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A POSTERIORI ERROR ESTIMATES FOR THE FINITE ELEMENT APPROXIMATION OF STEKLOV EIGENVALUE PROBLEM

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Abstract. This paper deals with *a posteriori* error estimators for the linear finite element approximation of the Steklov eigenvalue problem introduced by M. Armentano and C. Padra. We extend their results to any dimension and we obtain an optimal error estimates under the weaker regularity conditions.

Keywords: Steklov eigenvalue problem, a posteriori error estimate,finite elements.

1 INTRODUCTION

The Steklov eigenvalue problem is:

$$\begin{cases} -\Delta u + u = 0 & \text{in } \Omega \\ \frac{\partial u}{\partial n} = \lambda u & \text{on } \Gamma = \partial\Omega \end{cases} \tag{1}$$

In the bidimensional case, it is well known that eigenfunctions belong to $H^{1+r}(\Omega)$, where $r = 1$ if Ω is convex, and $r < \frac{\pi}{\alpha}$ (with α being the largest inner angle of Ω) otherwise. It is interesting to know the maximal regularity of the solution of the problem because it determines the rate of convergence in standard finite element approximations and the theorical efficience of *a posteriori* error estimators.

If $d > 2$ and $\Omega \subset \mathbb{R}^d$ is a polyhedron, the situation is rather more complicated (Babuška and Anderson). It is important to know what is the maximal regularity of the solution in general without assuming any particular knowledge about the regularity of the solution except those which yield the general theory of the Neumann problem.

An *a posteriori* error estimator of residual type for the linear finite element approximation of the Steklov eigenvalue problem was introduced in (Armentano and Padra, 2006) for bounded polygonal domains $\Omega \subset \mathbb{R}^2$.

The aim of this paper is to extend the *a posteriori* error estimator for domains $\Omega \subset \mathbb{R}^d$, $d > 2$, and to show that $r = 3/2$ is the minimal order of the rate of convergence, mainly because the solutions always belong to the Besov space $B^{3/2}_{2,\infty}(\Omega)$, even if they do not belong to $H^{3/2}(\Omega)$.

This regularity result is optimal under weak assumptions on the data and can be extended to the more general elliptic Steklov eigenvalue problem:

$$\begin{cases} div(A(x)\nabla u(x)) + c(x)u(x) = 0 & \text{in } \Omega, \\ u(x) = 0 & \text{on } \Gamma_{0,} \\ \frac{\partial u(x)}{\partial \nu_A} = \lambda u(x) & \text{on } \Gamma_n, \end{cases} \tag{2}$$

where $\Gamma_0 \subset \partial\Omega, \Gamma_0 \neq \emptyset$ and $\Gamma_0 \neq \partial\Omega$.

The *a posteriori* error estimator may also be extended to handle problem (2).

2 MODEL PROBLEM AND FINITE ELEMENT APPROXIMATION

Let $\Omega \subset \mathbb{R}^d$ be a bounded open polyhedron. Although results extend to general dimension d, we assume for simplicity that $d = 3$.

We use standard notation for Sobolev spaces, norms, and seminorms. Hence, the eigenvalues λ and normalized eigenfunctions u of the problem above satisfy $u \in H^1(\Omega)$ and

$$\begin{cases} a(u,v) = \lambda\, b(u,v) & \forall\, v \in H^1(\Omega) \\ ||\, u\, ||_{L^2(\Gamma)} = 1 \end{cases} \tag{3}$$

where $a(u,v) = \int_\Omega \nabla u \cdot \nabla v + \int_\Omega u\, v$, which is continuous and coercive on $H^1(\Omega)$, and $b(u,v) = \int_\Gamma uv$. The solution of this problem is given by a sequence of pairs (λ_j, u_j), such that $\lambda_1 \leq \lambda_2 \leq ... \leq \lambda_j \leq ... \to \infty$ (Auchmuty, 2004).

We consider the solution $u \in H^1(\Omega)$ of the Neumann problem

$$\begin{cases} -\Delta v + v = 0 & \text{in } \Omega \\ \frac{\partial v}{\partial n} = g & \text{on } \Gamma = \partial\Omega \end{cases} \tag{4}$$

with $g \in H^{-1/2}(\Gamma)$.

The solution v belongs to the interpolation Besov space

$$B^{3/2}_{2,\infty}(\Omega) = [H^1(\Omega), H^2(\Omega)]_{1/2,\infty},$$

if $g \in H^{1/2}(\Gamma)$ (Savaré, 1997, 1998). Furthermore, there is a constant $C = C(\Omega)$ such that

$$|| v ||^2_{B^{3/2}_{2,\infty}(\Omega)} \leq C || v ||_{H^1(\Omega)} \{|| g ||_{H^{1/2}(\Gamma)} + || v ||_{H^1(\Omega)}\} \tag{5}$$

for all $g \in H^{1/2}(\Gamma)$.

Remark 1 *We recall that $B^{3/2}_{2,\infty}(\Omega) \subset H^{3/2-\varepsilon}(\Omega)$, $\forall \varepsilon > 0$, but $B^{3/2}_{2,\infty}(\Omega)$ is not included in $H^{3/2}(\Omega)$.*

Remark 2 *For the basic fact of the theory of interpolation spaces see (Babuška and Osborn, 1991; Gergh and Löfström, 1976; Brenner and Scott, 1994).*

Let $\{\mathcal{T}_h\}$ be a family of conforming partitioning of $\overline{\Omega}$ formed by tetrahedra. Let h stand for the mesh-size; namely $h := \max_{T \in \mathcal{T}_h} h_T$, with h_T being the diameter of the tetrahedra T. For each $\mathcal{T}_h$, we denote with $V_h \subset H^1(\Omega)$ the standard finite element space of continuous piecewise linear elements. We assume that the meshes $\{\mathcal{T}_h\}$ satisfy some regularity condition in such a way that error estimates hold independently of h. For example, we may assume the following weak regularity assumptions:

Condition 3 *For $K \geq 1$ and $\epsilon > 0$, the mesh $\mathcal{T}_h$ satisfies*

$$m(T) \geq \epsilon\, h_T^d\,, \qquad \text{for all } T \in \mathcal{T}_h,$$

and

$$h_T \leq K\, h_S, \qquad \text{for all neighboring elements } T, S \in \mathcal{T}_h,$$

where $m(S)$ denotes the Lebesgue measure of a set S.

Then, the constant appearing in error estimates like Céa´s lemma, as well as the constants appearing in error estimates for the interpolation operator of Clément I_h, are computable and depend on K, ϵ.

Notation 4 *From now on, C denotes a generic positive constant which depends only on $C(\Omega)$ and on the regularity of the mesh.*

Let $u \in H^1(\Omega)$. A key result in finite element convergence is the estimate of

$$\inf_{v \in V_h} || u - v ||_{H^1(\Omega)}$$

under the precise characterization of the regularity of u.

Proposition 5 *There exists a constant C such that*

$$\inf_{v \in V_h} || u - v ||_{H^1(\Omega)} \leq C\, h^{1/2} || u ||_{B^{3/2}_{2,\infty}(\Omega)} \tag{6}$$

for all $u \in B^{3/2}_{2,\infty}(\Omega)$.

Proof. Let P_h be the projection w.r.t. the energy scalar product a and $Q_h : H^1(\Omega) \to H^1(\Omega)$ defined by $Q_h u := u - P_h u$. From standard finite element estimates, we know that

$$|| \, Q_h ||_{\mathcal{L}(H^1(\Omega),H^1(\Omega))} \leq C,$$

and

$$|| \, Q_h ||_{\mathcal{L}(H^2(\Omega),H^1(\Omega))} \leq C \, h \, .$$

Then, from real interpolation theory (Brenner and Scott, 1994; Gergh and Löfström, 1976), we have an interpolation operator

$$[P_h]_{1/2,\infty} : B^{3/2}_{2,\infty}(\Omega) \to [V_h, V_h]_{1/2,\infty} = V_h,$$

and

$$[Q_h]_{1/2,\infty} : B^{3/2}_{2,\infty}(\Omega) \to [H^1(\Omega), H^1(\Omega)]_{1/2,\infty} = H^1(\Omega),$$

such that

$$[Q_h]_{1/2,\infty} = I - [P_h]_{1/2,\infty}.$$

Moreover,

$$\begin{aligned} || \, [Q_h]_{1/2,\infty} ||_{\mathcal{L}(B^{3/2}_{2,\infty}(\Omega),H^1(\Omega))} &\leq || \, Q_h ||^{1/2}_{\mathcal{L}(H^1(\Omega),H^1(\Omega))} || \, Q_h ||^{1/2}_{\mathcal{L}(H^2(\Omega),H^1(\Omega))} \\ &\leq C \, h^{1/2}. \end{aligned}$$

■

Remark 6 *The rate of convergence of the finite element approximation is rather pessimistic. The mesh-dependent constant C grows as an inverse power of $h_{\min}$ (the global minimum mesh diameter). Since adaptive process may produce meshes with significant local variation of mesh size, this bound may be large and useless. In fact, h-weighted estimates are preferable and allow quite general mesh refinement procedures (Eriksson and Johnson, 1995).*

We shall follows now the approach in (Armentano, 2004). The finite element approximate solutions of the spectral problem are defined by $u_h \in V_h$ and,

$$\begin{cases} a(u_h, v_h) = \lambda_h \, b(u_h, v_h), & \forall \, v_h \in V_h \\ \quad || \, u_h \, ||_{L^2(\Gamma)} = 1. & \end{cases} \tag{7}$$

This problem reduces to a generalized eigenvalue problem involving positive (semi) definite matrix symmetric matrices. It attains a finite number of eigenpairs $(\lambda_{h,j}, u_{h,j})$ such that $\lambda_{h,1} \leq \lambda_{h,2} \leq ... \leq \lambda_{h,N(h)}$, where $N(h) = \dim(V_h)$.

The eigenmodes convergence can be analyzed by the compact resolvent operators. Let $T : H^1(\Omega) \to H^1(\Omega)$ defined by

$$a(Tf, v) = b(f, v), \qquad \forall \, v \in H^1(\Omega), \tag{8}$$

and $T_h := P_h T$. That is

$$a(T_h f, v) = b(f, v), \qquad \forall \, v \in V_h. \tag{9}$$

The non-zero eigenvalues of T are the reciprocals of λ_j and the non-zero eigenvalues of T_h are the reciprocals of $\lambda_{h,j}$. Furthermore, the eigenfunctions of T and T_h are u_j and $u_{h,j}$ respectively.

A direct proof using (5) gives

$$|| \, Tf \, ||_{B^{3/2}_{2,\infty}(\Omega)} \leq C \, || \, f \, ||_{H^1(\Omega)} \tag{10}$$

On the other hand,

$$a(Tf - T_h f, v) = 0, \quad \forall \, v \in V_h.$$

Therefore,

$$|| \, Tf - T_h f \, ||_{H^1(\Omega)} \leq \inf_{v \in V_h} || \, Tf - v \, ||_{H^1(\Omega)}.$$

Now, using (6) and (10), we get

$$|| \, Tf - T_h f \, ||_{H^1(\Omega)} \leq C \; h^{1/2} \, || \, f \, ||_{H^1(\Omega)},$$

or

$$|| \, T - T_h \, ||_{\mathcal{L}(H^1(\Omega), H^1(\Omega))} = \varepsilon(h) \leq C \; h^{1/2}.$$

Using the spectral approximation theory given in (Babuška and Osborn, 1991), we have

$$| \, \lambda_j - \lambda_{h,j} \, | \leq C \, h, \tag{11}$$

and

$$|| \, u_j - u_{h,j} \, ||_{H^1(\Omega)} \leq (C \, \lambda_j^{1/2}) \, h^{1/2} = C_j \, h^{1/2}. \tag{12}$$

Even if we shall not use the estimate of $e = u_j - u_{h,j}$ in the L^2 norm, we can estimate it by a standard duality argument. Let w be the solution of

$$\begin{cases} -\Delta w + w = e & \text{in } \Omega, \\ \frac{\partial w}{\partial n} = 0 & \text{on } \Gamma. \end{cases}$$

Using the theory in (Savaré, 1997, 1998), we have the estimate

$$\begin{aligned} || \, w \, ||^2_{B^{3/2}_{2,\infty}(\Omega)} &\leq C \, || \, w \, ||_{H^1(\Omega)} \{ || \, e \, ||_{L^2(\Omega)} + || \, w \, ||_{H^1(\Omega)} \} \\ &\leq C \, || \, e \, ||^2_{L^2(\Omega)} \end{aligned}$$

Therefore, since $a(e, v_h) = 0$, $\forall \, v_h \in V_h$, we have

$$\begin{aligned} || \, e \, ||^2_{L^2(\Omega)} &= (e, e)_{L^2(\Omega)} \\ &= a(w, e) \\ &= a(w - [P_h]_{1/2,\infty} w, e) \\ &\leq C \, || \, e \, ||_{H^1(\Omega)} || \, w - [P_h]_{1/2,\infty} w \, ||_{H^1(\Omega)} \\ &\leq C \, h^{1/2} \, || \, e \, ||_{H^1(\Omega)} || \, w \, ||_{B^{3/2}_{2,\infty}(\Omega)} \\ & \quad\; C \, h^{1/2} \, || \, e \, ||_{H^1(\Omega)} \, || \, e \, ||_{L^2(\Omega)} \end{aligned}$$

Hence,

$$|| \, e \, ||_{L^2(\Omega)} \leq C \, h^{1/2} \, || \, e \, ||_{H^1(\Omega)}.$$

Finally, (12) yields

$$|| \, e \, ||_{L^2(\Omega)} \leq C_j \, h \, .$$

We shall use the techniques given in (Armentano, 2004) to prove the following result.

Proposition 7 *There exists a constant* $C_j = C_j(C, \lambda_j)$, *such that*

$$|| \, u_j - u_{h,j} \, ||_{L^2(\Gamma)} \leq C_j \, h^{3/4}. \tag{13}$$

Proof. Let $e = u_j - u_{h,j}$. Using (12) and the trace operator, we have

$$|| \, e \, ||_{H^{1/2}(\Gamma)} \leq C_j \, h^{1/2}.$$

We consider the following auxiliary problem:

$$\begin{cases} -\Delta\phi + \phi = 0 & \text{in } \Omega, \\ \frac{\partial \phi}{\partial n} = e & \text{on } \Gamma \end{cases}$$

or, in variational form:

$$a(\phi, v) = \int_\Gamma e \, v, \qquad \forall \, v \in H^1(\Omega).$$

Then,

$$|| \, \phi \, ||_{H^1(\Omega)} \leq C \, || \, e \, ||_{L^2(\Gamma)}.$$

Now, let ϕ_h be the finite element solution of this problem. Subtracting (9) from (4) we have $a(e, \phi_h) = 0$. Then,

$$\begin{aligned} || \, e \, ||^2_{L^2(\Gamma)} &= \int_\Gamma e^2 = \int_\Gamma \frac{\partial \phi}{\partial n} e = a(\phi, e) \\ &= a(\phi - \phi_h, e) \leq || \, \phi - \phi_h \, ||_{H^1(\Omega)} || \, e \, ||_{H^1(\Omega)}. \end{aligned}$$

By (5) we have

$$\begin{aligned} || \, \phi \, ||_{B^{3/2}_{2,\infty}(\Omega)} &\leq C \, \left(|| \, \phi \, ||_{H^{-1}(\Omega)} \{ || \, e \, ||_{H^{1/2}(\Gamma)} + || \, \phi \, ||_{H^{-1}(\Omega)} \} \right)^{1/2} \\ &\leq C \, \left(|| \, e \, ||_{H^{1/2}(\Gamma)} \{ || \, e \, ||_{H^{1/2}(\Gamma)} + || \, e \, ||_{H^{1/-2}(\Gamma)} \} \right)^{1/2} \\ &\leq C \, || \, e \, ||_{H^{1/2}(\Gamma)} \leq C_j \, h^{1/2}. \end{aligned}$$

Using (6) we have

$$|| \, \phi - \phi_h \, ||_{H^1(\Omega)} \leq C \, h^{1/2} || \, \phi \, ||_{B^{3/2}_{2,\infty}(\Omega)} \leq C_j \, h.$$

Therefore

$$|| \, e \, ||^2_{L^2(\Gamma)} \leq C_j \, h^{3/2} \tag{14}$$

and the result follows. ■

3 THE *A POSTERIORI* ERROR ESTIMATOR

Following the approach given in (Durán and Rodriguez, 1999), it was introduced in (Armentano and Padra, 2006) an *a posteriori* error estimator for the Steklov eigenvalue problem restricted to the case $d = 2$. This approach can be followed almost word for word in the d-dimensional case taking into account the appropriate changes due to the regularity results below and. The triangle and edge buble functions necessary in the proof of efficiency of the estimators are $d-$dimensional standard bubble functions.

First we introduce some notation:

For each $T \in \mathcal{T}_h$ we denote by $\mathcal{F}_T$ the set of faces of T,

$$\mathcal{F} = \bigcup_{T \in \mathcal{T}_h} \mathcal{F}_T,$$

and we decompose $\mathcal{F} = \mathcal{F}_\Omega \cup \mathcal{F}_\Gamma$ where $\mathcal{F}_\Gamma := \{F \in \mathcal{F} \, : F \subset \Gamma\}$ and $\mathcal{F}_\Omega := \mathcal{F} \setminus \mathcal{F}_\Gamma$.

For each face $F \in \mathcal{F}_\Omega$ we choose a unit normal vector n_F and denote the two elements sharing this face T_{in} and T_{out}. For each $v_h \in V_h$, the jump of the normal derivative of v_h across F is:

$$\left] \frac{\partial v_h}{\partial n} \right[_F = \nabla(v_h|T_{out}) \cdot n_F - \nabla(v_h|T_{in}) \cdot n_F \, .$$

We consider a particular eigenpair (λ_j, u_j) and its corresponding finite element approximation $(\lambda_{h,j}, u_{h,j})$. From now on, we drop out the subindexes j in $\lambda_j, \lambda_{h,j}, u_j, u_{h,j}, C_j$ to simplify notation. The following results provide some error equations which will be the starting point of the error analysis.

Lemma 8 *The error $e = u - u_h$ satisfies*

$$\begin{aligned}\int_\Omega \nabla e \cdot \nabla v + \int_\Omega e\, v &= -\sum_T \int_T u_h v \\ &+ \sum_T \left\{ \sum_{F \in \mathcal{F}_T \cap \mathcal{F}_\Gamma} \int_F (\lambda_h u_h - \frac{\partial u_h}{\partial n}) v + \frac{1}{2} \sum_{F \in \mathcal{F}_T \cap \mathcal{F}_\Omega} \int_F \left] \frac{\partial u_h}{\partial n} \right[_F v \right\} \\ &+ \int_\Gamma \lambda u v - \int_\Gamma \lambda_h u_h v,\end{aligned}$$

for all $v \in H^1(\Omega)$.

Proof. For any $v \in H^1(\Omega)$ using (3), integrating by parts and using that $\Delta u_h = 0$ we have

$$\begin{aligned}\int_\Omega \nabla e \cdot \nabla v + \int_\Omega e\, v &= a(u, v) - a(u_h, v) \\ &= \int_\Gamma \lambda u v - \sum_T \left\{ \int_{\partial T} \frac{\partial u_h}{\partial n} v + \int_T u_h v \right\};\end{aligned}$$

and thus

$$\begin{aligned}\int_\Omega \nabla e \cdot \nabla v + \int_\Omega e\, v &= -\sum_T \int_T u_h v \\ &+ \sum_T \left\{ \sum_{F \in \mathcal{F}_T \cap \mathcal{F}_\Gamma} \int_F (\lambda_h u_h - \frac{\partial u_h}{\partial n}) v + \frac{1}{2} \sum_{F \in \mathcal{F}_T \cap \mathcal{F}_\Omega} \int_F \left] \frac{\partial u_h}{\partial n} \right[_F v \right\} \\ &+ \int_\Gamma \lambda u v - \int_\Gamma \lambda_h u_h v \, .\end{aligned}$$

■

Lemma 9 *There holds*

$$\int_\Gamma (\lambda u - \lambda_h u_h)\, e = (\lambda + \lambda_h) \left(1 - \int_\Gamma u\, u_h \right) = \frac{(\lambda + \lambda_h)}{2} \int_\Gamma e^2. \tag{15}$$

Proof. It follows easily using that $\int_\Gamma u^2 = \int_\Gamma u_h^2 = 1$. ■

We shall use the following well known localized error estimates for the interpolation operator of Clément $I_h : H^1(\Omega) \to V_h$

$$\begin{aligned} \|\, v - I_h v \,\|_{L^2(T)} &\leq c_T\, h_T \,\|\, v \,\|_{H^1(\widetilde{T})} && (16)\\ \|\, v - I_h v \,\|_{L^2(F)} &\leq c_T\, h_F^{1/2} \,\|\, v \,\|_{H^1(\widetilde{F})} && (17)\end{aligned}$$

where $\widetilde{T}$ is the union of all elements sharing a node with T, $\widetilde{F}$ is the union of all elements sharing a node with F, and h_F is the diameter of the facet F.

Remark 10 *The constant c_T appearing in these interpolation error estimates is of great importance for a correct calibration of a posteriori error estimator. It is of local character, reflect local geometric properties of the mesh, and can be effectively calculated (Verfürth, 1999). Using the regularity of the mesh, the local constants c_T could be replaced by a global constant C. A posteriori error estimators may be defined by taking care of the local constants c_T or in a simpler way by using the global constant C.*

For each $F \in \mathcal{F}$ we define J_F by

$$J_F(u_h) = \begin{cases} \frac{1}{2}\left]\frac{\partial u_h}{\partial n}\right[_F & F \in \mathcal{F}_\Omega \\ \lambda_h u_h - \frac{\partial u_h}{\partial n} & F \in \mathcal{F}_\Gamma \end{cases}.$$

Now, the local error indicator is defined by

$$\widehat{\eta}_T := c_T \left(h_T^2 \,\|\, u_h \,\|_{L^2(\Omega)}^2 + \sum_{F \in \mathcal{F}_T} h_F \,\|\, J_F(u_h) \,\|_{L^2(F)}^2 \right)^{1/2}, \tag{18}$$

and the global one by

$$\widehat{\eta}_\Omega := \left(\sum_{T \in \mathcal{T}_h} \widehat{\eta}_T^2 \right)^{1/2}.$$

The following theorem provides the upper bound on the error.

Theorem 11 *There exists a constant C such that*

$$\|\, e \,\|_{H^1(\Omega)} \leq \left\{ \widehat{\eta}_\Omega + C \frac{(\lambda + \lambda_h)}{2} \|\, e \,\|_{L^2(\Gamma)} \right\}$$

Proof. From (3) and (7) we know that for any $v_h \in V_h$ the error satisfies

$$\int_\Omega \nabla e \cdot \nabla v_h + \int_\Omega e\, v_h = \int_\Gamma \lambda u v_h - \int_\Gamma \lambda_h u_h v_h.$$

Using Lemma 8 we have

$$\begin{aligned} \int_\Omega |\nabla e|^2 + |e|^2 &= \int_\Omega \nabla e \cdot \nabla (e - I_h e) + \int_\Omega e(e - I_h e) + \int_\Gamma \lambda u I_h e - \int_\Gamma \lambda_h u_h I_h e \\ &= \sum_T \left\{ - \int_T u_h (e - I_h e) + \sum_{F \in \mathcal{F}_T} \int_F J_F(u_h)(e - I_h e) \right\} + \int_\Gamma (\lambda u - \lambda_h u_h) e \end{aligned}$$

Then,

$$\begin{aligned}
||\, e\,||^2_{H^1(\Omega)} &\leq \sum_T ||\, u_h\,||_{L^2(T)}||\, e - I_h e\,||_{L^2(T)} + \sum_T \sum_{F\in\mathcal{F}_T} ||\, J_F(u_h)\,||_{L^2(F)}||\, e - I_h e\,||_{L^2(F)} \\
&\quad + \int_\Gamma (\lambda u - \lambda_h u_h) e \\
&\leq \sum_T c_T\, h_T\, ||\, u_h\,||_{L^2(T)}||\, e||_{H^1(\widetilde{T})} + \sum_T \sum_{F\in\mathcal{F}_T} c_T\, h_F^{1/2}||\, J_F(u_h)\,||_{L^2(F)}||\, e\,||_{H^1(\widetilde{F})} \\
&\quad + \int_\Gamma (\lambda u - \lambda_h u_h) e.
\end{aligned}$$

Therefore, using lemma 9 we have

$$\begin{aligned}
||\, e\,||^2_{H^1(\Omega)} &\leq \left\{ \sum_T \left(c_T^2\, h_T^2\, ||\, u_h\,||^2_{L^2(T)} + \sum_{F\in\mathcal{F}_T} c_T^2\, h_F ||\, J_F(u_h)\,||^2_{L^2(F)} \right) \right\}^{1/2} ||\, e||_{H^1(\Omega)} \\
&\quad + \frac{(\lambda + \lambda_h)}{2} ||\, e||^2_{L^2(\Gamma)} \\
&\leq \left\{ \sum_T c_T^2 \left(h_T^2\, ||\, u_h\,||^2_{L^2(T)} + \sum_{F\in\mathcal{F}_T} h_F ||\, J_F(u_h)\,||^2_{L^2(F)} \right) \right\}^{1/2} ||\, e||_{H^1(\Omega)} \\
&\quad + C\frac{(\lambda + \lambda_h)}{2} ||\, e||_{L^2(\Gamma)}||\, e||_{H^1(\Omega)}
\end{aligned}$$

by the trace theorem, and the theorem follows. ■

As a consequence of the previous theorem and estimate (14), the global estimator provides an upper bound of the error in the energy norm up to a $h^{1/2}$ order term.

Corollary 12 *There exists a constant C_j such that*

$$||\, e\,||_{H^1(\Omega)} \leq \widehat{\eta}_\Omega + C_j\, h^{1/2}.$$

Nevertheless, it is customary to use the global constant C in (16) and (17). In this case, the local error indicator is defined by

$$\eta_T := \left(h_T^2\, ||\, u_h\,||^2_{L^2(T)} + \sum_{F\in\mathcal{F}_T} h_F\, ||\, J_F(u_h)\,||^2_{L^2(F)} \right)^{1/2}, \tag{19}$$

and the global one by

$$\eta_\Omega := \left(\sum_{T\in\mathcal{T}_h} \eta_T^2 \right)^{1/2}.$$

Certainly, we have

$$A\, \eta_\Omega \leq \widehat{\eta}_\Omega \leq B\, \eta_\Omega \tag{20}$$

with $A, B > 0$.

The result above can be stated for η_Ω.

Corollary 13 *There exist constants C, C_j such that*

$$|| e ||_{H^1(\Omega)} \leq C\, \eta_\Omega + C_j\, h^{1/2}.$$

Remark 14 *We recall that we use a subindex j in a constant which depends on the eigenpair (u_j, λ_j).*

4 THE FACE RESIDUAL ERROR ESTIMATOR

A new error indicator can be defined without involving the volumetric part of $\widehat{\eta}_\Omega$ (or η_Ω), since the face residuals dominate the volumetric ones up to higher order terms (Armentano and Padra, 2006; Carstensen and Verfürth, 1999; Durán and Rodriguez, 1999). We shall not come into proofs of these theorems mainly because, as we have already seen, is a matter of following almost word for word the approach in the cited works.

Let $\mathcal{N}$ be the set of interior nodes of the mesh $\mathcal{T}_h$. For $x \in \mathcal{N}$, let $\omega_x := \cup\{T \in \mathcal{T}_h : \ x \in T\}$. Let φ_x be the basis function with support equal to ω_x, and h_x the diameter of ω_x. Finally, let $\mathcal{T}_x \subset \mathcal{T}_h$ be the subset of elements containing the node x, let $\mathcal{F}_x \subset \mathcal{F}$ be the subset of faces containing the node x and $c_x := \max_{T\in\mathcal{T}_x}\{c_T\}$.

Lemma 15 *For all $x \in \mathcal{N}$, there holds*

$$-\int_{\omega_x} u_h \varphi_x = \sum_{F\in\mathcal{F}_x.} \frac{m(F)|}{d} J_F(u_h).$$

Lemma 16 *For all $x \in \mathcal{N}$, there holds*

$$\sum_{T\in\mathcal{T}_x} c_T^2\, h_T^2\, ||\, u_h\, ||^2_{L^2(T)} \leq C \left(c_x^2 \sum_{F\in\mathcal{F}_x.} h_F\, ||\, J_F(u_h)||^2_{L^2(F)} + c_x^2\, h_x^4\, ||\, \nabla u_h\, ||^2_{L^2(\omega_x)} \right),$$

where C is a constant depending only on the regularity of the mesh.

Now, a simpler indicator is defined by

$$\widetilde{\eta}_T := \widetilde{c}_T \left(\frac{1}{2} \sum_{F\in\mathcal{F}_T.} h_F\, ||\, J_F(u_h)||^2_{L^2(F)} \right)^{1/2},$$

where $\widetilde{c}_T := \max_{x\in T}\{c_x\}$, and the corresponding global error estimator is defined by

$$\widetilde{\eta}_\Omega := \left(\sum_{T\in\mathcal{T}_h.} \widetilde{\eta}_T^2 \right)^{1/2}.$$

With the same considerations as above, it could also be defined the simpler local indicator

$$\widehat{\widetilde{\eta}}_T := \left(\frac{1}{2} \sum_{F\in\mathcal{F}_T.} h_F\, ||\, J_F(u_h)||^2_{L^2(F)} \right)^{1/2},$$

and a global error estimator by

$$\widehat{\widetilde{\eta}}_\Omega := \left(\sum_{T\in\mathcal{T}_h.} \widehat{\widetilde{\eta}}_T^2 \right)^{1/2}.$$

The following result shows that this estimator is globally reliable up to higher order terms.

Theorem 17 *There exist constants $C, \overline{C}$ depending only on the regularity of $\mathcal{T}_h$, such that*

$$||\, e||_{H^1(\Omega)} \leq C\left(\widetilde{\eta}_\Omega^2 + \frac{(\lambda+\lambda_h)}{2}||\, e||_{L^2(\Gamma)}\right) + \overline{C}\,||\, \nabla u_h\,||_{L^2(\Omega)}\, h^2.$$

5 EFFICIENCY OF THE ESTIMATORS

It is well known that a local estimator must be *efficient* in the sense of pointing out which elements should be effectively refined because they support large local error (Ainsworth and Oden, 2000). Several results can be proved in this direction. For example, for the η_Ω estimator we have:

Theorem 18 *There exists a constant C such that, for every $T \in \mathcal{T}_h$ which satisfies $\partial T \cap \Gamma = \emptyset$, we have*

$$\eta_T \leq C\,||\, e\,||_{H^1(T^*)},$$

where T^* is the union of all elements sharing a face with T.

Theorem 19 *There exists a constant C such that, for every $T \in \mathcal{T}_h$ which satisfies $\partial T \cap \Gamma \neq \emptyset$, we have*

$$\eta_T \leq C\left\{||\, e\,||_{H^1(T)} + \sum_{F\in\mathcal{F}_T\cap\mathcal{F}_\Gamma} h_F\,||\,\lambda u - \lambda_h u_h\,||_{L^2(F)}\right\}.$$

Remark 20 *Using the fact that*

$$||\,\lambda u - \lambda_h u_h\,||_{L^2(\Gamma)} \leq \lambda\,||\, u - u_h\,||_{L^2(\Gamma)} + |\lambda - \lambda_h|\,||\, u_h\,||_{L^2(\Gamma)},$$

and previous a priori error estimates (11) and (13), we have

$$||\,\lambda u - \lambda_h u_h\,||_{L^2(\Gamma)} \leq C\, h^{3/4}.$$

Therefore, the term

$$\sum_{F\in\mathcal{F}_T\cap\mathcal{F}_\Gamma} h_F\,||\,\lambda u - \lambda_h u_h\,||_{L^2(F)}$$

is an $O(h^{7/4})$-term.

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REFERENCES

M. Ainsworth and J. T. Oden. *A Posteriori Error Estimates in Finite Element Analysis*. Wiley, 2000.

M. G. Armentano. The effect of reduced integration in the steklov eigenvalue problem. *Mathematical Modelling and Numerical Analysis (M2AN)*, pages 27–36, 2004.

M. G. Armentano and C. Padra. A posteriori error estimates for the steklov eigenvalue problem. *to appear in Applied Numerical Mathematics*, 2006.

G. Auchmuty. Steklov eigenproblems and the representation of solutions of elliptic boundary value problems. *Numerical Functional Analysis and Optimization*, 25:321–348, 2004.

I. Babuška and J. Osborn. *Eigenvalue Problems, Handbook of Numerical Analysis, Finite Elements Methods*, volume 2. 1991.

Von Petersdorff T. Babuška, I. and B. Anderson. Numerical treatement of vertex singularities and intensity factors for mixed boundary value problems for the laplace equations in.

S. C. Brenner and L. R. Scott. *The Mathematic Theory of Finite Element Methods*. Springer-Verlag, 1994.

C. Carstensen and R. Verfürth. Edge residuals dominate a posteriori error estimates for low order finite element methods. *SIAM J. Numer. Anal*, 36:1571–1587, 1999.

Padra C. Durán, R. G. and R. Rodriguez. A posteriori error estimates for the finite element approximation of eigenvalue problems. *Math. Mod. & Met. in Appl. Sc. (M^3AS)*, 13:1165–1178, 1999.

Estep D. Hansbo P. Eriksson, K. and C. Johnson. Introduction to adaptive methods for differential equations. *Acta Numerica*, pages 105–158, 1995.

J. Gergh and J. Löfström. *Interpolation Spaces*. Springer-Verlag, 1976.

G. Savaré. Regularity and perturbation results for mixed second order elliptic problems. *Communications in Partial Differential Equations*, 22:869–899, 1997.

G. Savaré. Regularity results for elliptic equations in lipschitz domains. *Journal of Functional Analysis*, 152:176–201, 1998.

R. Verfürth. Error estimates for some quasi-interpolation operators. *Math. Mod. & Num. An. (M^2AN)*, 33:695–713, 1999.