tau$, where $j = 1, 2, \ldots, N$:

$$\begin{aligned} I_S(t)_j &= \int_0^t h_S(t-\tau) \overset{(j-1)}{f}(\tau)\,d\tau \\ &= \frac{1}{\omega}\left\{\begin{aligned} &\overset{(j-1)}{f}(t) - \left[\zeta\, h_S(t) + h_C(t)\right] \overset{(j-1)}{f}(0) \\ &-\int_0^t \left[\zeta\, h_S(t-\tau) + h_C(t-\tau)\right] \overset{(j)}{f}(\tau)\,d\tau \end{aligned}\right\}, \end{aligned} \tag{18}$$

$$\begin{aligned} I_C(t)_j &= \int_0^t h_C(t-\tau) \overset{(j-1)}{f}(\tau)\,d\tau \\ &= \frac{1}{\omega}\left\{\begin{aligned} &\zeta \overset{(j-1)}{f}(t) - \left[\zeta\, h_C(t) + \left(\zeta^2-1\right) h_S(t)\right] \overset{(j-1)}{f}(0) \\ &- \int_0^t \left[\zeta\, h_C(t-\tau) + \left(\zeta^2-1\right) h_S(t-\tau)\right] \overset{(j)}{f}(\tau)\,d\tau \end{aligned}\right\}. \end{aligned} \tag{19}$$

Eqs. (18) and (19) are useful tools to integrate Duhamel's integral N^{th} times by parts. Even though each successive integration generates more intricate expressions, a proper pattern analysis, based on mathematical induction, yields the following, compact, general expression after N^{th} integrations by parts (Maldonado, 1992):

$$D_N(t) = \frac{1}{k}\left[\sum_{j=1}^{N}\left(\frac{r_j \overset{(j-1)}{f}(t) - h_j(t) \overset{(j-1)}{f}(0)}{\omega^{j-1}}\right) - \frac{1}{\omega^{N-1}}\int_0^t h_N(t-\tau) \overset{(N)}{f}(\tau)\,d\tau\right], \tag{20}$$

where functions $h_j(t)$ are linear combinations of $h_S(t)$ and $h_C(t)$:

$$h_j(t) = \left(\zeta\ r_j + r_{j-1}\right) h_S(t) + r_j\, h_C(t)\,. \tag{21}$$

All coefficients r_j , for $j = 1, 2, \ldots, N$, are obtained by using the following second order, three-term, linear recursion:

$r_{j+1} = -2\zeta\, r_j - r_{j-1}$ with $j = 1, 2, \ldots, (N-1)$ and starting values $r_0 = 0$ and $r_1 = 1$ (22)

or

$r_j = -2\zeta\, r_{j-1} - r_{j-2}$ with $j = 1, 2, \ldots, N$ and starting values $r_0 = 0$ and $r_{-1} = -1$ (23)

Alternatively, two closed-form expressions for r_j are presented below. One of them is given in terms of trigonometric functions, and is based on the solution of the above recursion. Its expression is:

$$r_j = \left\{ \sin\left[j \cos^{-1}(-\zeta) \right] \right\} \Big/ \sin\left[\cos^{-1}(-\zeta) \right]. \tag{24}$$

The second closed-form expression is given in terms of factorials:

$$r_j = \sum_{q=1}^{q_{\max}} \frac{(-1)^{j+q}\,(j-q)!}{(q-1)!\,(j-2q+1)!}\,(2\zeta)^{j+1-2q} \quad \text{with} \quad q_{\max} = \frac{2j+1+(-1)^{j+1}}{4} \quad \text{and} \quad j = 1, 2, \ldots, N. \tag{25}$$

As previously indicated, all expressions presented in this work are valid for subcritically damped oscillators, including the undamped case. That is, they are valid for $0 \le \zeta < 1$.

Eq. (20) presents three distinctive components:

$$D_N(t) = D_N^{[f]}(t) + D_N^{[0]}(t) + D_N^{[I]}, \tag{26}$$

where

$$D_N^{[f]}(t) = \frac{1}{k} \sum_{j=1}^{N} \frac{r_j}{\omega^{j-1}} \overset{(j-1)}{f}(t), \tag{27}$$

$$D_N^{[0]}(t) = -\frac{1}{k} \sum_{j=1}^{N} \frac{h_j(t)}{\omega^{j-1}} \overset{(j-1)}{f}(0), \tag{28}$$

$$D_N^{[I]}(t) = -\frac{1}{k\,\omega^{N-1}} \int_0^t h_N(t-\tau) \overset{(N)}{f}(\tau)\, d\tau \ . \tag{29}$$

$D_N^{[f]}(t)$ is the *force-derivative* component. It involves the original forcing function and its time derivatives. $D_N^{[0]}(t)$ is the *initial-force-derivative* component. It contains transient terms associated to the initial values of the forcing function and its derivatives at time zero. $D_N^{[I]}(t)$ is the *integral* component. It contains the remaining integral from the successive integration-by-part process. It represents the contribution of the N^{th} derivative of the forcing function to the dynamic response.

2.4 Closed-Form Expressions with $\overset{(N)}{f}(t) = 0$

An important simplification of Eq. (20) occurs if $\overset{(N)}{f}(t) = 0$. In such a case, the integral component vanishes and $D_N(t)$ reduces to:

$$D_N(t) = \frac{1}{k} \sum_{j=1}^{N} \frac{1}{\omega^{j-1}} \left(r_j \overset{(j-1)}{f}(t) - h_j(t) \overset{(j-1)}{f}(0) \right) \quad \text{with} \quad \overset{(N)}{f}(t) = 0. \tag{30}$$

Consequently, if $f(t)$ is a function of time, expressed in closed form, continuous and continuously differentiable *N* times, with respect to time, with $\overset{(N)}{f}(t) = 0$, Eq. (30) constitutes a closed-form solution of Duhamel's integral. In this case, $D_N(t)$ possesses only two components:

$$D_N(t) = D_N^{[f]}(t) + D_N^{[0]}(t) \quad \text{with} \quad \overset{(N)}{f}(t) = 0\,. \tag{31}$$

As previously indicated, the relative displacement response of the oscillator can be written in terms of the original Duhamel's integral: $u(t) = u_H(t) + D_0(t)$. After N integrations by parts, $D_0(t)$ can be substituted by $D_N(t)$ to get:

$$u(t) = u_H(t) + D_N(t)\,. \tag{32}$$

If $D_N(t)$ is given in closed form, as in Eq. (30), $u(t)$ is also a closed-form expression. In this case, substitution of Eqs. (7) and (30) into (32) produces the following exact, closed-form expression for the relative displacement of the oscillator:

$$u(t) = u_0\, h_C(t) + \left(u_0\, \zeta + \frac{\overset{(1)}{u}_0}{\omega} \right) h_S(t) + \frac{1}{k} \sum_{j=1}^{N} \frac{1}{\omega^{\,j-1}} \left(r_j \overset{(j-1)}{f}(t) - h_j(t) \overset{(j-1)}{f}(0) \right) \quad \text{with} \quad \overset{(N)}{f}(t) = 0. \tag{33}$$

Its three distinctive components are:

$$u(t) = u_H(t) + D_N^{[f]}(t) + D_N^{[0]}(t) \quad \text{with} \quad \overset{(N)}{f}(t) = 0\,, \tag{34}$$

where

$$u_H(t) = u_0\, h_C(t) + \left(u_0\, \zeta + \frac{\overset{(1)}{u}_0}{\omega} \right) h_S(t)\;, \tag{35}$$

$$D_N^{[f]}(t) = \frac{1}{k} \sum_{j=1}^{N} \frac{r_j}{\omega_n^{\,j-1}} \overset{(j-1)}{f}(t)\,, \tag{36}$$

$$D_N^{[0]}(t) = -\frac{1}{k} \sum_{j=1}^{N} \frac{h_j(t)}{\omega_n^{\,j-1}} \overset{(j-1)}{f}(0)\,. \tag{37}$$

The corresponding exact, closed-form expression for the relative velocity $\overset{(1)}{u}(t)$ is:

$$\overset{(1)}{u}(t) = u_0\, \overset{(1)}{h}_C(t) + \left(u_0\, \zeta + \frac{\overset{(1)}{u}_0}{\omega} \right) \overset{(1)}{h}_S(t) + \frac{1}{k} \sum_{j=1}^{N} \frac{1}{\omega^{\,j-1}} \left(r_j \overset{(j)}{f}(t) - \overset{(1)}{h}_j(t) \overset{(j-1)}{f}(0) \right) \quad \text{with} \quad \overset{(N)}{f}(t) = 0, \tag{38}$$

where the first derivatives of the exponentially decaying sinusoidal functions can be written in terms of $h_C(t)$ and $h_S(t)$ as follows:

$$\overset{(1)}{h}_C(t) = \omega\left[\left(\zeta^2 - 1\right) h_S(t) - \zeta\, h_C(t) \right], \tag{39}$$

$$\overset{(1)}{h}_S(t) = \omega\left[h_C(t) - \zeta\, h_S(t) \right], \tag{40}$$

$$\overset{(1)}{h}_j(t) = \omega\left[r_{j-1}\, h_C(t) - \left(\zeta\, r_{j-1} + r_j \right) h_S(t) \right]. \tag{41}$$

Substitution of Eqs. (39), (40) and (41) into (38), and rearrangement of terms, yileds:

$$\overset{(1)}{u}(t) = \overset{(1)}{u}_0\, h_C(t) - \left[u_0\, \omega + \overset{(1)}{u}_0\, \zeta \right] h_S(t) + \frac{1}{k} \sum_{j=1}^{N-1} \frac{1}{\omega^{j-1}}\, r_j\, \overset{(j)}{f}(t)$$
$$- \frac{1}{k} \sum_{j=1}^{N} \frac{1}{\omega^{j-2}} \left[r_{j-1}\, h_C(t) - \left(\zeta\, r_{j-1} + r_j \right) h_S(t) \right] \overset{(j-1)}{f}(0) \quad \text{with} \quad \overset{(N)}{f}(t) = 0, \tag{42}$$

where the upper limit of the first summation was reduced from N to $N-1$ (because $\overset{(N)}{f}(t) = 0$). In this expression, there are also three distinctive components:

$$\overset{(1)}{u}(t) = \overset{(1)}{u}_H(t) + \overset{(1)}{D}{}_N^{[f]}(t) + \overset{(1)}{D}{}_N^{[0]}(t)\,, \tag{43}$$

where

$$\overset{(1)}{u}_H(t) = \overset{(1)}{u}_0\, h_C(t) - \left[u_0\, \omega + \overset{(1)}{u}_0\, \zeta \right] h_S(t)\,, \tag{44}$$

$$\overset{(1)}{D}{}_N^{[f]}(t) = \frac{1}{k} \sum_{j=1}^{N-1} \frac{r_j}{\omega^{j-1}}\, \overset{(j)}{f}(t)\,, \tag{45}$$

$$\overset{(1)}{D}{}_N^{[0]}(t) = -\frac{1}{k} \sum_{j=1}^{N} \frac{\left[r_{j-1}\, h_C(t) - \left(\zeta\, r_{j-1} + r_j \right) h_S(t) \right]}{\omega^{j-2}}\, \overset{(j-1)}{f}(0)\,. \tag{46}$$

3 EXAMPLE

Closed-Form Solution for a Quartic Polynomial Forcing Function.

This example considers the motion of a SDF, linear oscillator subjected to zero initial conditions, $u_0 = \overset{(1)}{u}_0 = 0$, and to a quartic polynomial forcing function $f(t) = \sum_{j=1}^{N} a_j\, t^{\,j-1}$, with $N = 5$. The normalized equation of motion, with respect to the stiffness k, is:

$$\frac{\overset{(2)}{u}(t)}{\omega^2} + \frac{2\zeta}{\omega}\overset{(1)}{u}(t) + u(t) = \frac{f(t)}{k} = \frac{1}{k} \sum_{n=1}^{5} a_n\, t^{\,n-1} \quad ; \quad u_0 = 0 \quad ; \quad \overset{(1)}{u}_0 = 0\,. \tag{47}$$

The quartic polynomial pulse is selected to be similar in shape to the haversine pulse, $\mathrm{hav}(2\pi\, t_\eta) = \sin^2(\pi\, t_\eta)$, where $t_\eta = t/T_d$ is the normalized time, and T_d is the pulse duration. The haversine pulse is commonly used in automobile crash and safety studies (Varat, 2003). It models the main dynamic force produced in frontal barrier collisions. Figure 1 shows the normalized shapes of both, the quartic and the haversine pulses. The quartic pulse, $p(t_\eta)$, is herein defined with amplitude A and time duration T_d. Its expression, in terms of t_η, is:

$$p(t_\eta) = 16\, A \left[t_\eta^2 - 2\, t_\eta^3 + t_\eta^4 \right] \quad \text{with} \quad 0 \le t_\eta \le 1\,. \tag{48}$$

The corresponding polynomial forcing function, in terms of time t, is:

$$f(t) = \sum_{n=1}^{5} a_n\, t^{\,n-1} = 16\, A \left[\left(\frac{t}{T_d} \right)^2 - 2 \left(\frac{t}{T_d} \right)^3 + \left(\frac{t}{T_d} \right)^4 \right] \quad \text{with} \quad 0 \le t \le T_d\,. \tag{49}$$

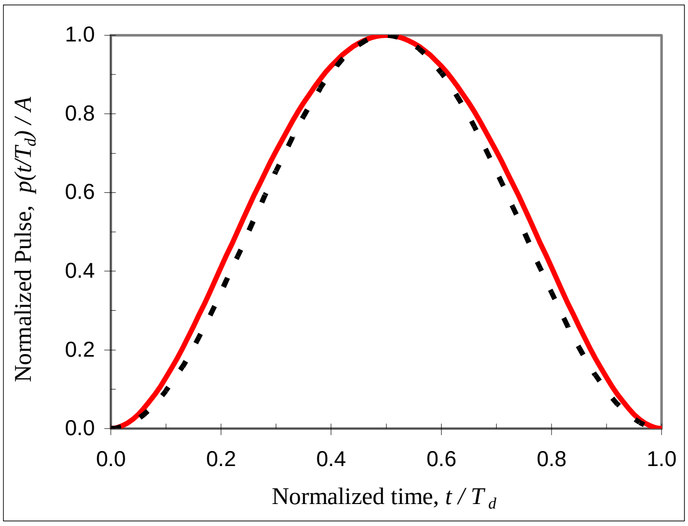

Figure 1: Quartic (solid line) and haversine (traced line) pulses.

All five coefficients a_n are identified in the following expanded form of Eq. (49):

$$f(t)=\sum_{n=1}^{5} a_n\,t^{n-1}=(0)\,t^0+(0)\,t^1+\left(\frac{16A}{T_d^2}\right)t^2+\left(-\frac{32A}{T_d^3}\right)t^3+\left(\frac{16A}{T_d^4}\right)t^4, \tag{50}$$

$$a_1=a_2=0 \quad , \quad a_3=\frac{16\,A}{T_d^2} \quad , \quad a_4=-\frac{32\,A}{T_d^3} \quad , \quad a_5=\frac{16\,A}{T_d^4}. \tag{51}$$

Table 1 shows the characteristics of the forcing pulse, its derivatives and their corresponding values at time zero. In this table, the expressions are presented in such a form that the normalized time $t_\eta=t/T_d$ is easily identified. Table 2 shows the recursive coefficients r_j and their combinations used in the expressions of the response components.

The general closed-form solution for the relative displacement $u(t)$ is given by Eqs. (34) to (37). In this particular example, $N=5$, $u_0=\overset{(1)}{u_0}=0$, and component $u_H(t)=0$. Therefore, Eq. (34) reduces to:

$$u(t)=D_5^{[f]}(t)+D_5^{[0]}(t). \tag{52}$$

In order to analyze these two components, each of their terms is written as a single entity, $\Delta_j^{[f]}(t)$ or $\Delta_j^{[0]}(t)$, as follows:

$$D_5^{[f]}(t)=\frac{1}{k}\sum_{j=1}^{5}\frac{r_j}{\omega^{j-1}}\overset{(j-1)}{f}(t)=\sum_{j=1}^{5}\Delta_j^{[f]}(t) \quad \text{where} \quad \Delta_j^{[f]}(t)=\frac{r_j}{\omega^{j-1}}\frac{\overset{(j-1)}{f}(t)}{k}, \tag{53}$$

$$D_5^{[0]}(t)=-\frac{1}{k}\sum_{j=1}^{5}\frac{h_j(t)}{\omega^{j-1}}\overset{(j-1)}{f}(0)=\sum_{j=1}^{5}\Delta_j^{[0]}(t) \quad \text{where} \quad \Delta_j^{[0]}(t)=-\frac{h_j(t)}{\omega^{j-1}}\frac{\overset{(j-1)}{f}(0)}{k}. \tag{54}$$

The individual response terms $\Delta_j^{[f]}(t)$ and $\Delta_j^{[0]}(t)$, due to the quartic pulse, are shown in Tables 3 and 4, respectively.

j	*Polynomial Coefficients* a_j	*Forcing Function and its Time Derivatives* $\overset{(j-1)}{f}(t)$	*Evaluated Forcing Function & Derivatives at Time Zero* $\overset{(j-1)}{f}(0)$
1	0	$f(t)=16A\left[\left(\frac{t}{T_d}\right)^2-2\left(\frac{t}{T_d}\right)^3+\left(\frac{t}{T_d}\right)^4\right]$	$f(0)=0$
2	0	$\overset{(1)}{f}(t)=\frac{16A}{T_d}\left[2\left(\frac{t}{T_d}\right)-6\left(\frac{t}{T_d}\right)^2+4\left(\frac{t}{T_d}\right)^3\right]$	$\overset{(1)}{f}(0)=0$
3	$\frac{16A}{T_d^2}$	$\overset{(2)}{f}(t)=\frac{16A}{T_d^2}\left[2-12\left(\frac{t}{T_d}\right)+12\left(\frac{t}{T_d}\right)^2\right]$	$\overset{(2)}{f}(0)=\frac{32A}{T_d^2}$
4	$-\frac{32A}{T_d^3}$	$\overset{(3)}{f}(t)=\frac{16A}{T_d^3}\left[-12+24\left(\frac{t}{T_d}\right)\right]$	$\overset{(3)}{f}(0)=-\frac{192A}{T_d^3}$
5	$\frac{16A}{T_d^4}$	$\overset{(4)}{f}(t)=\frac{16A}{T_d^4}[24]$	$\overset{(4)}{f}(0)=\frac{384A}{T_d^4}$

Table 1: Characteristics of the Selected Quartic Polynomial Forcing Function: $f(t)=p(t/T_d)$.

j	*Coefficients* $r_j=-2\zeta\, r_{j-1}-r_{j-2}$ *with* $r_0=0$ *and* $r_{-1}=-1$	*Combined Coefficients in Expression for* $\Delta_j^{[0]}(t)$ $(\zeta\ r_j+r_{j-1})$	*Combined Coefficients in Expression for* $\overset{(1)}{\Delta}{}_j^{[0]}(t)$ $(\zeta\ r_{j-1}+r_j)$
1	1	(ζ)	(1)
2	(-2ζ)	$(-2\zeta^2+1)$	$(-\zeta)$
3	$(4\zeta^2-1)$	$(4\zeta^3-3\zeta)$	$(2\zeta^2-1)$
4	$(-8\zeta^3+4\zeta)$	$(-8\zeta^4+8\zeta^2-1)$	$(-4\zeta^3+3\zeta)$
5	$(16\zeta^4-12\zeta^2+1)$	$(16\zeta^5-20\zeta^3+5\zeta)$	$(8\zeta^4-8\zeta^2+1)$

Table 2: Coefficients Used in the Expressions for the Relative Displacement and Velocity Responses.

j	$D_N^{[f]}(t) = \dfrac{1}{k}\sum_{j=1}^{N}\dfrac{r_j}{\omega^{j-1}}\overset{(j-1)}{f}(t) = \sum_{j=1}^{N}\Delta_j^{[f]}(t)$ *where* $\Delta_j^{[f]} = \dfrac{r_j}{\omega^{j-1}}\dfrac{\overset{(j-1)}{f}(t)}{k}$	
1	$\Delta_1^{[f]}(t) =$	$\dfrac{f(t)}{k} = 16\left[\left(\dfrac{t}{T_d}\right)^2 - 2\left(\dfrac{t}{T_d}\right)^3 + \left(\dfrac{t}{T_d}\right)^4\right]\dfrac{A}{k}$
2	$\Delta_2^{[f]}(t) =$	$64\dfrac{(-\zeta)}{\omega T_d}\left[\left(\dfrac{t}{T_d}\right) - 3\left(\dfrac{t}{T_d}\right)^2 + 2\left(\dfrac{t}{T_d}\right)^3\right]\dfrac{A}{k}$
3	$\Delta_3^{[f]}(t) =$	$32\dfrac{\left(-1+4\zeta^2\right)}{\omega^2 T_d^2}\left[1 - 6\left(\dfrac{t}{T_d}\right) + 6\left(\dfrac{t}{T_d}\right)^2\right]\dfrac{A}{k}$
4	$\Delta_4^{[f]}(t) =$	$768\dfrac{\left(\zeta - 2\zeta^3\right)}{\omega^3 T_d^3}\left[-1 + 2\left(\dfrac{t}{T_d}\right)\right]\dfrac{A}{k}$
5	$\Delta_5^{[f]}(t) =$	$384\dfrac{\left(16\zeta^4 - 12\zeta^2 + 1\right)}{\omega^4 T_d^4}\dfrac{A}{k}$

Table 3: Individual Force-Derivative Terms Contributing to the Relative Displacement Response of an Oscillator Subjected to Pulse $p(t/T_d)$.

j	$D_N^{[0]}(t) = -\dfrac{1}{k}\sum_{j=1}^{N}\dfrac{h_j(t)}{\omega^{j-1}}\overset{(j-1)}{f}(0) = \sum_{j=1}^{N}\Delta_j^{[0]}(t)$ *where* $\Delta_j^{[0]}(t) = -\dfrac{h_j(t)\overset{(j-1)}{f}(0)}{\omega^{j-1}k} = -\dfrac{\left[\left(\zeta\ r_j + r_{j-1}\right)h_S(t) + r_j\, h_C(t)\right]}{\omega^{j-1}}\dfrac{\overset{(j-1)}{f}(0)}{k}$	
1	$\Delta_1^{[0]}(t) =$	0
2	$\Delta_2^{[0]}(t) =$	0
3	$\Delta_3^{[0]}(t) =$	$(-32)\dfrac{\left[\left(4\zeta^3 - 3\zeta\right)h_S(t) + \left(4\zeta^2 - 1\right)h_C(t)\right]}{\omega^2 T_d^2}\dfrac{A}{k}$
4	$\Delta_4^{[0]}(t) =$	$(192)\dfrac{\left[\left(-8\zeta^4 + 8\zeta^2 - 1\right)h_S(t) + \left(-8\zeta^3 + 4\zeta\right)h_C(t)\right]}{\omega^3 T_d^3}\dfrac{A}{k}$
5	$\Delta_5^{[0]}(t) =$	$(-384)\dfrac{\left[\left(16\zeta^5 - 20\zeta^3 + 5\zeta\right)h_S(t) + \left(16\zeta^4 - 12\zeta^2 + 1\right)h_C(t)\right]}{\omega^4 T_d^4}\dfrac{A}{k}$

Table 4: Individual Initial-Force-Derivative Terms Contributing to the Relative Displacement Response of an Oscillator Subjected to Quartic Pulse $p(t/T_d)$.

For zero initial conditions of the motion, the relative velocity response of an oscillator, subjected to the above quartic pulse ($N = 5$), is given by the following two main components:

$$\overset{(1)}{u}(t) = \overset{(1)}{D}{}_5^{[f]}(t) + \overset{(1)}{D}{}_5^{[0]}(t)\,, \tag{55}$$

where $\overset{(1)}{D}{}_5^{[f]}(t)$ is the first derivative of the force-derivative component, and $\overset{(1)}{D}{}_5^{[0]}(t)$ is the first derivative of the initial-force-derivative component. To analyze their individual terms, the following expressions are considered:

$$\overset{(1)}{D}{}_5^{[f]}(t) = \frac{1}{k}\sum_{j=1}^{5-1}\frac{r_j}{\omega^{j-1}}\overset{(j)}{f}(t) = \sum_{j=1}^{5-1}\overset{(1)}{\Delta}{}_j^{[f]} \quad \text{where} \quad \overset{(1)}{\Delta}{}_j^{[f]} = \sum_{j=1}^{5-1}\frac{r_j}{k\,\omega^{j-1}}\overset{(j)}{f}(t)\,, \tag{56}$$

and

$$\begin{aligned}\overset{(1)}{D}{}_5^{[0]}(t) &= -\frac{1}{k}\sum_{j=1}^{5}\frac{\overset{(1)}{h}_j(t)}{\omega^{j-2}}\overset{(j-1)}{f}(0)\\ &= -\frac{1}{k}\sum_{j=1}^{5}\frac{\left[r_{j-1}\,h_C(t) - \left(\zeta\, r_{j-1} + r_j\right)h_S(t)\right]}{\omega^{j-2}}\overset{(j-1)}{f}(0) = \sum_{j=1}^{5}\overset{(1)}{\Delta}{}_j^{[0]}(t)\\ &\text{where} \quad \overset{(1)}{\Delta}{}_j^{[0]}(t) = \frac{\left[\left(\zeta\, r_{j-1} + r_j\right)h_S(t) - r_{j-1}\,h_C(t)\right]}{\omega^{j-2}}\frac{\overset{(j-1)}{f}(0)}{k}.\end{aligned} \tag{57}$$

In order to graph the relative velocity response in normalized fashion, it is more convenient to rewrite Eqs. (56) and (57) in terms of the damping coefficient c. Hence, the stiffness coefficient k is substituted by the following expression:

$$k = \left(\omega c\right)\big/\left(2\zeta\right). \tag{58}$$

After this substitution, Eqs. (56) and (57) become:

$$\overset{(1)}{D}{}_5^{[f]}(t) = \frac{2\zeta}{\omega c}\sum_{j=1}^{5-1}\frac{r_j}{\omega^{j-1}}\overset{(j)}{f}(t) = \sum_{j=1}^{5-1}\overset{(1)}{\Delta}{}_j^{[f]} \quad \text{where} \quad \overset{(1)}{\Delta}{}_j^{[f]} = 2\zeta\,\frac{r_j}{\omega^{j}}\frac{\overset{(j)}{f}(t)}{c}\,, \tag{59}$$

$$\begin{aligned}\overset{(1)}{D}{}_5^{[0]}(t) &= -\frac{2\zeta}{\omega c}\sum_{j=1}^{5}\frac{\overset{(1)}{h}_j(t)}{\omega^{j-2}}\overset{(j-1)}{f}(0)\\ &= -\frac{2\zeta}{\omega c}\sum_{j=1}^{5}\frac{\left[r_{j-1}\,h_C(t) - \left(\zeta\, r_{j-1} + r_j\right)h_S(t)\right]}{\omega^{j-2}}\overset{(j-1)}{f}(0) = \sum_{j=1}^{5}\overset{(1)}{\Delta}{}_j^{[0]}(t)\\ &\text{where} \quad \overset{(1)}{\Delta}{}_j^{[0]}(t) = 2\zeta\,\frac{\left[\left(\zeta\, r_{j-1} + r_j\right)h_S(t) - r_{j-1}\,h_C(t)\right]}{\omega^{j-1}}\frac{\overset{(j-1)}{f}(0)}{c}.\end{aligned} \tag{60}$$

The individual response terms $\overset{(1)}{\Delta}{}_j^{[f]}(t)$ and $\overset{(1)}{\Delta}{}_j^{[0]}(t)$, due to the quartic pulse, are shown in Tables 3 and 4, respectively.

j	$\overset{(1)}{D}{}_N^{[f]}(t) = \frac{2\zeta}{\omega c}\sum_{j=1}^{N-1}\frac{r_j}{\omega^{j-1}}\overset{(j)}{f}(t) = \sum_{j=1}^{N-1}\overset{(1)}{\Delta}{}_j^{[f]}$ *where* $\overset{(1)}{\Delta}{}_j^{[f]} = 2\zeta\frac{r_j}{\omega^j}\frac{\overset{(j)}{f}(t)}{c}$	
1	$\overset{(1)}{\Delta}{}_1^{[f]}(t) =$	$\frac{64\zeta}{\omega T_d}\left[\left(\frac{t}{T_d}\right) - 3\left(\frac{t}{T_d}\right)^2 + 2\left(\frac{t}{T_d}\right)^3\right]\frac{A}{c}$
2	$\overset{(1)}{\Delta}{}_2^{[f]}(t) =$	$128\frac{\zeta^2}{\omega^2 T_d^2}\left[-1 + 6\left(\frac{t}{T_d}\right) - 6\left(\frac{t}{T_d}\right)^2\right]\frac{A}{c}$
3	$\overset{(1)}{\Delta}{}_3^{[f]}(t) =$	$384\frac{\left(-\zeta + 4\zeta^3\right)}{\omega^3 T_d^3}\left[-1 + 2\left(\frac{t}{T_d}\right)\right]\frac{A}{c}$
4	$\overset{(1)}{\Delta}{}_4^{[f]}(t) =$	$3072\frac{\left(\zeta^2 - 2\zeta^4\right)}{\omega^4 T_d^4}\frac{A}{c}$

Table 5: Individual Force-Derivative Terms Contributing to the Relative Velocity Response of an Oscillator Subjected to Quartic Pulse $p(t/T_d)$.

j	$\overset{(1)}{D}{}_5^{[0]}(t) = -\frac{2\zeta}{\omega c}\sum_{j=1}^{N}\frac{\overset{(1)}{h}_j(t)}{\omega^{j-2}}\overset{(j-1)}{f}(0) = \sum_{j=1}^{N}\overset{(1)}{\Delta}{}_j^{[0]}(t)$ *where* $\overset{(1)}{\Delta}{}_j^{[0]}(t) = 2\zeta\frac{\left[\left(\zeta r_{j-1} + r_j\right)h_S(t) - r_{j-1}h_C(t)\right]}{\omega^{j-1}}\frac{\overset{(j-1)}{f}(0)}{c}$	
1	$\overset{(1)}{\Delta}{}_1^{[0]}(t) =$	0
2	$\overset{(1)}{\Delta}{}_2^{[0]}(t) =$	0
3	$\overset{(1)}{\Delta}{}_3^{[0]}(t) =$	$64\zeta\frac{\left[\left(2\zeta^2 - 1\right)h_S(t) + 2\zeta h_C(t)\right]}{\omega^2 T_d^2}\frac{A}{c}$
4	$\overset{(1)}{\Delta}{}_4^{[0]}(t) =$	$384\zeta\frac{\left[\left(4\zeta^3 - 3\zeta\right)h_S(t) + \left(4\zeta^2 - 1\right)h_C(t)\right]}{\omega^3 T_d^3}\frac{A}{c}$
5	$\overset{(1)}{\Delta}{}_5^{[0]}(t) =$	$768\zeta\frac{\left[\left(8\zeta^4 - 8\zeta^2 + 1\right)h_S(t) + \left(8\zeta^3 - 4\zeta\right)h_C(t)\right]}{\omega^4 T_d^4}\frac{A}{c}$

Table 6: Individual Initial-Force-Derivative Terms Contributing to the Relative Velocity Response of an Oscillator Subjected to Quartic Pulse $p(t/T_d)$.

In order to compare the response components listed in Tables 3 to 6, the following normalization process is performed. Firstly, the normalized time, t_η, and normalized frequency, Ω, are defined as:

$$t_\eta = \frac{t}{T_d} \quad \text{and} \quad \Omega = \frac{\omega}{\left(2\pi/T_d\right)}. \tag{61}$$

The associated normalized damped frequency is $\Omega_d = \Omega\sqrt{1-\zeta^2}$. Secondly, the following substitutions are performed in the expressions of the responses:

$$t \to t_\eta T_d \quad ; \quad \omega \to 2\pi\Omega/T_d \quad ; \quad \omega t \to 2\pi\Omega t_\eta \quad ; \quad \omega_d t \to 2\pi\Omega_d t_\eta \quad ; \quad \omega T_d \to 2\pi\Omega. \tag{62}$$

Then, responses are written in the following form:

$$u(t) = u(t_\eta T_d) = d(t_\eta)\left(A/k\right), \tag{63}$$

$$\overset{(1)}{u}(t) = \overset{(1)}{u}(t_\eta T_d) = v(t_\eta)\left(A/c\right). \tag{64}$$

where $d(t_\eta)$ is the normalized relative displacement, $u(t)/\left(A/k\right)$, and $v(t_\eta)$ is the normalized relative velocity, $\overset{(1)}{u}(t)/\left(A/c\right)$. Both, $d(t_\eta)$ and $v(t_\eta)$, are written in terms of normalized time and normalized frequency.

At $t_\eta = 1$, the forcing pulse ceases. Hence, there is no forcing action on the oscillator at normalized times $t_\eta > 1$. During this stage, the free vibration is caused by the displacement and velocity conditions attained at the end of the pulse (i.e., at $t = T_d$ or $t_\eta = 1$):

$$u(T_d) = d(1)\left(A/k\right), \tag{65}$$

$$\overset{(1)}{u}(T_d) = v(1)\left(A/c\right). \tag{66}$$

where all quantities are constant values.

For $t > T_d$ (or $t_\eta > 1$), the relative displacement is denoted $u_{free}(t_s)$, and is obtained by using Eqs. (7), (65), and (66). Its expression is:

$$u_{free}(t_s) = u(T_d)\, h_C(t_s) + \left(\zeta\, u(T_d) + \frac{\overset{(1)}{u}(T_d)}{\omega}\right) h_S(t_s), \tag{67}$$

where $t_s = t - T_d$ is the shifted time. Substitution of Eqs. (65) and (66) into (67), and substitution of the damping coefficient c by $2\zeta k/\omega$, results in the following expression for $u_{free}(t_s)$:

$$u_{free}(t_s) = \left[d(1)\; h_C(t_s) + \left(\zeta\, d(1) + \frac{v(1)}{2\zeta}\right) h_S(t_s)\right]\left(\frac{A}{k}\right), \tag{68}$$

This expression can be rewritten in terms of normalized time t_η and frequency Ω. For this, t_s is substituted by $T_d\left(t_\eta - 1\right)$, ωt_s by $2\pi\Omega\left(t_\eta - 1\right)$, and $\omega_d\, t_s$ by $2\pi\Omega_d\left(t_\eta - 1\right)$. This normalization process is defined by the following expression:

$$u_{free}(t_s) = u_{free}(T_d\,[t_\eta - 1]) = d_{free}(t_\eta)\left(A/k\right), \tag{69}$$

where $d_{free}(t_\eta)$ is the normalized relative displacement, $u_{free}(t_s)/\left(A/k\right)$, of the freely vibrating oscillator, for $t_\eta > 1$.

Figure 2 shows the normalized relative displacement response of an oscillator subjected to the quartic pulse. The selected oscillator has normalized frequency $\Omega = 1$, and damping ratio $\zeta = 0.1$. The response is shown normalized with respect to A/k, the static displacement due to the pulse amplitude A.

Part (a) of Figure 2 shows the contributions to the response due to the forcing function and its derivatives. They are the polynomial functions listed in Table 3 (quartic, cubic, quadratic, linear, and constant terms). The curve indicated as *All* contains the sum of all these terms. Since these contributions are associated to Duhamel's integral, they cease when the pulse does, i.e., at normalized time $t_\eta = 1$.

Part (b) of Figure 2 presents contributions to the response due to the initial values of the pulse and its times derivatives at time zero. They correspond to the transient sinusoidal terms listed in Table 4. The contribution indicated as *Quadratic(0)* is a sinusoidal function due to the initial value of the second derivative of the quartic pulse. The contribution indicated as *Linear(0)* is a sinusoidal function due to the initial value of the third derivative of the quartic pulse. The *Constant* contribution is also a sinusoidal function. It is due to the initial value of the fourth derivative of the quartic pulse. Since these contributions are associated to Duhamel's integral, they terminate when the pulse finish. That is, they are zero at $t_\eta > 1$.

Part (c) of Figure 2 shows the total relative displacement response. From normalized times $t_\eta = 0$ to $t_\eta = 1$, this response is the sum of all individual contributions shown in parts (a) and (b). On the other hand, for $t_\eta > 1$, it shows the free vibration response due to the displacement and velocity conditions attained at the end of the pulse.

Figures 3 and 4 are similar to Figure 2. They are presented for comparison purposes. Both are for oscillators with the same damping ratio as that of Figure 2, $\zeta = 0.1$. Figure 3 shows the response of an oscillator with normalized frequency $\Omega = 0.5$, and Figure 4 presents the response of an oscillator with $\Omega = 2$.

It is observed that, for the same damping ratio, $\zeta = 0.1$, the force-derivative contribution corresponding to $\Omega = 0.5$ is larger in normalized magnitude than the one corresponding to $\Omega = 1$ and, in turn, this is larger than the one for $\Omega = 2$. This is also the case for the contribution due to the initial value of the force and its derivatives at time zero. However, even though the force-derivative and the initial-force-derivative contributions present larger normalized magnitudes for smaller frequencies, the maximum normalized peak response occurs at frequency values close to $\Omega = 1$.

For $\zeta = 0.1$, the maximum normalized peak displacement response, to the above quartic pulse, corresponds to an oscillator with $\Omega = 0.993$. The non-dimensional value of this maximum is 1.51. Therefore, it is 51% larger than the corresponding static response A/k.

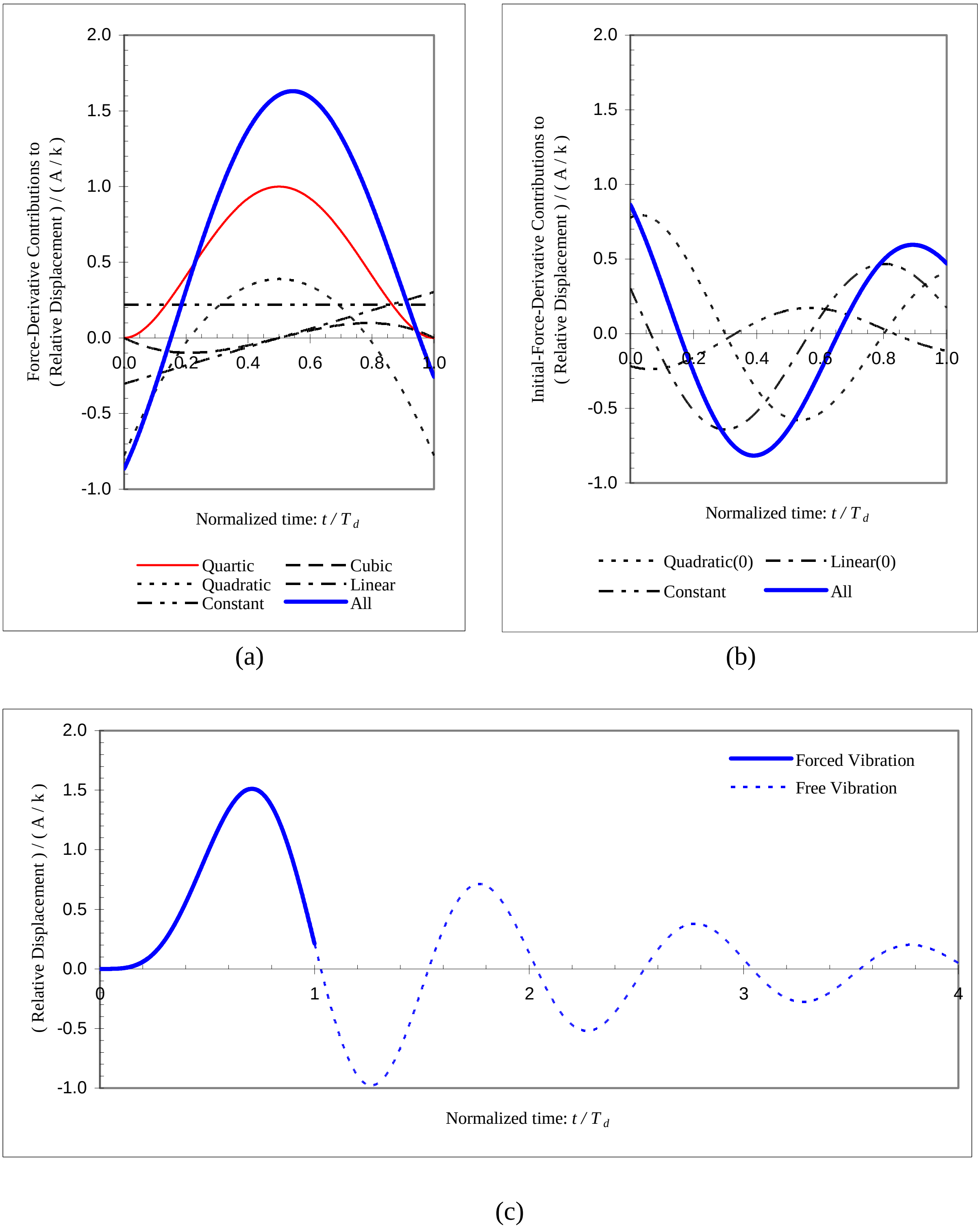

Figure 2: Displacement Response for Oscillator with $\Omega = \omega/(2\pi/T_d) = 1$ and $\zeta = 0.1$.

(a) Response Contributions due to the Force and its Derivatives.

(b) Response Contributions Due to the Initial Conditions of the Force and its Derivatives.

(c) Normalized, Total Relative Displacement Response. It Includes the Free Vibration Response for Normalized Time $t_\eta = t/T_d > 1$.

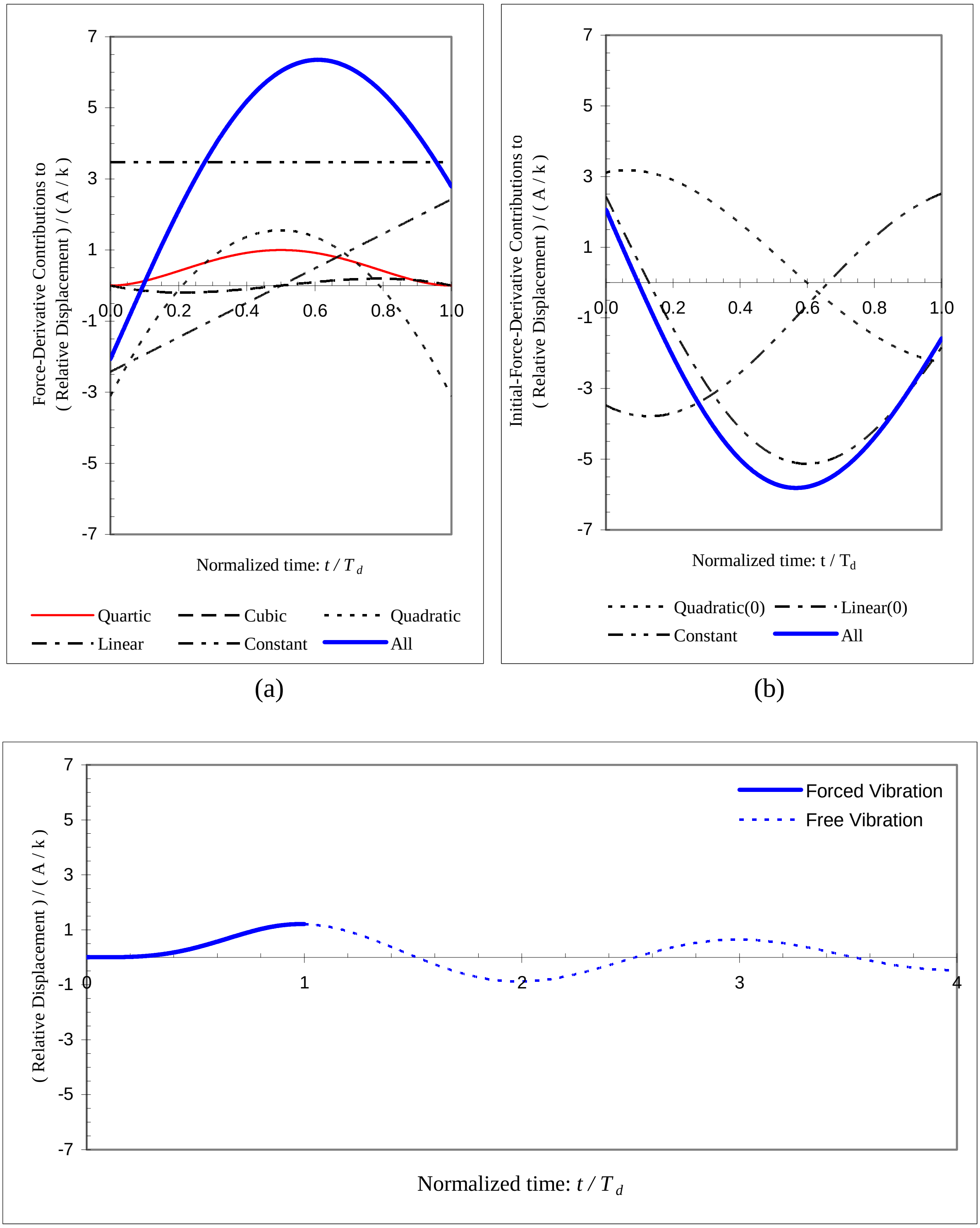

Figure 3: Displacement Response for Oscillator with $\Omega = 0.5$ and $\zeta = 0.1$.
(a) Response Contributions due to the Force and its Derivatives.
(b) Response Contributions Due to the Initial Conditions of the Force and its Derivatives.
(c) Normalized, Total Relative Displacement Response. It Includes the Free Vibration Response for Normalized Time $t_\eta = t/T_d > 1$.

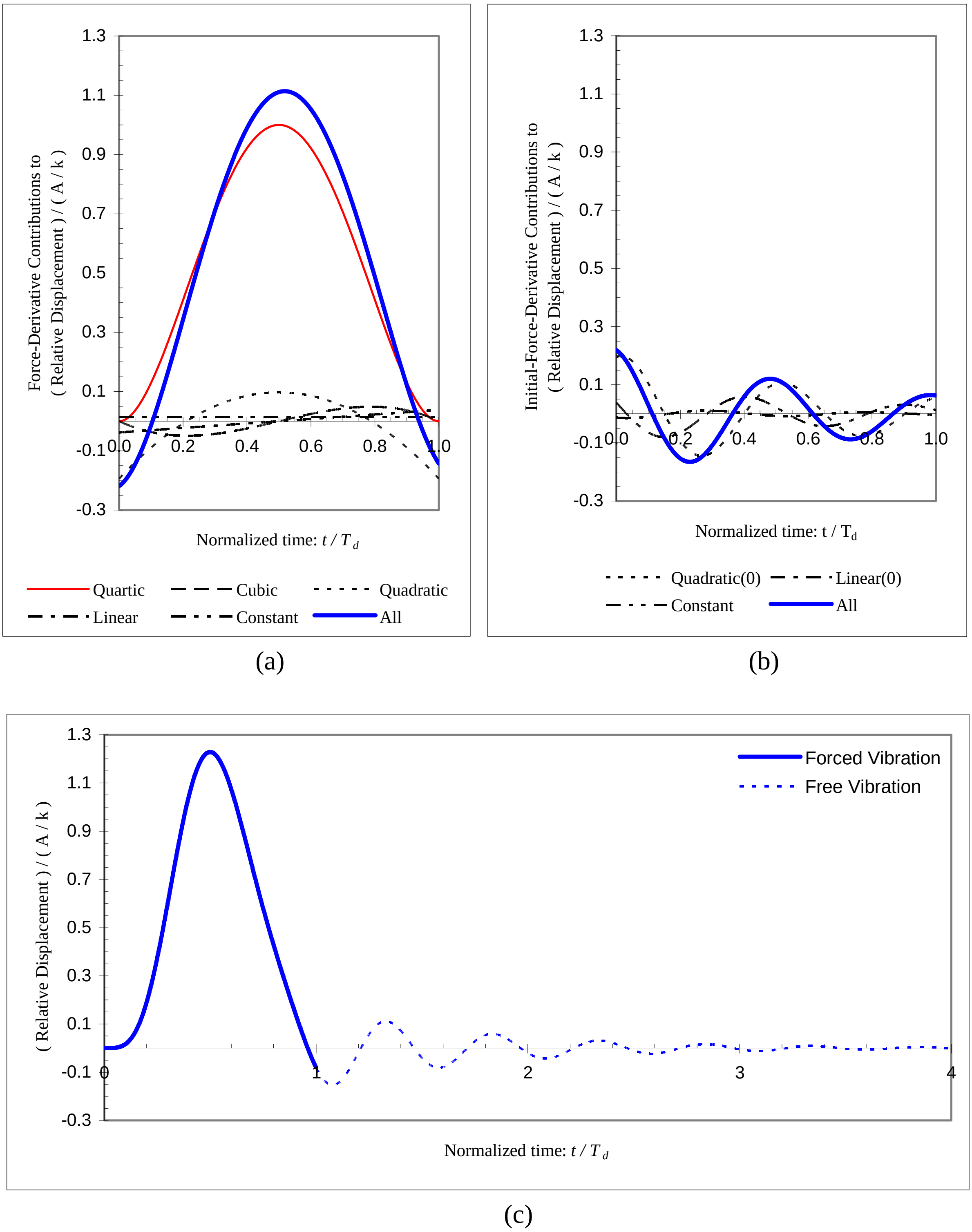

Figure 4: Displacement Response for Oscillator with $\Omega = 2$ and $\zeta = 0.1$.
(a) Response Contributions due to the Force and its Derivatives.
(b) Response Contributions Due to the Initial Conditions of the Force and its Derivatives.
(c) Normalized, Total Relative Displacement Response. It Includes the Free Vibration Response for Normalized Time $t_\eta = t/T_d > 1$.

4 CONCLUSIONS

This conference article presents closed-form expressions corresponding to N^{th} integrations by parts of Duhamel's integral and to the response of SDF linear oscillators with subcritical damping. Due to lack of space, only final expressions are presented. However, their derivation process is explained. Complete derivations are presented in a Journal article that is being submitted for possible publication.

The first proposed expression corresponds to N^{th} integrations by parts of Duhamel's integral. It is based on the use of recursive coefficients that can be written in closed form. It requires that the forcing function be continuous and N -times continuously differentiable. It is not necessarily given in closed form. However, if the N^{th} derivative of the forcing function is zero, the proposed expression is a closed-form solution of Duhamel's integral. Since polynomial forcing functions of $(N-1)^{th}$ degree satisfy the above conditions, they are used to obtain closed-form expressions for Duhamel's integral and for the response of SDF, linear oscillators.

The characteristics of the proposed expression are such that they allow for proper identification of the different components contributing to the oscillator response. These components are identified as force-derivative and as initial-force-derivative terms. It is shown that the first ones are due to the forcing function and its derivatives. They transfer their shapes to the oscillator response. For polynomial forcing functions, the force-derivative terms are also polynomials. The initial-force-derivative terms are due to the initial values of the force and its derivatives at time zero. They always possess sinusoidal shapes.

The use of the proposed expressions is presented in an example. It employs a quartic polynomial pulse as a forcing function. The pulse and its induced closed-form responses are completely defined. All response components, for relative displacement and velocity, are identified in tables. Also, for comparison purposes, three figures show normalized time histories of displacement responses for several oscillators subjected to the quartic pulse.

Since the proposed expressions are in closed form, they constitute a ready and fast alternative to the classical methodologies used to solve second order, linear differential equations, with constant coefficients, and polynomial forcing functions.

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