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EXPLICIT SOLUTIONS FOR TWO ONE-PHASE UNIDIMENSIONAL STEFAN PROBLEMS FOR A NON-CLASSICAL HEAT EQUATION

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Abstract. We consider two free boundary problems (one-phase non-classical unidimensional Stefan problems) for a non classical source function F depends on the heat flux or the total heat flux on the fixed face $x = 0$. An explicit solution of a similarity type is obtained in both cases and the behavior of the first explicit solution is studied with respect to the time t and a dimensionless parameter λ of the system.

1. INTRODUCTION

The one-phase Stefan problem for a semi-infinite material is a free boundary problem for the classical heat equation which requires the determination of the temperature distribution u of the liquid phase (melting problem) or the solid phase (solidification problem) and the evolution of the free boundary $x = s(t)$. Phase change problems appear frequently in industrial processes and other problems of technological interest (Alexiades and Solomon, 1983; Crank, 1984; Lunardini, 1991).

Non-classical heat conduction problem for a semi-infinite material were studied in (Berrone, Tarzia and Villa, 2000; Cannon and Yin, 1989; Glashoff and Sprekels, 1982; Kenmochi and Primicerio, 1988; Tarzia and Villa, 2000; Villa, 1986). A problem of this type is the following

$$\begin{cases} u_t - u_{xx} = -F\left(W(t),t\right), & x>0, \quad t>0, \\ u(0,t) = f(t), & t>0, \\ u(x,0) = h(x), & x>0 \end{cases} \tag{1}$$

where $f = f(t), h = h(x)$ are continuous real functions, and $F = F(W(t),t), t > 0$ is a given function of two variables. Some particular and interesting cases are the following:

$$F\left(W(t),t\right) = \frac{\lambda_0}{\sqrt{t}} W(t), \quad (\lambda_0 > 0) \tag{2}$$

and

$$F\left(\int_0^t W(\tau)\,d\tau, t\right) = \frac{\lambda_0}{t^{3/2}} \int_0^t W(\tau)\,d\tau, \quad (\lambda_0 > 0) \tag{3}$$

where in any case $W(t)$ represents the heat flux on the boundary $x = 0$.

Non-classical free boundary problems of the Stefan type were recently studied in (Briozzo and Tarzia, 2006b, a; Tarzia, 2001) from a theoretical point of view by using an equivalent formulation through a system of second kind Volterra integral equations (Friedman, 1959; Rubinstein, 1971; Sherman, 1967). A large bibliography on free boundary problems for the heat equation was given in (Tarzia, 2000).

In this paper, two free boundary problems (one-phase non-classical Stefan problem) which consist in determining the temperature $u = u(x,t)$ and the free boundary $x = s(t)$ such that the following conditions are satisfied, i.e.

$$\rho c u_t - k u_{xx} = -\gamma F\left(W(t),t\right), \quad 0 < x < s(t), \quad t > 0, \tag{4}$$

$$u(0,t) = f > 0, \quad t > 0, \tag{5}$$

$$u(s(t),t) = 0, \quad t > 0, \tag{6}$$

$$k u_x\left(s(t),t\right) = -\rho l \dot{s}(t), \quad t > 0, \tag{7}$$

$$s(0) = 0, \tag{8}$$

where the thermal coefficients $k, \rho, c, l, \gamma > 0$, and the control function F depends on the evolution of the heat flux at the boundary $x = 0$ as follow

$$W(t) = u_x(0,t) \qquad \text{and} \qquad F(W(t),t) = F(u_x(0,t),t) = \frac{\lambda_0}{\sqrt{t}} u_x(0,t), \tag{9}$$

or

$$W(t) = \int_0^t u_x(0,\tau)\,d\tau \qquad \text{and} \qquad F(W(t),t) = F\left(\int_0^t u_x(0,\tau)\,d\tau, t\right) = \frac{\lambda_0}{t^{3/2}} \int_0^t u_x(0,\tau)\,d\tau, \tag{10}$$

with $\lambda_0 > 0$.

In Section 2 we show an explicit solution of a similarity type for the one-phase Stefan problem (4)-(8) for a non classical control function F given by (9).

In Section 3 we consider the same one-phase Stefan problem (4)-(8) but now we consider that the non classical control function F is given by (10) instead of (9) which takes into account the total heat flux on the face $x = 0$. We also obtain an explicit solution of a similarity type for this problem which is related to the explicit solution obtained in Section 2.

In Section 4 we study the behavior of the explicit solution given in Section 2 with respect to the time t and the dimensionless parameter λ defined by (21).

2. EXPLICIT SOLUTION TO A ONE-PHASE STEFAN PROBLEM FOR A NON-CLASSICAL HEAT EQUATION WITH CONTROL FUNCTION OF THE TYPE $F(u_x(0,t),t) = \frac{\lambda_0}{\sqrt{t}} u_x(0,t)$

The free boundary problem consists in determining the temperature $u = u(x,t)$ and the free boundary $x = s(t)$ with a control function F which depends on the evolution of the heat flux at the extremum $x = 0$ given by the following conditions.

$$\rho c u_t - k u_{xx} = -\gamma F(u_x(0,t),t), \quad 0 < x < s(t), \quad t > 0, \tag{11}$$

$$u(0,t) = f > 0, \quad t > 0, \tag{12}$$

$$u(s(t),t) = 0, \quad t > 0, \tag{13}$$

$$k u_x(s(t),t) = -\rho l \dot{s}(t), \quad t > 0, \tag{14}$$

$$s(0) = 0, \tag{15}$$

where the thermal coefficients k, ρ, c, l, γ are positive and the control function F is given by (9).

In order to obtain an explicit solution of a similarity type, we define

$$\Phi(\eta) = u(x,t), \qquad \eta = \frac{x}{2a\sqrt{t}} \tag{16}$$

where $a^2 = \frac{k}{\rho c}$ is the diffusion coefficient of the phase change material.

The problem (11)-(15) and (9) is equivalent to the following one:

$$\Phi''(\eta) + 2\eta\Phi'(\eta) = 2\lambda\Phi'(0), \qquad 0 < \eta < \eta_0, \tag{17}$$

$$\Phi(0) = f, \tag{18}$$

$$\Phi(\eta_0) = 0, \tag{19}$$

$$\Phi'(\eta_0) = -\frac{2l}{c}\eta_0 \tag{20}$$

where the dimensionless parameter λ is defined by

$$\lambda = \frac{\gamma\lambda_0}{\rho c a} > 0, \tag{21}$$

and

$$s(t) = 2a\eta_0\sqrt{t} \tag{22}$$

is the free boundary where η_0 is an unknown parameter to be determined.

After some elementary computations, from (17), (18) and (19) we obtain

$$\Phi(\eta) = f\left[1 - \frac{E(\eta,\lambda)}{E(\eta_0,\lambda)}\right], \qquad 0 < \eta < \eta_0, \tag{23}$$

where

$$E(x,\lambda) = erf(x) + \frac{4\lambda}{\sqrt{\pi}}\int_0^x f_1(r)\,dr \tag{24}$$

and

$$f_1(x) = \exp(-x^2)\int_0^x \exp(r^2)\,dr \tag{25}$$

is the Dawson's integral (Abramowitz and Stegun, 1972; Petrova, Tarzia and Turner, 1994). Taking into account the condition (20), the unknown parameter $\eta_0 = \eta_0(\lambda, Ste)$ must be the solution of the following equation

$$\frac{Ste}{\sqrt{\pi}}\left[\exp(-x^2) + 2\lambda f_1(x)\right] = x\left[erf(x) + \frac{4\lambda}{\sqrt{\pi}}\int_0^x f_1(z)\,dz\right], \qquad x > 0 \tag{26}$$

where $Ste = \dfrac{fc}{l} > 0$ is the Stefan's number.

The Eq. (26) is equivalent to the following one

$$W_1(x) = 2\lambda W_2(x), \quad x > 0 \tag{27}$$

where the real functions W_1 and W_2 are defined by

$$W_1(x) = Ste\exp(-x^2) - \sqrt{\pi}\,x\,erf(x) \tag{28}$$

$$W_2(x) = 2x\int_0^x f_1(r)\,dr - Ste\, f_1(x) \tag{29}$$

with $erf\left(x\right)=\frac{2}{\sqrt{\pi}}\int_0^x \exp\left(-z^2\right)dz$.

Remark 1 If $\lambda=0$ (i.e. $\lambda_0=0$) we have the classical Lamé-Clapeyron solution and there exists a unique solution η_{00} of the Eq. (26) given by

$$F_0\left(x\right)=\frac{Ste}{\sqrt{\pi}}, \qquad x>0 \tag{30}$$

where

$$F_0\left(x\right)=x erf\left(x\right)\exp\left(x^2\right). \tag{31}$$

In order to solve the Eq.(27) we obtain previously some preliminary properties.

Lemma 1 The Dawson's integral satisfies the following properties:

$$(i)\ f_1\left(0\right)=0, \qquad (ii)\ f_1\left(+\infty\right)=0,$$

$$(iii)\ f_1'\left(x\right)=1-2x f_1\left(x\right)=\begin{cases} >0 & if \quad 0<x<x_1 \\ =0 & if \quad x=x_1 \\ <0 & if \quad x>x_1 \end{cases}$$

where

$$x_1\simeq 0.924, \qquad f_1\left(x_1\right)\simeq 0.541.$$

$$(iv)\ f_1''\left(x\right)=-2\left[1+f_1\left(x\right)\left(1-2x^2\right)\right]=\begin{cases} <0 & if \quad 0<x<x_2 \\ =0 & if \quad x=x_2 \\ >0 & if \quad x>x_2 \end{cases}$$

where

$$x_2\simeq 1.502, \qquad f_1\left(x_2\right)\simeq 0.428.$$

$$(v)\ \lim_{x\to+\infty} 2x f_1\left(x\right)=1$$

Lemma 2 The functions $W_1(x)$ and $W_2(x)$ defined by (28) and (29) respectively satisfy the following properties

a) Properties of function W_1 :

$$(i) W_1\left(0\right)=Ste, \qquad (ii) W_1\left(+\infty\right)=-\infty, \qquad (iii)\ \lim_{x\to+\infty}\frac{W_1\left(x\right)}{x}=-\sqrt{\pi},$$

$$(iv)\ \lim_{x\to+\infty}\left(W_1(x)+\sqrt{\pi}x\right)=0, \qquad (v) W_1'(x)<0, \quad \forall x>0, \qquad (vi) W_1\left(\eta_{00}\right)=0,$$

where η_{00} is the unique solution of the Eq. (30).

$$(vii) W_1''(x) = \begin{cases} > 0 & if \quad 0 < x < x_0 \\ = 0 & if \quad x = x_0 \\ < 0 & if \quad x > x_0 \end{cases}, \quad \text{where} \quad x_0 = \sqrt{\frac{3+2Ste}{4(1+Ste)}},$$

$$(viii) W_1''(0^+) = -2(3+2Ste) < 0.$$

b) Properties of function W_2:

$$(i) W_2(0) = 0, \qquad (ii) W_2(+\infty) = +\infty,$$

$$(iii) \text{ there exists a unique } x_4 > 0 \text{ such that } W_2(x_4) = 0,$$

$$(iv) W_2'(x) = 2\int_0^x f_1(r)dr + 2x f_1(x)(1+Ste) - Ste,$$

$$(v) \text{ there exists a unique } x_3 > 0 \text{ such that } W_2'(x_3) = 0 \text{ and } W_2(x_3) < 0,$$

$$(vi) W_2'(0^+) = -Ste < 0, \qquad (vii) W_2'(+\infty) = +\infty,$$

$$(viii) W_2''(x) = 2(1+Ste)x + 2 f_1(x)\left[2+Ste-2(1+Ste)x^2\right],$$

$$(ix) W_2''(0^+) = 0, \qquad (x) W_2(\eta_{00}) < 0.$$

Lemma 3 For each $\lambda > 0$ there exists a unique solution η_0 of the Eq.(27). This solution $\eta_0 = \eta_0(\lambda)$ satisfies the following properties

$$\begin{aligned} &(i) \eta_0(0^+) = \eta_{00}, \\ &(ii) \eta_0(+\infty) = x_4, \\ &(iii) \eta_0 = \eta_0(\lambda) \text{ is an increasing function on } \lambda. \end{aligned} \tag{32}$$

Then we have proved the following

Theorem 4 For each $\lambda > 0$ the free boundary problem (11)-(15) where F is defined by (9) has a unique similarity solution of the type

$$\begin{cases} u(x,t,\lambda) = f\left[1 - \dfrac{E(\eta,\lambda)}{E(\eta_0(\lambda),\lambda)}\right], & 0 < \eta = \dfrac{x}{2a\sqrt{t}} < \eta_0(\lambda) \\ s(t,\lambda) = 2a\eta_0(\lambda)\sqrt{t} \end{cases} \tag{33}$$

where

$$E(\eta,\lambda) = erf(\eta) + \frac{4\lambda}{\sqrt{\pi}}\int_0^\eta f_1(r)\,dr \tag{34}$$

and $\eta_0 = \eta_0(\lambda)$ is the unique solution of the Eq.(27) with $\eta_{00} < \eta_0(\lambda) < x_4$.

3. EXPLICIT SOLUTION TO A ONE-PHASE STEFAN PROBLEM FOR A NON-CLASSICAL HEAT EQUATION WITH CONTROL FUNCTION OF THE TYPE $F\left(\int_0^t u_x(0,\tau)\,d\tau, t\right) = \frac{\lambda}{t^{3/2}}\int_0^t u_x(0,\tau)\,d\tau$

In this section we shall consider an analogous problem to the free boundary problem studied in Section 2 , that is

$$\rho c u_t - k u_{xx} = -\gamma F\left(\int_0^t u_x(0,\tau)\,d\tau, t\right), \quad 0 < x < s(t), \quad t > 0, \tag{35}$$

$$u(0,t) = f > 0, \quad t > 0, \tag{36}$$

$$u(s(t),t) = 0, \quad t > 0, \tag{37}$$

$$k u_x(s(t),t) = -\rho l \dot{s}(t), \quad t > 0, \tag{38}$$

$$s(0) = 0, \tag{39}$$

where the control function F is defined by (10) which takes into account the total heat flux on the face $x = 0$.

In order to obtain the explicit solution corresponding to the problem (35)-(39) and (10) we will consider the same kind of transformations used in Section 2. Then, we solve the equivalent problem

$$\Phi''(\eta) + 2\eta\Phi'(\eta) = 2\lambda^*\Phi'(0), \qquad 0 < \eta < \eta^*_0, \tag{40}$$

$$\Phi(0) = f, \tag{41}$$

$$\Phi(\eta^*_0) = 0, \tag{42}$$

$$\Phi'(\eta^*_0) = -\frac{2l}{c}\eta^*_0 \tag{43}$$

where the new dimensionless parameter λ^* is defined by $\lambda^* = 2\lambda = \frac{2\gamma\lambda_0}{\rho c a} > 0$, and $s(t) = 2a\eta^*_0\sqrt{t}$. Therefore, we obtain the following results:

Theorem 5: For each $\lambda^* > 0$ the free boundary problem (35)-(39) has a unique similarity solution of the type

$$\begin{cases} u(x,t,\lambda^*) = f\left[1 - \dfrac{E(\eta,\lambda^*)}{E(\eta_0(\lambda^*),\lambda^*)}\right], & 0 < \eta = \dfrac{x}{2a\sqrt{t}} < \eta_0(\lambda^*) \\ s(t,\lambda^*) = 2a\eta_0(\lambda^*)\sqrt{t} \end{cases} \tag{44}$$

where $\eta_0^* = \eta_0^*(\lambda^*)$ is a unique solution of equation

$$W_1(x) = 2\lambda^* W_2(x), \quad x > 0$$

where the functions $W_1(x)$ and $W_2(x)$ are defined by (28) and (29) respectively.

Corollary 6 For each $\lambda_0 > 0$ the solution to the problem (35)-(39) for the non classical control function (10) is the solution to the problem (11)-(15) for the control function

$$F(u_x(0,t),t) = \frac{2\lambda_0}{\sqrt{t}} u_x(0,t).$$

Reciprocally, the solution to the problem (11)-(15) for the non classical control function given by (9) is the solution to the problem (35)-(39) for the control function

$$F\left(\int_0^t u_x(0,\tau)\,d\tau, t\right) = \frac{\lambda_0}{2t^{3/2}} \int_0^t u_x(0,\tau)\,d\tau.$$

Remark 2 Taking into account Lemma 4 (32) (iii) we have

$$\eta_0(\lambda^*) = \eta_0(2\lambda) > \eta_0(\lambda).$$

Moreover

$$s(t,\lambda^*) = s(t,2\lambda) \geq s(t,\lambda).$$

4. BEHAVIOR OF THE SOLUTION OF SECTION 2 WITH RESPECT TO THE TIME t AND THE DIMENSIONLESS PARAMETER λ

Now we will prove a result concerning the behavior of the solution of the free boundary problem obtained in Section 2 with respect to the time t and the dimensionless parameter λ.

Theorem 7 The explicit solution (33) of the problem (11)-(15) has the following properties:

$$(i)\ u_x(0,t,\lambda) = \frac{-f}{aE(\eta_0(\lambda),\lambda)\sqrt{\pi t}} < 0, \qquad \forall t > 0$$

$$\text{(ii)} \begin{cases} u(x,t,\lambda) \geq u_0(x,t), & \forall 0 \leq x \leq s_0(t), \quad t > 0 \\ s(t,\lambda) = s_0(t), & \forall t > 0 \end{cases}$$

where

$$\begin{cases} u_0(x,t) = f\left[1 - \frac{erf(\eta)}{erf(\eta_{00})}\right], & 0 < \eta = \frac{x}{2a\sqrt{t}} < \eta_{00}, \quad t > 0 \\ s_0(t) = s_0(t,0) = 2a\eta_{00}\sqrt{t} \end{cases}$$

$$(iii) \quad 1 \le \frac{u(x,t,\lambda)}{u_0(x,t)} \le \frac{1}{1-\frac{\eta(x,t)}{\eta_{00}}}\left[1-\frac{2}{Ste}\frac{\eta_0(\lambda)\left(1+2\lambda\|f_1\|_\infty\right)}{\exp\left(-\eta_0^2(\lambda)\right)+2\lambda f_1\left(\eta_0(\lambda)\right)}\eta(x,t)\right]$$

$$(iv) \quad \lim_{t\to+\infty}\frac{u(x,t,\lambda)}{u_0(x,t)}=1 \quad \text{uniformly } \forall x\in \text{ compacts sets } \subset\left[0,s_0(t)\right)$$

Proof. (i)We have

$$u_x(x,t,\lambda)=f\,\frac{-1}{E\left(\eta_0(\lambda),\lambda\right)}\frac{\partial E}{\partial\eta}(\eta,\lambda)\frac{1}{2a\sqrt{t}}$$

$$=\frac{-f}{a\sqrt{\pi t}E\left(\eta_0(\lambda),\lambda\right)}\left[\exp\left(-\eta^2\right)+2\lambda f_1(\eta)\right]=\frac{-f\exp\left(-\eta^2\right)}{a\sqrt{\pi t}E\left(\eta_0(\lambda),\lambda\right)}\left[1+2\lambda F(\eta)\right]$$

then we get

$$u_x(0,t,\lambda)=\frac{-f}{a\sqrt{\pi t}E\left(\eta_0(\lambda),\lambda\right)}<0.$$

In particular if $\lambda_0=0$ then we get $\lambda=0$ and

$$\left\{u_0(x,t)=f\left[1-\frac{erf(\eta)}{erf(\eta_{00})}\right],\qquad 0<\eta=\frac{x}{2a\sqrt{t}}<\eta_{00},\qquad t>0\right.$$

which is the Lamé-Clapeyron classical solution (see Remark 1).
To prove (ii) we apply the maximum principle. Let $v(x,t)=u(x,t,\lambda)-u_0(x,t)$, $0\le x\le s_0(t),\quad t>0,$ which satisfies

$$\rho c v_t-kv_{xx}=\frac{\gamma\lambda_0 f}{a\sqrt{\pi t}}\frac{1}{E\left(\eta_0(\lambda),\lambda\right)}>0,\qquad v(0,t)=0,\qquad v\left(s_0(t),t\right)=u\left(s_0(t),t,\lambda\right)>0.$$

Then we get $v(x,t)\ge 0$ and (ii) holds.
To prove (iii) we take into account the following properties:

$$(a)\, erf(\eta)<\frac{2}{\sqrt{\pi}}\eta,\qquad (b)\int_0^\eta f_1(r)\,dr\le\|f_1\|_\infty\,\eta,\qquad \left(\|f_1\|_\infty=f_1(x_1)\right)$$

$$(c)\,\eta<\eta_{00}\Rightarrow\frac{erf(\eta)}{erf(\eta_{00})}\le\frac{\eta}{\eta_{00}}\Rightarrow\frac{1}{1-\frac{erf(\eta)}{erf(\eta_{00})}}\le\frac{1}{1-\frac{\eta}{\eta_{00}}}$$

(d) $\eta_0(\lambda)$ satisfies the relation

$$erf\left(\eta_0(\lambda)\right)+\frac{4\lambda}{\sqrt{\pi}}\int_0^{\eta_0(\lambda)}f_1(r)\,dr=\frac{Ste}{\sqrt{\pi}}\frac{\exp\left(-\eta^2{}_0(\lambda)\right)+2\lambda f_1\left(\eta_0(\lambda)\right)}{\eta_0(\lambda)}.$$

Then we get

$$1 \le \frac{u(x,t,\lambda)}{u_0(x,t)} = \frac{1-\dfrac{E(\eta(\lambda),\lambda)}{E(\eta_0(\lambda),\lambda)}}{1-\dfrac{erf(\eta)}{erf(\eta_{00})}} = \frac{1}{1-\dfrac{erf(\eta)}{erf(\eta_{00})}} \frac{E(\eta_0(\lambda),\lambda)-E(\eta(\lambda),\lambda)}{E(\eta_0(\lambda),\lambda)}$$

$$= \frac{1}{1-\dfrac{erf(\eta)}{erf(\eta_{00})}} \left[1 - \frac{\sqrt{\pi}}{Ste}\eta_0(\lambda) \frac{erf(\eta)+\dfrac{4\lambda}{\sqrt{\pi}}\int_0^\eta f_1(r)\,dr}{\exp\left(-\eta^2{}_0\right)+2\lambda f_1(\eta_0(\lambda))} \right]$$

$$\le \frac{1}{1-\dfrac{\eta(x,t)}{\eta_{00}}} \left[1 - \frac{2}{Ste} \frac{\eta_0(\lambda)\left(1+2\lambda\|f_1\|_\infty\right)}{\exp\left(-\eta^2{}_0(\lambda)\right)+2\lambda f_1(\eta_0(\lambda))} \eta(x,t) \right]$$

(iv) If we let $t \to +\infty$, we obtain that

$\eta(x,t) = \dfrac{x}{2a\sqrt{t}} \to 0^+$ uniformly $\forall x \in$ compact sets $\subset [0, s_0(t))$ and

$$\frac{1}{1-\dfrac{\eta(x,t)}{\eta_{00}}} \to 1, \qquad 1 - \frac{2}{Ste} \frac{\eta_0(\lambda)\left(1+2\lambda\|f_1\|_\infty\right)}{\exp\left(-\eta^2{}_0(\lambda)\right)+2\lambda f_1(\eta_0(\lambda))} \eta(x,t) \to 1$$

then we obtain

$$\lim_{t\to+\infty} \frac{u(x,t,\lambda)}{u_0(x,t)} = 1 \quad \text{uniformly } \forall x \in \text{ compacts sets } \subset [0, s_0(t)).$$

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