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COMPARISON AND EVALUATION OF TWO APPROACHES OF UNCERTAINTY MODELING IN DYNAMCAL SYSTEMS

Rubens Sampaio[a], Thiago Ritto[a] and Edson Cataldo[b]

[a]*Mechanical Engineering Department – PUC-Rio, Rua Marquês de São Vicente, 225, Gávea, RJ, CEP: 22453-900, Brazil, rsampaio@mec.puc-rio.br, thiagoritto@gmail.com,*
http://www.mec.puc-rio.br
[b]*Applied Mathematics Department – Universidade Federal Fluminense, Graduate Program in Telecommunications Engineering, Graduate Program in Mechanical Engineering, Rua Mário Santos Braga, S/N, Centro, Niterói, CEP: 24020-140, RJ, Brazil, ecataldo@im.uff.br*

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Abstract. When mechanical systems are modeled, uncertainties should be taken into account for improving the predictability of the model. In this work a two d.o.f. (degrees of freedom) dynamical system is used to compare two strategies to model uncertainties in structural dynamics. Uncertainties are considered present only on the spring stiffnesses. In the first approach, uncertainties are inserted into each spring stiffness. A probabilistic model is constructed for each random variable associated to each spring stiffness. In the second approach, uncertainties are considered in a global way, that is, a probability model is constructed for the stiffness matrix. In both approaches, the probability density functions are deduced from the Maximum Entropy Principle, using only the available information. The simple example used is helpful to assure a better understanding of the two approaches. The event space generated by each strategy will be shown and it will be discussed how good they are to predict data uncertainties and model uncertainties.

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1 INTRODUCTION

In order to improve the predictability of a complex dynamical system uncertainties should be taken into account. Probability Theory and Statistics give helpful tools to model random data. Randomness was introduced in the analysis of mechanical systems at the beginning of the XXth century and, at that time, only external forces were considered as random. A good review of the random vibration history can be found in Lin (2002); Paez (2006); a more modern approach is presented in Sampaio and Soize (2007) Cataldo et al. (2007).

However, to take into account uncertainties in the modeling of dynamical systems, a strategy for constructing the probability density functions for the random variables associated to these uncertainties must be chosen and this choice is a difficult task. An idea, unfortunately still common, is to use the normal distribution for all the random variables. But, this approach can be dangerous because the errors cannot be controlled, and it may even be against the physics of the problem. For example, a positive random variable cannot, of course, assume negative values; when one says that its distribution is normal it is against the physics since negative values are given a non-zero probability.

In this paper an approach based on the Maximum Entropy Principle is used to construct the probability density functions of the random variables modeling the uncertainties. With this strategy, only the available information is used and among all of the possible probability density functions, that one with the maximum entropy (or uncertainty) is chosen. The concept of entropy is the one used by Shannon (1948) and some applications of this method can be found in Kapur and K.Kesavan (1992).

In this work, a two d.o.f. system is used for discussing these two types of uncertainties. Two approaches are used: (1) *Scalar Random Variable (SRV)*, when scalar parameters are modeled as random variables, and (2) *Matrix Random Variable (MRV)*, when system matrices are modeled as random variables. We will see why the first approach (SRV) is good to model data uncertainties while the second one (MRV) is capable of taking into account model uncertainties.

The simple lumped parameter system studied is used in order that the attention is directed to the modeling and not to the complexity of the system. We will be able to construct the event space to each of the two approaches performed.

The organization of this article is as following: in Sec. 2, the deterministic dynamical system that will be studied is presented, and this system will be taken as the mean model for the corresponding stochastic problem. The procedure to build the corresponding stochastic problem is presented in Sec. 3, for both approaches (SRV and MRV). In section 4 it is presented the numerical simulations, and the two approaches are compared. Finally, in Sec. 6 concluding remarks are outlined.

1.1 Definitions and notations used

Some definitions and notations that will be used in the text are described in the following:

(1) The deterministic spring stiffnesses are represented by lowercase letters (k_1 and k_2).

(2) Deterministic vectors are represented by boldface lowercase letters (e.g. $\mathbf{f} = (f_1, f_2)$).

(3) The random variables K_1 and K_2 represented by uppercase letters are related to the spring stiffnesses (k_1 and k_2).

(4) Random vectors are represented by boldface upper case letters (e.g. $\mathbf{Q} = (q_1, q_2)$).

(5) Deterministic matrices are represented by upper case letters between brackets (e.g. $[K]$).

(6) Random matrices are represented by boldface upper case letters between brackets (e.g. $[\mathbf{K}]$).

(7) Mean values are represented underlying the corresponding random variables (e.g. $\underline{k}$, $[\underline{K}]$, $\underline{\mathbf{Q}}$).

The following matrix sets will be used: $\mathbb{M}_{n,m}(\mathbb{R})$ is the set of all $n \times m$ real matrices; $\mathbb{M}_n(\mathbb{R}) = \mathbb{M}_{n,n}(\mathbb{R})$ is the set of all square $(n \times n)$ real matrices; $\mathbb{M}_n^S(\mathbb{R})$ is the set of all real symmetric matrices; $\mathbb{M}_n^{+0}(\mathbb{R})$ is the set of all real symmetric semipositive–definite matrices; $\mathbb{M}_n^+(\mathbb{R})$ is the set of all real symmetric positive–definite matrices; and $\mathbb{M}_n^D(\mathbb{R})$ is the set of all positive diagonal matrices.
Then,

$$\mathbb{M}_n^D(\mathbb{R}) \subset \mathbb{M}_n^+(\mathbb{R}) \subset \mathbb{M}_n^{+0}(\mathbb{R}) \subset \mathbb{M}_n^S(\mathbb{R}) \subset \mathbb{M}_n(\mathbb{R}) \,. \tag{1}$$

Let $[K_n]$ be a real square $n \times n$ matrix. The Frobenius norm of $[K_n]$ is defined by $|||[K_n]|||_F = (tr\{[K_n]^T[K_n]\})^{1/2}$ and the matrix norm of $[K_n]$ is defined by $|||[K_n]||| = \max_{\mathbf{b}\in\mathbb{R}^n, ||\mathbf{b}||=1} ||[K_n]\mathbf{b}||$, where $||\mathbf{b}||$ is the Euclidian norm of the vector $\mathbf{b}$. These norms satisfy $|||[K_n]||| \leq |||[K_n]|||_F \leq \sqrt{n}\,|||[K_n]||| \,.$

2 MEAN MODEL

In this section the mean model is presented. The two d.o.f. system represented in Fig. 1 is used to compare the two probabilistic approaches (SRV and MRV).

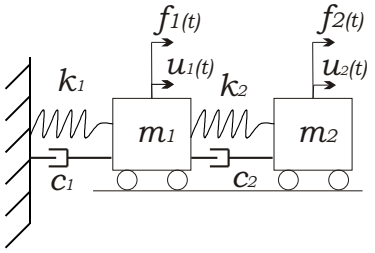

Figure 1: Two d.o.f. system used.

The system dynamics is given by:

$$[M]\ddot{\mathbf{u}} + [C]\dot{\mathbf{u}} + [K]\mathbf{u} = \mathbf{f}(t) \tag{2}$$

in which

$$[M] = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}; \quad [C] = \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 \end{bmatrix}; \quad [K] = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \tag{3}$$

and

$$\mathbf{f}(t) = \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix}; \quad \mathbf{u}(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} \,. \tag{4}$$

The mass, damping and stiffness matrices are real symmetric and positive-definite and they are denoted by $[M]$, $[C]$ and $[K]$, respectively. The external force is represented by vector $\mathbf{f} = (f_1, f_2)^T$, in which the functions f_1 and f_2 represent the forces applied on masses m_1 and m_2, respectively. The displacements of the masses are denoted by the functions u_1 and u_2, which are the components of the vector $\mathbf{u}$.

2.1 Frequency Response Function (FRF) calculated from the mean model

In this work, the response to be observed and analyzed will be in the frequency domain. For this simple linear system, the frequency domain gathers all the information of interest.

Let $\mathbf{f}(\mathbf{t}) = (f_1(t), 0)^T$ be the input force applied on the system, and let $\mathbf{u}(\mathbf{t}) = (u_1(t), u_2(t))^T$ be the correspondent output. Let $\widehat{f_1}$ be the Fourier transform of f_1, $\widehat{u}_1$ be the Fourier transform of u_1 and $\widehat{u}_2$ be the Fourier transform of u_2. In this paper, it is considered the FRF (H) defined by

$$H(\omega) = \frac{\widehat{u}_2(\omega)}{\widehat{f}_1(\omega)}\,. \tag{5}$$

The values of the parameters used for the simulations are $m_1 = 1.5\,\text{kg}$, m_2 = 0.75, kg, $k_1 = 1000\,\text{N/m}$, $k_2 = 150\,\text{N/m}$, $c_1 = 0.5\,\text{N.s/m}$, $c_2 = 0.05\,\text{N.s/m}$ (data were taken from Aguiar and Weber (2007)). Figure 2 shows the FRF calculated from the mean model. The two peaks correspond to the two natural frequencies of the system: 2.055 and 4.502 Hz.

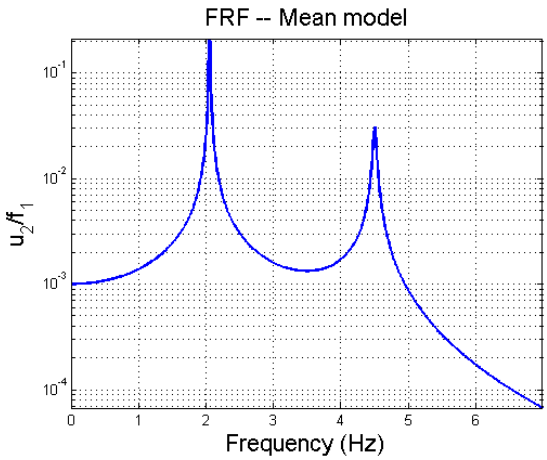

Figure 2: FRF of the mean model.

In the deterministic system the response is simply the one showed above, i.e., if another numerical simulation is performed, the results will be exactly the same. There is no uncertainty.

3 STOCHASTIC MODEL

In order to study the system behavior when uncertainties are present, some parameters, or matrices, must be considered as uncertain. As the final objective is to compare two different approaches, the uncertainties will be considered concentrated only in the stiffnesses. Modeling as SRV, the parameters k_1 and k_2 will be considered uncertain; meaning that, in the stochastic model these parameters will be substituted for random variables, K_1 and K_2, and for each one of these random variables a probability density function will be constructed. Modeling as MRV, matrix $[K]$ is globally considered as uncertain, and a probability density function will be directly associated to the corresponding random matrix, denoted by $[\mathbf{K}]$.

Clearly, for either approaches used, the stiffness matrix will be uncertain and it will be represented by a random matrix. Consequently, the functions u_1 and u_2, which represent the displacements, will not be deterministic anymore. Instead they will be random processes which will be denoted by U_1 and U_2, respectively.

3.1 Probabilistic model for Scalar Random Variables (SRV)

In this case, the parameters k_1 and k_2 are modeled by random variables K_1 and K_2, respectively. Then, an appropriate probabilistic model for each random variable must be constructed taking into account only the available information. The Maximum Entropy Principle (Shannon, 1948; Jaynes, 1957a,b) is used and it consists in maximizing the entropy subjected to constraints defined by the available information. The measure of entropy used is (Shannon):

$$S(P_K) = -\int_{\mathbb{R}} P_K(k) ln(P_K(k)) dk \tag{6}$$

The available information used for both random variables K_1 and K_2 will be described in the following:

(1) The random variables K_1 and K_2 are positive random variables, so their supports are equal to $]0, +\infty[$.
(2) The expected values of K_1 and K_2 are known and given by $E\{K_1\} = \underline{K_1}$ and $E\{K_2\} = \underline{K_2}$.
(3) K_1 and K_2 are second order random variables, that is, $E\{K_1^2\} < +\infty$ and $E\{K_2^2\} < +\infty$.
(4) As the random variables K_1 and K_2 represent stiffnesses they must satisfy $E\{1/K_1^2\} = c_1'$, with $c_1' < +\infty$ and $E\{1/K_2^2\} = c_2'$, with $c_2' < +\infty$. This constraint is taken into account by requiring that $E\{ln(K_1)\} = c_1$ with $|c_1| < +\infty$ and $E\{ln(K_2)\} = c_2$ with $|c_2| < +\infty$.

The probability density functions obtained for K_1 and K_2 are given by:

$$P_{K_1}(k_1) = \mathbf{1}_{]0,+\infty[}(k_1) \frac{1}{\underline{K_1}} \left(\frac{1}{\delta_{K_1}^2}\right)^{\frac{1}{\delta_{K_1}^2}} \frac{1}{\Gamma\left(1/\delta_{K_1}^2\right)} \left(\frac{k_1}{\underline{K_1}}\right)^{\frac{1}{\delta_{K_1}^2}-1} exp\left(-\frac{k_1}{\delta_{K_1}^2 \underline{K_1}}\right) , \tag{7}$$

and

$$P_{K_2}(k_2) = \mathbf{1}_{]0,+\infty[}(k_2) \frac{1}{\underline{K_2}} \left(\frac{1}{\delta_{K_2}^2}\right)^{\frac{1}{\delta_{K_2}^2}} \frac{1}{\Gamma\left(1/\delta_{K_2}^2\right)} \left(\frac{k_2}{\underline{K_2}}\right)^{\frac{1}{\delta_{K_2}^2}-1} exp\left(-\frac{k_2}{\delta_{K_2}^2 \underline{K_2}}\right) , \tag{8}$$

where δ_{K_1} and δ_{K_2} are the dispersion parameters and $\Gamma(z)$ is the Gamma function defined for $z > 0$ as $\Gamma(z) = \int_0^{+\infty} t^{z-1} e^{-1} dt$. There are limits for the dispersion parameters: they must be greater than zero and they should be less than $1/\sqrt{3}$.

Considering the random variables K_1 and K_2, the corresponding random matrix $[\mathbf{K}]$ becomes:

$$[\mathbf{K}] = \begin{bmatrix} K_1 + K_2 & -K_2 \\ -K_2 & K_2 \,. \end{bmatrix} \tag{9}$$

The realizations for each simulation can be performed with the aid of Matlab statistics toolbox.

3.2 Probabilistic model for Matrix Random Variables (MRV)

In this approach matrix $[K]$ is globally considered as uncertain. So, a probability density function will be constructed directly for the corresponding random matrix $[\mathbf{K}]$. This will be done following the ideas given in Soize (2001), but here the steps will be explained in detail.

3.2.1 Probability density function of a random matrix

Let $[\mathbf{K}]$ be a random matrix, $[\mathbf{K}] \subset \mathbb{M}_2(\mathbb{R})$. Let $P_{[\mathbf{K}]}([K])$ be the probability density function of $[\mathbf{K}]$. The normalization condition must be verified:

$$\int_{\mathbb{M}_2(\mathbb{R})} P_{[\mathbf{K}]}([K])dK = 1 \tag{10}$$

Where $dK = \prod_{i,j=1}^{2} dK_{ij}$ with $[K] = [K_{ij}]$.

The integral presented in eq. 10 is calculated considering the following. Let $\widetilde{\mathbf{K}} = (K_{11}, K_{12}, K_{21}, K_{22})$ be the random vector associated to the random matrix $[\mathbf{K}]$. The probability density function of $\widetilde{\mathbf{K}}$ is given by $P_{[\widetilde{\mathbf{K}}]}(\widetilde{\mathbf{k}})$. One has:

$$P_{[\widetilde{\mathbf{K}}]}(\widetilde{\mathbf{k}}) = P_{[\mathbf{K}]}([K]) \tag{11}$$

Then,

$$\int_{\mathbb{M}_2(\mathbb{R})} P_{[\mathbf{K}]}([K])dK = \int_{\mathbb{R}^4} P_{[\widetilde{\mathbf{K}}]}(\widetilde{\mathbf{k}})d\widetilde{\mathbf{k}} = 1 \tag{12}$$

3.2.2 Using the Maximum Entropy Principle to built the probability density function of a random matrix

The Maximum Entropy Principle states that within all the probability density functions ($P_{[\mathbf{K}]}$) that satisfy the available information, one searches the one with maximum entropy ($S(P_{[\mathbf{K}]})$). The entropy is a measure of uncertainty and is given by the expression below (Shannon):

$$S(P_{[\mathbf{K}]}) = -\int_{\mathbb{M}_2(\mathbb{R})} P_{[\mathbf{K}]}([K])ln(P_{[\mathbf{K}]}([K]))dK \tag{13}$$

In this work a probability density function ($P_{[\mathbf{K}]}$) of a random matrix ($[\mathbf{K}]$) will be built, using the Maximum Entropy Principle, with the following available information (represent the information that one is sure about):

(1) The matrix is positive-definite, i.e., $[\mathbf{K}]$ is in the space of the symmetric positive-definite matrices, so $[\mathbf{K}] \in \mathbb{M}_2^+(\mathbb{R})$. The probability density function may be written as:

$$\int_{\mathbb{M}_2(\mathbb{R})} P_{[\mathbf{K}]}([K])dK = 1 \tag{14}$$

where $dK = \sqrt{2}\prod_{1\leq i\leq j\leq 2} dK_{ij} = \sqrt{2}dK_{11}dK_{12}dK_{22}$. Note that the factor $\sqrt{2}$ appears due to the symmetry of $[\mathbf{K}]$.

(2) The expected value of $[\mathbf{K}]$ is known, so:

$$E\{[\mathbf{K}]\} = \int_{\mathbb{M}_2(\mathbb{R})} [K]P_{[\mathbf{K}]}([K])dK = [\underline{K}] \tag{15}$$

(3) Writing Eq. 2 in the frequency domain, it can be proved that the corresponding random equation has a unique second-order random solution if and only if $E\{||[\mathbf{K}^{-1}]^2||_F\} < +\infty$ (Soize, 2001).

To construct the probability density function of $[\mathbf{K}]$, one must take into account all the available information listed above. Using the Maximum Entropy Principle, one gets, (Soize, 1999):

$$P_{[\mathbf{K}]}([K]) = 1_{\mathbb{M}_2^+(\mathbb{R})}([K]) \times C_{\mathbf{K}} \times det([K])^{\lambda-1} \times \exp\left\{\frac{-(2\lambda+1)}{2} \ll [K]^{-1}, [K] \gg\right\} \quad (16)$$

where

$$C_{\mathbf{K}} = \frac{(2\pi)^{-1/2}\left(\frac{1+2\lambda}{\lambda}\right)^{1+2\lambda}}{\Gamma\left(\frac{1+2\lambda}{2}\right)\Gamma(\lambda)(det[K])^{(1+2\lambda)/2}} \quad \text{and} \quad \ll [A], [B] \gg = tr\{[A][B]^T\} \quad (17)$$

λ is a positive real number, it appears in the optimization problem due to the information (3): $E\{||[\mathbf{K}^{-1}]^2||_F\}$ is not known, but it is finite. The parameter λ has to do with the dispersion of the distribution function. In the MRV strategy the generation of the realizations for the Monte Carlo simulation is not straightforward. Some manipulations are required.

The Cholesky decomposition can be performed since $[\mathbf{K}]$ is positive-definite:

$$[\underline{\mathbf{K}}] = [\underline{L}_K]^T[\underline{L}_K] \quad (18)$$

where $[L_K]$ is an upper triangular matrix. A random matrix $[\mathbf{K}]$ will be written as:

$$[\mathbf{K}] = [\underline{L}_K]^T[\mathbf{G}][\underline{L}_K] \quad (19)$$

with $[\mathbf{G}]$ being a random matrix with characteristics described in the following:
(1) The matrix is positive-definite ($[\mathbf{G}] \in \mathbb{M}_2^+(\mathbb{R})$);
(2) its mean value is the identity matrix ($E\{[\mathbf{G}]\} = [I]$); and
(3) the mean square value of its inverse is finite. It can be shown (Soize, 1999) that:

$$P_{[\mathbf{G}]}([G]) = 1_{\mathbb{M}_2^+(\mathbb{R})}([G]) \times C_{\mathbf{G}} \times det([G])^{\lambda-1} \times \exp\left\{\frac{-(2\lambda+1)}{2} tr[G]\right\} \quad (20)$$

where

$$C_{\mathbf{G}} = \frac{(2\pi)^{-1/2}\left(\frac{1+2\lambda}{2}\right)^{1+2\lambda}}{\Gamma\left(\frac{1+2\lambda}{2}\right)\Gamma(\lambda)}\,. \quad (21)$$

It is still necessary one more step so the realizations of the entries of the matrix can be performed independently. The Cholesky decomposition can be performed since $[\mathbf{G}]$ is positive-definite:

$$[\underline{\mathbf{G}}] = [L]^T[L] \quad (22)$$

where $[L] \subset \mathbb{M}_2^+(\mathbb{R})$ is an upper triangular matrix:

$$[\mathbf{L}] = \begin{bmatrix} L_{11} & L_{12} \\ 0 & L_{22} \end{bmatrix} \quad (23)$$

The probability density functions of the entrances of $[\mathbf{L}]$, ($P_{[L_{11}]}, P_{[L_{12}]}, P_{[L_{22}]}$ can be calculated using eq. (20) and the definitions of the probability density functions of the marginal distributions. It can be concluded then:

$P_{L_{11}L_{12}L_{22}}(l_{11}l_{12}l_{22}) = P_{L_{11}}(l_{11})P_{L_{12}}(l_{12})P_{L_{22}}(l_{22})$, i.e., the random variables L_{11}, L_{12} and L_{22} are statistically independents.

λ parameter will be replaced by δ. It is written as:

$$\delta = \left\{ \frac{E\{||\underline{\mathbf{K}} - [\underline{\mathbf{K}}]||_F^2\}}{||[\underline{\mathbf{K}}]||_F^2} \right\}^{\frac{1}{2}} = \left\{ \frac{1}{2} E\{||\underline{\mathbf{K}} - [I]||_F^2\} \right\}^{\frac{1}{2}} \tag{24}$$

Where $0 < \delta < \sqrt{\frac{n+1}{n+5}}$ and δ is related to λ: $\delta = \left\{ \frac{1}{1+2\lambda} \left(1 + \frac{(tr[\underline{K}])^2}{tr([\underline{K}]^2)} \right) \right\}^{1/2}$

To build the matrix $[\mathbf{G}]$ for each realization of matrix $[\mathbf{K}]$, one needs to follow the steps described below.

(1) Decomposing (Cholesky decomposition) $[\mathbf{G}]$: $[\mathbf{G}] = [\mathbf{L}]^T[\mathbf{L}]$;
(2) $[\mathbf{L}]$ is an upper triangular random matrix with values in $\mathbb{M}^+(\mathbb{R})$ such that:
• The random variables $\{[\mathbf{L}]_{jj'}, j \leq j'\}$ are independents.
• For $j < j'$ the real-valued random variable $[\mathbf{L}]_{jj'} = \sigma V_{jj'}$, in which $\sigma = \delta(n+1)^{-1/2}$ and $V_{jj'}$ is a real-valued gaussian random variable with zero mean and unit variance.
• For $j = j'$ the real-valued random variable $[\mathbf{L}]_{jj'} = \sigma\sqrt{2V_j}$. In which V_j is a real-valued Gamma random variable with probability density function:

$$P_{V_j}(v) = \mathbf{1}_{\mathbb{R}^+}(v) \frac{1}{\Gamma\left(\frac{n+1}{2\delta^2} + \frac{1-j}{2}\right)} v^{\frac{n+1}{2\delta^2} - \frac{1+j}{2}} \exp(-v) \tag{25}$$

The random matrix $[\mathbf{K}]$ becomes:

$$[\mathbf{K}] = [L_K]^T \begin{bmatrix} 2\sigma^2 V_1 & \sigma^2\sqrt{2V_1}V_{12} \\ \sigma^2\sqrt{2V_1}V_{12} & \sigma^2 V_{12}^2 + 2\sigma^2 V_2 \end{bmatrix} [L_K] \tag{26}$$

Now the realizations for each simulation can be performed with the aid of Matlab statistics toolbox.

3.3 Approximation of the stochastic system

To construct an approximation of the corresponding stochastic system, the matrix $[K]$ in Eq. 2 is substituted by the random matrix $[\mathbf{K}]$. Then, the function $\mathbf{u}$ is not deterministic anymore; it is a random process, denoted by $\mathbf{U}$. Eq. 2 should be rewritten as

$$[M]\ddot{\mathbf{U}} + [C]\dot{\mathbf{U}} + [\mathbf{K}]\mathbf{U} = \mathbf{f}(t)\,. \tag{27}$$

It should be noted that the matrix $[\mathbf{K}]$ is constructed in a different way, depending on the probabilistic approach used, if SRV or MRV.

Let $\widehat{\mathbf{U}} = (\widehat{U}_1(\omega), \widehat{U}_2(\omega))^T$ and $\widehat{\mathbf{f}}(\omega) = (\widehat{f}_1(\omega), \widehat{f}_2(\omega))^T$ be the Fourier transforms of $\mathbf{U}$ and $\mathbf{f}$, respectively. Then, Eq. 27 can be written, in the frequency domain, as:

$$(-\omega^2[M] + i\omega[C] + [\mathbf{K}])\widehat{\mathbf{U}}(\omega) = \widehat{\mathbf{f}}(\omega)\,. \tag{28}$$

So, the response for the stochastic system, in the frequency domain ($\widehat{\mathbf{U}}(\omega)$) is given by

$$\widehat{\mathbf{U}}(\omega) = (-\omega^2[M] + i\omega[C] + [\mathbf{K}])^{-1}\widehat{\mathbf{f}}(\omega)\,. \tag{29}$$

Defining B as the frequency band, that in this work will be taken as $[0, 7]$ Hz, in this work, $\widehat{\mathbf{f}}(\omega) = (\mathbf{1}_B(\omega), 0)$, where $\mathbf{1}_B(\omega) = 1$, if $\omega \in B$ and $\mathbf{1}_B(\omega) = 0$, if $\omega \notin B$.

Let $[\mathbf{U}(\theta, \omega)]$ be the response of the stochastic system calculated for a realization θ. The mean-square convergence analysis with respect to independent realizations of random variable $\widehat{\mathbf{U}}$, denoted by $\widehat{\mathbf{U}}_j(\theta, \omega)$, is carried out studying the function $n_s \mapsto \text{conv}(n_s)$ defined by

$$\text{conv}(n_s) = \frac{1}{n_s} \sum_{j=1}^{n_s} \int_B ||\widehat{\mathbf{U}}_j(\theta, \omega) - \underline{\widehat{\mathbf{U}}}(\omega)||^2 d\omega \tag{30}$$

where $\underline{\widehat{\mathbf{U}}}(\omega)$ is the response calculated for the corresponding mean model.

For a realization θ, the FRF $\mathrm{H}(\theta, \omega)$, according to Eq. 5, is given by

$$\mathrm{H}(\theta, \omega) = \frac{\widehat{u}_2(\theta, \omega)}{\widehat{f}_1(\omega)} = \widehat{u}_2(\theta, \omega)\,. \tag{31}$$

3.4 Convergence of the stochastic approximation

As the objective is to compare the two approaches, it is important to know the number of simulations n_s (Eq. 30) that assures the convergence of the approximations. This convergence analysis is performed for different values of the dispersion parameter (that will be discussed later) and it is verified that for $n_s = 600$ the approximation always converges.

Figure 3 shows an example for the function $conv$, considering the SRV approach, for $\delta_{K_1} = \delta_{K_2} = 0.3$. The figure shows that for $n_s = 600$ one gets a good enough convergence.

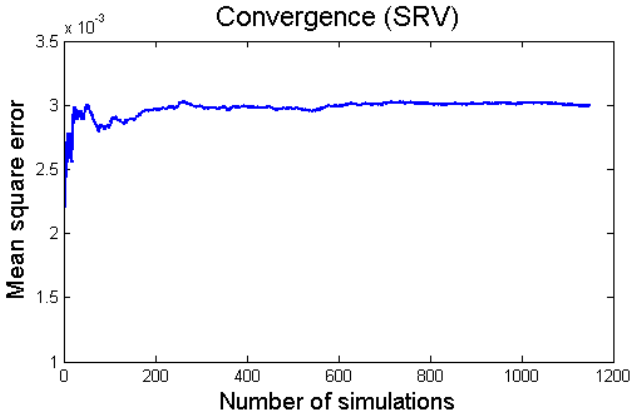

Figure 3: Convergence of the mean square error.

Confidence regions will be constructed, for a probability level $P_c = 0.95$, using the quantiles (Serfling, 1980).

4 NUMERICAL SIMULATIONS

The objective is to compare the two different approaches, but using the same values of the coefficient of dispersion.

Then, $\delta_{K_1} = \delta_{K_2} = \delta_{[\mathbf{K}]} = \delta$ and δ assumes different values.

Figure 4 shows the confidence region for the FRFs calculated as discussed, for $\delta = 0.05$, 0.1, 0.2, and 0.3.

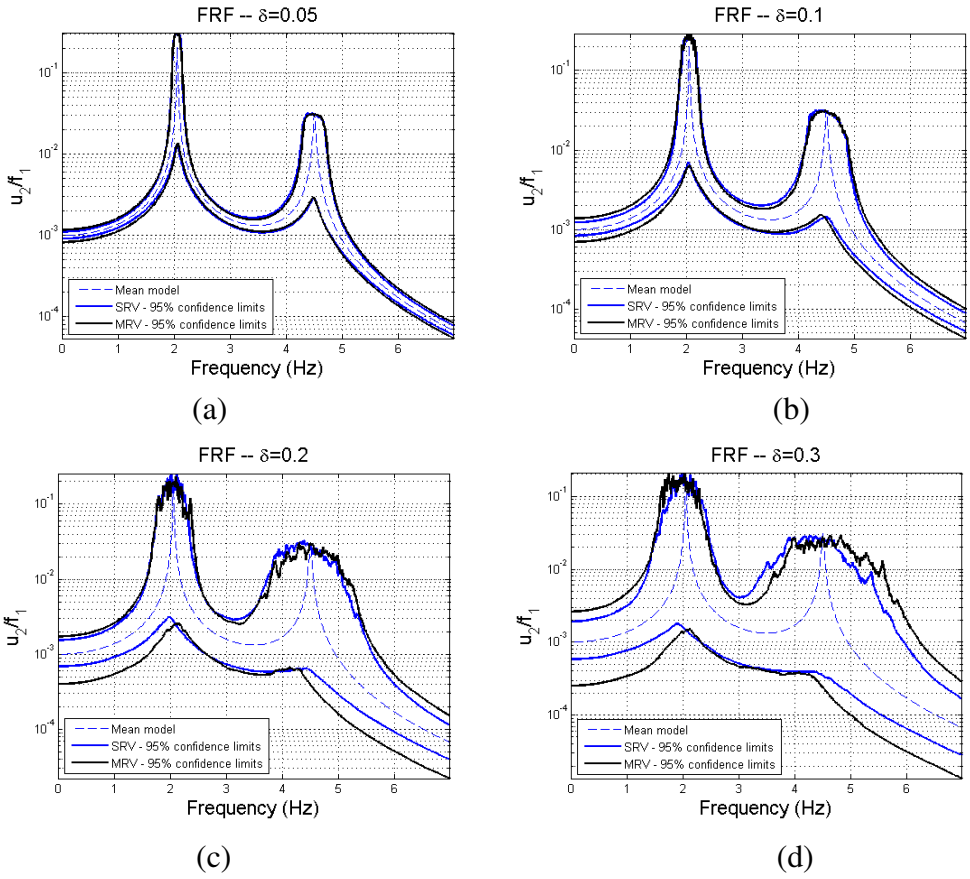

Figure 4: Results for the FRF of the mean model with 95% confidence limits for both approaches (SRV MRV). (a) $\delta_{K_1} = \delta_{K_2} = \delta_{[\mathbf{K}]} = 0.05$, (b) $\delta_{K_1} = \delta_{K_2} = \delta_{[\mathbf{K}]} = 0.1$, (c) $\delta_{K_1} = \delta_{K_2} = \delta_{[\mathbf{K}]} = 0.2$, (d) $\delta_{K_1} = \delta_{K_2} = \delta_{[\mathbf{K}]} = 0.3$.

For a low dispersion parameter, $\delta = 0.05$, Figure 4(a), the results are very similar for the two approaches (SRV and MRV). As the dispersion parameter increases: , $\delta = 0.1$, 0.2, and 0.3, Figure 4(b),(c), and (d), the difference between the results of the two models increases as well. The response using MRV has a wider range of possible outcomes comparing with the SRV approach.

For $\delta = 0.1$ the difference is still small. For $\delta = 0.2$ the difference can be noticed, specially for the low values. For $\delta = 0.3$ the difference is evident.

The values of the coefficients of dispersion are the greatest possible. That is, $\delta_{K_1} = \delta_{K_2} = \frac{1}{\sqrt{3}} = 0.58$ and $\delta_{[\mathbf{K}]} = \sqrt{\frac{n+1}{n+5}} = \sqrt{\frac{3}{7}} = 0.65$, because $n = 2$ is the dimension of the random matriz $[\mathbf{K}]$.

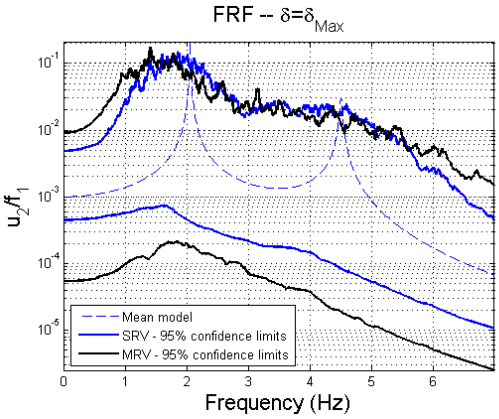

Figure 5: Results for the FRF of the mean model with 95% confidence limits for both approaches (SRV and MRV). $\delta_{K_1} = \delta_{K_2} = 0.58$, $\delta_{[\mathbf{K}]} = 0.65$.

5 UNDERSTANDING THE DIFFERENCE BETWEEN THE TWO APPROACHES

To understand why the MRV approach leads to a wider spectrum, let us decouple the system by means of the modal coordinates, $\mathbf{q} = [\Phi]\mathbf{u}$, where matrix $[\Phi]$ is composed by the normalized normal modes.

$$[\Phi]^T[M][\Phi]\ddot{\mathbf{q}} + [\Phi]^T[C][\Phi]\dot{\mathbf{q}} + [\Phi]^T[K][\Phi]\mathbf{q} = [\Phi]^T\mathbf{f}(t) \tag{32}$$

$$[\Phi]^T[M][\Phi] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; \quad [\Phi]^T[C][\Phi] = \begin{bmatrix} 2\xi_1\omega_1 & 0 \\ 0 & 2\xi_2\omega_2 \end{bmatrix}; \quad [\Phi]^T[K][\Phi] = \begin{bmatrix} \omega_1^2 & 0 \\ 0 & \omega_2^2 \end{bmatrix} \tag{33}$$

Where ω_1 and ω_2 are the natural frequencies, and ξ_1 and ξ_2 ($\xi = c/(2m\omega)$ are the damping rates. This will help to understand how good the two strategies are to model uncertainties on a dynamical system.

We can divide the study of uncertainties in two parts: (1) *data uncertainties*, in which the values of the parameters used in the model are uncertain; and (2) *model uncertainties*, in which the uncertainties of the mathematical-mechanical model used for representing the real system are analyzed.

Making $[M_r] = [\Phi]^T[M][\Phi]$, $[C_r] = [\Phi]^T[C][\Phi]$ and $[K_r] = [\Phi]^T[K][\Phi]$, equation (32) might be rewritten:

$$[M_r]\ddot{\mathbf{q}} + [C_r]\dot{\mathbf{q}} + [K_r]\mathbf{q} = [\Phi]^T\mathbf{f}(t) \tag{34}$$

This is the same system analyzed in the previous section, but now in modal coordinates that decouples it. Any uncertainty in k_1 or k_2 will affect $[K_r]$. Matrix $[C_r]$ will not be changed because $2\xi\omega = c/m$. When the probabilistic approaches are taken into account, the matrix $[K_r]$ should be substituted by $[\mathbf{K}_r]$.

For the mean model, one can write

$$[\mathbf{K}_r] = \begin{bmatrix} \omega_1^2 & 0 \\ 0 & \omega_2^2 \end{bmatrix} \tag{35}$$

where ω_1 and ω_2 are the natural frequencies of the system.

For the probabilistic approaches, the random variables Ω_1 and Ω_2 represent the natural frequencies of the system. Then, the two cases in the following should be considered.

(1) In the SRV approach, the random matrix $[\mathbf{K}_r]$ can be written as:

$$[\mathbf{K}_r] = \begin{bmatrix} \Omega_1^2 & 0 \\ 0 & \Omega_2^2 \end{bmatrix} \tag{36}$$

The system remains decoupled, so the uncertainties stay within the parameters. This type of modeling is helpful for data uncertainties because the things that vary are only the parameters of the system.

(2) In the MRV strategy, the random matrix becomes:

$$[\mathbf{K}_r] = \begin{bmatrix} \Omega_1^2 & \Phi_{12} \\ \Phi_{12} & \Omega_2^2 \end{bmatrix} \tag{37}$$

The random variable Φ_{12} appears due to how $[\mathbf{K}_r]$ is built. Although the mean model is decoupled, the realizations may be coupled, because a new random variable appear in the secondary diagonal. So, by using MRV approach, it is possible to analyze uncertainties within the model, and not only data (parameters) uncertainties.

5.1 Event space

Visualizing the event space might be helpful to see the difference between the two approaches. Let $\mathbf{K}_r$ be a positive-definite matrix:

$$[\mathbf{K}_r] = \begin{bmatrix} a & b \\ b & c \end{bmatrix} \tag{38}$$

Is is known that $a > 0$ and $ac - b^2 > 0$, and, consequently $c > 0$ and $b = \pm\sqrt{ac}$. For SRV approach $b = 0$, then the event space is bi-dimensional:

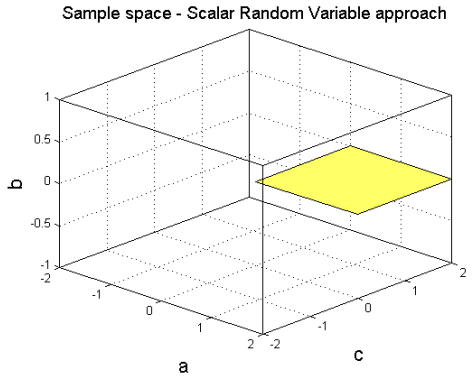

Figure 6: Event space for SRV approach

In MRV approach the event space is three-dimensional because b can assume values from $-\sqrt{ac}$ to $+\sqrt{ac}$. Figure 7 shows the projection in the $a \times c$ plane of the resulting geometry:

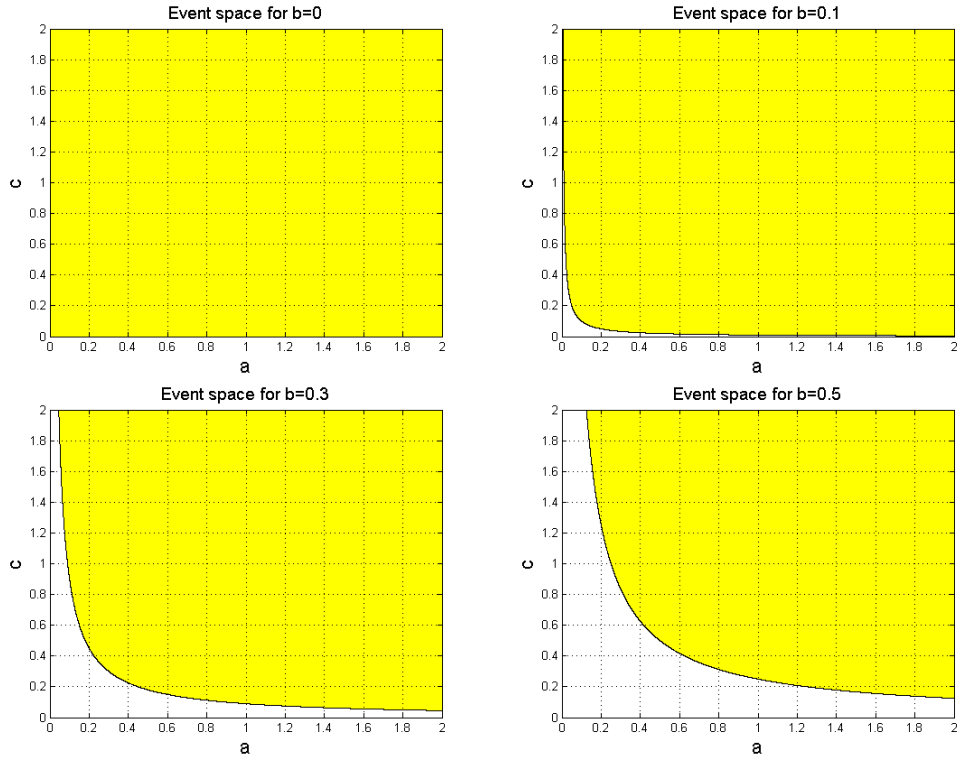

Figure 7: Event space for MRV approach. Projection on $a \times c$ plane for different b's

Note that, for $b = 0$, the event space of both approaches are the same, $\mathcal{S}_{SRV,n}|_{b=0} = \mathcal{S}_{MRV,n}|_{b=0}$, but, in general, $\mathcal{S}_{SRV,n} \subset \mathcal{S}_{MRV,n}$ since it has one extra dimension.

Taking a look in another perspective, one notes that the randomness of $\mathbf{K}_r$ in the SRV approach stays inside the matrix set $\mathbb{M}_n^D(\mathbb{R})$ (positive diagonal matrices), while the randomness of $\mathbf{K}_r$ in the MRV approach stays inside the matrix set $\mathbb{M}_n^+(\mathbb{R})$ (symmetric positive-definite matrices), see Figure 8.

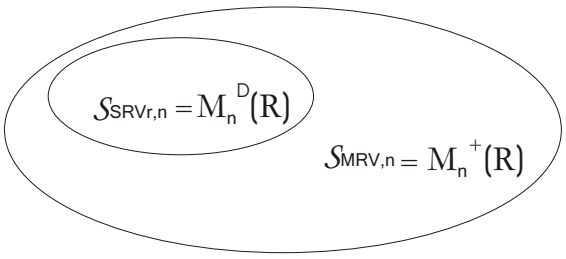

Figure 8: Space of matrices. $\mathcal{S}_{\text{MRV}}$ contains $\mathcal{S}_{\text{SRV}}$

Another interesting point is that, for the same δ it is not guaranteed that the MRV approach will cover all the possible outcomes of the SRV approach. If $\delta_{K_1} = \delta_{K_2} = \delta_{[K]}$, the outcome considering 95% confidence limits, $\mathcal{S}_{MRV,n}|_{95\%}$ does not contain $\mathcal{S}_{SRV,n}|_{95\%}$, see Figure 9.

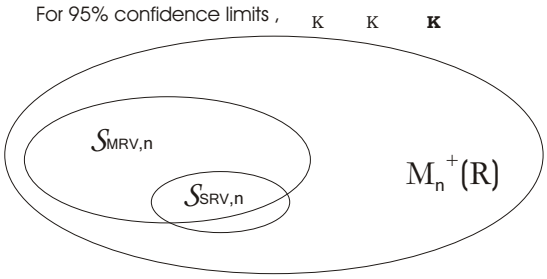

Figure 9: Space of matrices. If $\delta_{K_1} = \delta_{K_2} = \delta_{[K]}$, $\mathcal{S}_{\text{MRV}}|_{95\%}$ will not contain $\mathcal{S}_{\text{SRV}}|_{95\%}$

For example, using $\delta = 0.2$ and 95% confidence limits, matrix $[\mathbf{K}]$ presents the following variation, depending on the method used:

$$[\mathbf{K}_{\text{SRV}}] = \begin{bmatrix} (964\ to\ 1362) & (-180\ to\ -122) \\ (-180\ to\ -122) & (122\ to\ 180) \end{bmatrix}$$

$$[\mathbf{K}_{\text{MRV}}] = \begin{bmatrix} (974\ to\ 1342) & (-204\ to\ -101) \\ (-204\ to\ -101) & (126\ to\ 173) \end{bmatrix}$$

The limits of the entrance of matrix $[\mathbf{K}]$ are different for each approach. Note that the limits for the diagonal entrances are greater for SRV approach, while, the limits for the entrances outside the diagonal are greater for the MRV approach. It is clear, in this case, that $\mathcal{S}_{\text{MRV}}|_{95\%}$ does not englobe $\mathcal{S}_{\text{SRV}}|_{95\%}$. This means that the parameter δ can not be directly compared between the approaches.

To see more clearly this fact, lets take a look in the probability density functions of the first entrance of matrix $[\mathbf{K}]$.

$$[\mathbf{K}_{\text{SRV}}] = \begin{bmatrix} K_1 + K_2 & .. \\ .. & .. \end{bmatrix} \quad \text{, where,} \quad K_1 \sim \text{Gamma}(\delta_{K_1}^{-2}, \underline{k_1}\delta_{K_1}^2) \text{ and } K_2 \sim \text{Gamma}(\delta_{K_2}^{-2}, \underline{k_2}\delta_{K_2}^2)$$

$$[\mathbf{K}_{\text{MRV}}] = \begin{bmatrix} (2\underline{k_1} + 2\underline{k_2})\sigma^2 V_1 & .. \\ .. & .. \end{bmatrix} \quad \text{, where,} \quad V_1 \sim \text{Gamma}(3/(2\delta_{[\mathbf{K}]}^2), 1)$$

For MRV approach it is possible to calculate an analytical expression for the distribution, since $[\mathbf{K}_{11}]_{MRV} = CV_1$, where C is a constant, $C = (2\underline{k_1} + 2\underline{k_2})\sigma^2$. For SRV approach there is no close form because $[\mathbf{K}_{11}]_{SRV} = K_1 + K_2$. Note: if K_1 and K_2, which are Gamma random variables, had the same scale parameter, then $K_1 + K_2$ would also be a Gamma random variable, but this is not the case.

Figure 10 shows the density functions of $[\mathbf{K}_{11}]$. 4000 monte carlo simulation were performed. The green curve is very similar to the red curve as it should be, because the green curve is the analytical expression and the red curve is the result of monte carlo simulation for MRV approach. The blue curve is the result of monte carlo simulation for SRV approach. The dotted lines in the graphic shows the 95% confidence limits. Therefore, for the first entrance of $[\mathbf{K}]$, when a confidence region is established and the δ is the same, one sees a greater possibility of outcomes for SRV approach.

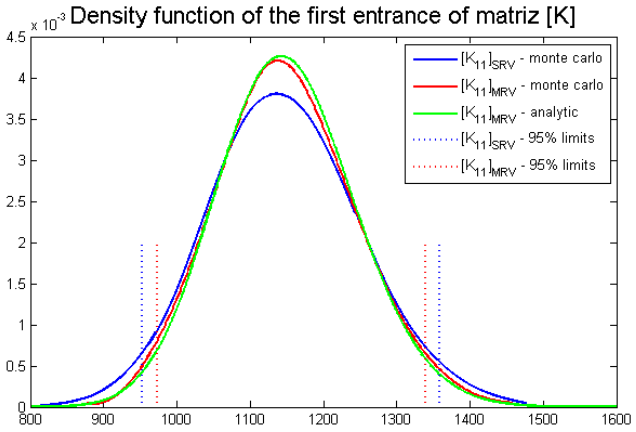

Figure 10: Density functions of the first entrance of matriz $[\mathbf{K}]$

5.2 Mean square error

The aim here is to compare the mean-square error between the random matrix $[\mathbf{K}]$, calculated using both approaches, SRV and MRV, and the matrix $[K]$ used in the mean model.

Let $[K]$ be the stiffness matrix used in Eq. 2. Let $[\mathbf{K}_{\text{SRV}}]$ be the random stiffness matrix used in Eq. 27, calculated using the SRV approach and let $[\mathbf{K}_{\text{MRV}}]$ be the random stiffness matrix used in Eq. 27, calculated using the MRV approach.

The mean-square error between the random matrices $[\mathbf{K}_{\text{SRV}}]$ and $[\underline{K}]$ is given by:

$$e_{\text{SRV}} = E\{\| \, [\mathbf{K}_{\text{SRV}}] - [\underline{K}] \, \|_F^2\} = \int_{\mathbb{M}_2^+(\mathbb{R})} \| \, [K_{\text{SRV}}] - [\underline{K}] \, \|_F^2 \; P_{[\mathbf{K}_{\text{SRV}}]}(\tilde{d}[K]) < +\infty \, . \quad (39)$$

and the mean-square error between the random matrices $[\mathbf{K}_{\text{MRV}}]$ and $[K]$ is given by:

$$e_{\text{MRV}} = E\{\| \, [\mathbf{K}_{\text{MRV}}] - [\underline{K}] \, \|_F^2\} = \int_{\mathbb{M}_2^+(\mathbb{R})} \| \, [K_{\text{MRV}}] - [\underline{K}] \, \|_F^2 \; P_{[\mathbf{K}_{\text{MRV}}]}(\tilde{d}[K]) < +\infty \, . \quad (40)$$

One wonders if:

$$E\{\| \, [\mathbf{K}_{\text{MRV}}] - [\underline{K}] \, \|_F^2\} < E\{\| \, [\mathbf{K}_{\text{SRV}}] - [\underline{K}] \, \|_F^2\} \quad (41)$$

The values of e_{SRV} and e_{MRV} for different values of the dispersion coefficients for 1000 simulations are presented in the following table.

δ	e_{SRV}	e_{MRV}
0.1	1.0695e4	1.0024e4
0.2	4.3410e4	4.0113e4
0.3	1.0525e5	8.6692e4
0.4	1.8416e5	1.6487e5
Max	3.3605e5	4.4289e5

Table 1: Mean square error

The error increases with δ as expected. But it is not guaranteed that $e_{\text{MRV}} < e_{\text{SRV}}$. For sure there is a distribution for $[\mathbf{K}]$ that minimizes the error, but the one calculated by means of the Principle of Maximum Entropy may not be the one.

6 CONCLUDING REMARKS

This article presented and compared two approaches to model uncertainties in structural dynamics: the *Scalar Random Variable-SRV approach*, in which the uncertainties are inserted into the parameters and the *Matrix Random Variable-MRV approach*, in which the uncertainties are inserted into the matrices of the dynamical system. Both strategies use the Maximum Entropy Principle to construct the probability density functions of the random variables associated to the uncertainties. Some points need to be remarked:

1. SRV and MRV approaches are different strategies to model uncertainties in a dynamical system;
2. SRV approach is good to model data uncertainties, while MRV approach is capable of taking into account model uncertainties;
3. MRV approach is able to take into account model uncertainties because some realizations of the random matrix couple the system modes (in SRV approach it does not happen);
4. The event space resulting from SRV approach is smaller than the event space resulting from MRV approach, what means that in MRV approach more cases might be considered;
5. It can be noted that the choice of the levels of dispersion for comparing the two approaches is not immediate. When the same values for the coefficients of dispersion are considered for both approaches, one can note that it is not clear that the MRV approach englobes the SRV approach.

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