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SENSITIVITY ANALYSIS ON A SIMPLIFIED MODEL OF THE EEG INVERSE PROBLEM

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Abstract. The electroencephalography (EEG) inverse problem consists in finding the location of a source inside the brain from measurements of the potential collected via electrodes placed on the scalp. This method provides a noninvasive technique that would contribute in the treatment of neurological diseases such as epilepsy. The electric activity in the head is usually modeled by an elliptic equation with interfaces on a bounded domain with Cauchy data on the boundary. The source is often assumed to be a dipole where its location is a parameter of the model.
Inspired in the EEG problem, we define a parametric second order ordinary differential equation defined on a real bounded interval with an interface where Dirichlet and interface conditions are imposed. The 1D inverse problem we are interested in consists in estimating the location of the source from measurements of the solution near to the endpoint of the interval. In this work sensitivity analysis is conducted and the impact of the results in the IP for different models for the source is discussed.

1 INTRODUCTION

This work is motivated by the interest in developing non invasive techniques to localize brain sources that give rise to spikes in EEG recordings. A second order PDE model may be used to describe the electric activity within the human head and brain sources are usually described by mathematical dipoles (de Munck et al., 1988). The location and strength of these dipoles are parameters of the model. The estimation of these parameters from measured EEG activity is known as the Inverse Problem in EEG (see Hamalainen et al. (1993); Mosher et al. (1999); Sarvas (1987)). Accurate solutions of the IP may contribute to medical diagnosis of neurological diseases.

In this work we consider a 1D second order differential equation with an interface that resembles the partial differential system used to simulate the electric activity inside the head. Although the results obtained in this work can not be directly applied to the 3D case, it is an interesting problem that may help to understand common issues.

In previous works (Burns et al. (2005) and Burns et al. (2006)) the sensitivity with respect to some parameters of the model such as the interface position, the endpoint of the domain and the value of the solution at the endpoint of the domain, were considered for the homogeneous case.

Here we focus on only one parameter model q which is the location of a single dipolar source. The sensitivities for a non-homogeneous equation are computed for different mathematical source models that approximate a dipole. The results are discussed and conclusions are presented.

2 THE EEG PROBLEM

The EEG Problem is usually modeled by a Poisson-type equation evolving in a bounded domain D that represents the human head. The head is modeled as three nested sets denoted by D_1, D_2 and D_3 which represent the brain, the skull and the scalp (see figure 1).

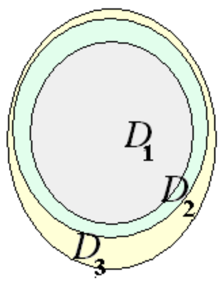

Figure 1: Head modeled as three nested sets.

The resulting PDE system is of the form

$$\nabla \cdot (\kappa(x)\nabla u(x)) = \nabla \cdot F(x) \qquad x \in D \tag{1}$$

with Neumann boundary condition

$$\frac{\partial u(x)}{\partial \eta} = 0, \qquad x \in \partial D \tag{2}$$

and interface conditions

$$\begin{aligned} u(x)\Big|_{S_i^-} &= u(x)\Big|_{S_i^+}, \\ \kappa(x)\frac{\partial u}{\partial \eta}(x)\Big|_{S_i^-} &= \kappa(x)\frac{\partial u}{\partial \eta}(x)\Big|_{S_i^+}, \end{aligned} \tag{3}$$

where S_i, i=1,2,3 are the interface surfaces with $S_3 = \partial D$ and η is the outward unit normal vector. The function u represents the electric potential and κ is a positive function that represents the electric conductivity. Usually the conductivity is taken to be constant on each partition of the head. Finally, $F(x) = J_i(x)$ is the impressed current which is often modeled as a dipole of current density (see Hamalainen et al. (1993)),

$$J_i(x) = a\delta(x - q), \tag{4}$$

where a is its magnitude, q the dipole location and $\delta(.)$ denotes the delta Dirac distribution.

In this frame the Forward Problem of EEG (FP) consists in finding the electric potential u for a given $F(x)$ while the Inverse Problem (IP) consists in finding the location and strength of the source J_i from measured values of the potential distribution u on the scalp surface. The existence and regularity of the weak solutions of (1)- (3) were stated in Troparevsky and Rubio (2003) and Troparevsky and Rubio (2005).

3 THE 1D PROBLEM

We consider a 1D parametric problem in the real interval $(0, 1)$ with an interface at $x = x_l$. We try to keep the main features that have the domain of the 3D model. We assume that u and $\kappa u'$ are continuous at the interface point while κ is not. The source is mainly localize between the origin and the interface point x_l, as shown in the figure 2.

Denoting by $'$ the spatial derivative, we may write

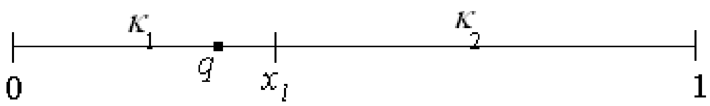

Figure 2: Domain for the 1D parametric problem.

$$(\kappa u')'(x) = F'(x) \qquad x \in (0, x_l) \cup (x_l, 1) \tag{5}$$

the boundary conditions and interface conditions are

$$\begin{array}{rclrcl} u(0) & = & 0 & u(1) & = & 1 \\ \kappa u'|_{x_l^-} & = & \kappa u'|_{x_l^+}, & u(x_l^-) & = & u(x_l^+), \end{array} \tag{6}$$

where $q \in (0, x_l)$ denotes the location of the source.

We let κ to be constant on each subinterval and independent of q, that is

$$\kappa(x) = \begin{cases} \kappa_1 & 0 < x < x_l \\ \kappa_2 & x_l < x < 1 \end{cases}. \tag{7}$$

In this frame the IP consists in finding q from observations of u at one or more points x close to 1. In Burns et al. (2005) and Burns et al. (2006) the homogeneous equations is analyzed and the sensitivities with respect to the interface point, the endpoint and the values of κ on each subinterval were studied. In this work we are concerned with the sensitivity of u with respect to the location of the source of the non-homogeneous case.

In order to conduct numerical examples we set $\kappa_1 = 1, \kappa_2 = 2$ and the interface point $x_l = 0.4$. As mathematical models for the source, different functions F concentrated around a point $q \in (0, 0.4)$ are proposed.

4 THE SENSITIVITY FUNCTIONS

From (5)- (6) we derive the differential equations for the sensitivity $s(x) = u_q(x) = \dfrac{\partial u(x)}{\partial q}$ by formal differentation (see Stanley (1999))

$$(\kappa s')'(x) = G(x) \qquad x \in (0, 0.4) \cup (0.4, 1). \tag{8}$$

The resulting boundary conditions and interface conditions

$$\begin{array}{rclcrcl} s(0) & = & 0 & \qquad & s(1) & = & 0 \\ (\kappa_1 s')|_{0.4^-} & = & (\kappa_2 s')|_{0.4^+}, & & s(0.4^-) & = & s(0.4^+), \end{array} \tag{9}$$

where $G(x) = F'_q(x) = \dfrac{\partial^2 F(x)}{\partial x \partial q}$.

5 THE SOURCE MODELS

In this section we present different models for the source F in (5) considering nearly pointwise functions depending on the position x and on the parameter q. We include different approximations of a dipole as well as the δ-Dirac distribution centered in q. Sensitivity equations are derived in each case.

5.1 A C^∞ Approximation of the δ-Dirac Distribution

First we consider the C^∞ approximation of a dipole given by

$$F_1(x) = Me^{-(\frac{x-q}{\sigma})^2}. \tag{10}$$

This function is defined over the whole domain $(0, 1)$ but it is concentrated around $q \in (0, x_l)$. The analytic solution to (8)- (9) is

$$s(x) = \begin{cases} -Me^{-(\frac{x-q}{\sigma})^2} + C_1 x + Me^{-(\frac{q}{\sigma})^2} & x \in (0, 0.4) \\ -\dfrac{M}{2}e^{-(\frac{x-q}{\sigma})^2} + C_2(x-1) + \dfrac{M}{2}e^{-(\frac{1-q}{\sigma})^2} & x \in (0.4, 1) \end{cases}, \tag{11}$$

where $C_1 = 2C_2$ and $C_2 = \frac{M}{2.8}e^{-(\frac{0.4-q}{\sigma})^2} - \frac{M}{1.4}e^{-(\frac{0-q}{\sigma})^2} + \frac{M}{2.8}e^{-(\frac{1-q}{\sigma})^2}$.

In figure 3 we plot the sensitivity s as a function of x for different values of q, $M = 10$ and $\sigma = 0.002$.

Remark Notice that the sensitivity s is nearly zero for $x \in (0.4, 1)$. This fact indicates the difficulty in the estimation of q from values of u in the interval $(0.4, 1)$.

5.2 The *Peskin's Discrete Delta Function*

As a second approximation to a dipole we consider the *Peskin's discrete delta function* (see Leveque and Zhilin (1994)) defined by

$$F_2(x) = \frac{M}{4h}\left(1 + cos(\frac{\pi(x-q)}{2h})\right)\chi_{\{|x-q|<2h\}} \tag{12}$$

where χ denotes the characteristic function of the indicated subset. Note that in this case the support of F_2 is contained in the subset $(0, 0.4)$, thus u is linear in that interval.

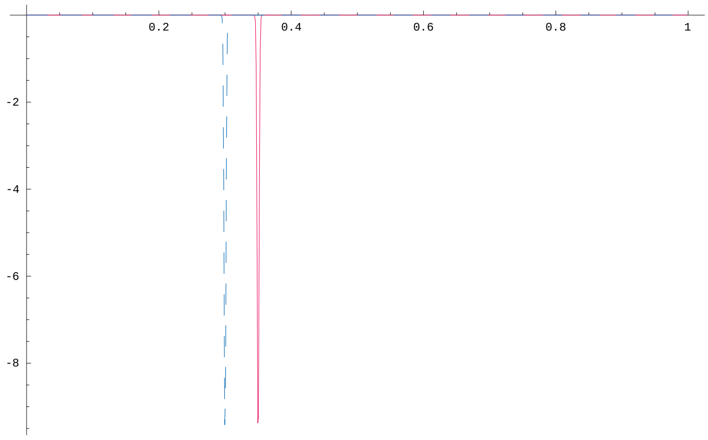

Figure 3: Sensitivity $s(x)$ for $q = 0.35$ (red solid line) and $q = 0.3$ (blue dashed line) when the source is modeled by $F_1(x) = 10e^{-(\frac{x-q}{0.002})^2}$.

The solution to (5)- (6) is the function

$$u = \begin{cases} mx & 0 < x < q-2h \\ (\frac{M}{4h}+m)x + \frac{M}{2h}\mathrm{sen}(\frac{\pi}{2h}(x-q)) + \frac{M}{4h}\mathrm{sen}(\frac{\pi}{2h}) + \gamma & q-2h < x < q+2h \\ m(x+1) & q+2h < x < 0.4 \\ \frac{1-M}{1.4}(x-1)+1 & 0.4 < x < 1 \end{cases} \tag{13}$$

where $m = (1-M)/0.7$, $\gamma = -\frac{M}{4h}(q-2h) - \frac{M}{4h}\mathrm{sen}(\frac{\pi q}{2h})$. As we pointed out above, u is linear in $(0.4, 1)$ and since the data would be given in this interval, we look for the derivative of its slope with respect to q that is the sensitivity we are interested in. This slope is $\frac{1-M}{1.4}$. It does not depend on the parameter q. Consequentely the solution u to the boundary interface problem (5)- (6) in $(0.4, 1)$ is independent of the source location. This is not an unexpected fact. Due to the continuity condition at the interface, the value of $u(0.4)$ could be affected by q. However, since $\frac{\partial F_2}{\partial x}$ is symmetric with respect to q, this influence vanishes. In consequence the function u on $(0.4, 1)$ does not depend on q.
We can make the same observation by solving to the equations (8)- (9) for the sensitivity,

$$s(x) = \begin{cases} -\dfrac{M}{4h}\left[\cos\left(\dfrac{\pi}{2h}x - q\right) - 1\right] & x \in (q-2h, q+2h) \\ 0 & x \notin (q-2h, q+2h) \end{cases}, \tag{14}$$

The sensitivty as a function of x is shown in figure 4 considering different values of q, $M = 10$ and $h = 0.0001$. We observe that the plot of sensitivity look almost identical as for the previous case.

5.3 A Perturbated *Peskin's Discrete Delta Function*

We consider the continuos C^1 approximation of the source proposed in (12) where a perturbation $\rho(x)$ is added

$$F_3(x) = F_2(x) + \rho(x) \tag{15}$$

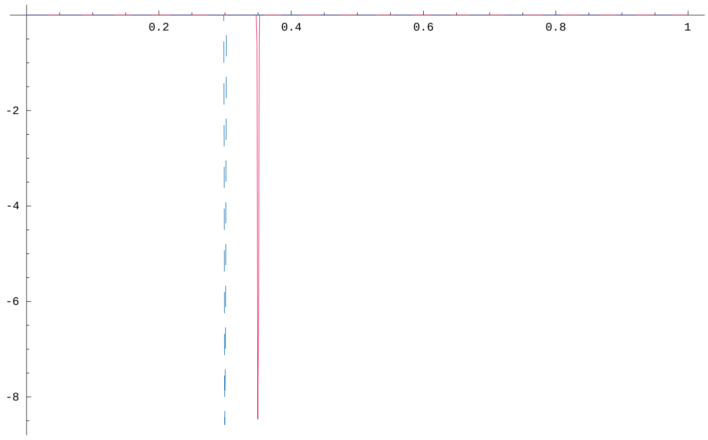

Figure 4: Sensitivity $s(x)$ for $q = 0.35$ (red solid line) and $q = 0.3$ (blue dashed line) when the source is modeled by $F_2(x) = \frac{1}{0.004}\left(1 + cos(\frac{\pi(x-q)}{0.0002})\right)\chi_{\{|x-q|<0.0002\}}$.

In this case the sensitivity of u with respect to q in $(0.4, 1)$ is independent of the amplitude M but it depends only on the sensitivity of the perturbation $\rho(x)$ with respect to q. Therefore perturbation is what is mostly reflected in the sensitivity function and in consequence we could not expect accurate estimates for q from values of u in (0.4,1) .

5.4 A Pure Dipole Source

We consider the source F as the delta distribution centered at q. In this case the equations results

$$(\kappa u')' = M\delta' \tag{16}$$

where M is the amplitude. The solution to (16) is

$$u(x) = \begin{cases} MH(x) + \frac{1-M}{0.7} & 0 < x < 0.4 \\ \frac{M}{2}H(x) + \frac{(1-M)}{1.4}(x-1) + 1 - \frac{M}{2} & 0.4 < x < 1 \end{cases} \tag{17}$$

where $H' = \delta$, is the Heaviside function with jump at q

$$H(x) = \begin{cases} 0 & 0 \le x < q \\ 1 & q \le x \le 1 \end{cases} \tag{18}$$

The values of u in $(0.4, 1)$ are independent of q, in consequence we could not recover the location of the source from data in $(0.4, 1)$. The sensitivity $s(x)$ is zero $\forall x \in (0.4, 1)$.

6 CONCLUSIONS

In this work we present a parametric second order ordinary differential equation on a real interval. The model (that resembles the one corresponding to EEG Problem) contains a source, assumed to be a *dipole*, that we are interested in estimating. The parameter of the model, q, is the location of the source. We intend to find q from observed values of the solution at points close to the right endpoint of the domain. With this purpose we study the sensitivity of the solution u with respect to q for different source models, $F_i, i = 1,,4$. When the source is modeled by $F_1 = Me^{-(\frac{x-q}{\sigma})^2}$ the sensitivity s is nearly zero on (0.4,1). When considering the Peskin discrete delta function $F_2(x) = \frac{M}{4h}\left(1 + cos(\frac{\pi(x-q)}{2h})\right)\chi_{\{|x-q|<2h\}}$ as the model of the pure dipole, the sensitivity s is zero everywhere in (0.4,1). For a perturbed case $F_3(x) = F_2(x) + \rho(x)$, the values on (0.4,1) depend on $\rho(x)$ but not on q. Finally, when the delta-Dirac function is considered, the sensitivity vanish in the whole subinterval (0.4,1). Moreover, for this kind of sources the sensitivities $s(x)$ vanish (or almost vanish) on the whole domain except for a small interval containing the source location q. The analysis of the sensitivities suggests that we could not trust on the estimation of q from data $u(x)$ for x near 1. Hence, the proposed IP cannot be solved without additional information.

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