Asociación Argentina de Mecánica Computacional

Mecánica Computacional Vol XXVII, págs. 985-992 (artículo completo)
Alberto Cardona, Mario Storti, Carlos Zuppa. (Eds.)
San Luis, Argentina, 10-13 Noviembre 2008

NUMERICAL IMPLEMENTATION OF EXTENDED MACARI, RUNESSON, STURE, LADE (MRS-LADE) MODEL FOR UNSATURATED SOIL FOR FINITE STRAIN

Ricardo Schiava[a] **, Guillermo Etse**[b]

[a]*Grupo de Mecánica Computacional, Universidad Nacional de Santiago del Estero (4200) Santiago del Estero, Argentina, schiava@unse.edu.ar*
[b]*CEMNCI, Universidad Nacional de Tucumán, Tucumán, Argentina, getse@herrera.unt.edu.ar*

Keywords: Finite strain, unsaturated soils

Abstract. Mechanical behaviour of geomaterials have high complexity and variability, witch have been reported a large number of constitutive models, depending on different mathematical idealizations.
The pressure dependent elasto-plastic models are considered as the most appropriate representations this behaviour because of the existence of inelastic deformation of the materials subjected to shear stress or/and volumetric pressure. Because the mathematical complexities this models, such as nonlinear elastic law and work plastic dissipation dependent hardening/softening law, resulting from the complex nature implementation is not trivial.
This paper present an elastic-plastic model for unsaturated soils within the regime of finite deformation based in an extension of model's of Macari, Runesson and Sture originally devised by Lade et al. (MRS-Lade). An overall implicit return method for non-associative elastic-plastic model of geomaterials is presented within the regime of large strains. The mathematical model formulated in the framework of additive elastic-plastic decomposition of the rate deformation tensor is briefly summarized. Representative numerical simulation was performed under axial symmetric compression conditions and the results of the simulations are discussed to assess the performance of the unsaturated soils model.

1 INTRODUCTION

In the last years, several models were implemented for geomaterials formulated for the case of small deformations and more recently elastoplastics models was applied in finite deformations to approaches of failure surface type Cam-clay carried out by Simo and Meshke (1993) and and Meschke et al.(1996). In this work the implementation of an elastoplastic model is presented for frictional cohesive materials into the finite deformations with development of a total implicit integration.

In the finite deformation regime the rate of deformation tensor $\mathbf{d}$ is decomposed into an elastic part $\mathbf{d}^e$ and plastic part $\mathbf{d}^p$ by means of the additive decomposition: $\mathbf{d} = \mathbf{d}^e + \mathbf{d}^p$. The stress response is obtained by virtue of the hipoelastic constitutive relation, where the components of the rate stress tensor are lineal functions of the components of the rate of deformation tensor Garino (1993).

In the specific model under consideration, the elastic domain is assumed to be defined by: $f(\boldsymbol{\sigma}', \kappa, s) < 0$, $f(\boldsymbol{\sigma}', \kappa, s) = 0$ determines a convex failure surface in the stress space.

2 CONSTITUTIVE MODEL

2.2 In the original configuration

The gradient of deformation:

$$\mathbf{F} = \frac{\partial \mathrm{x}}{\partial \mathrm{X}} \tag{1}$$

The Green Lagrange tensor:
$$\mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{I}) \tag{2}$$

The symmetric right Cauchy-Green tensor:

$$\mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F} \tag{3}$$

$$\mathbf{E} = \mathbf{E}^{e} + \mathbf{E}^{P} \tag{4}$$

The material is defined as hipoelastic, where the components of the rate stress tensor are lineal functions of the components of the rate strain tensor.

The “constitutive” second Piola-Kirchhoff stress tensor:

$$\dot{\mathbf{S}}' = \mathbf{D} : \dot{\mathbf{E}} \tag{5}$$

Where, $\mathbf{D}$ is the tangent elasticity module in the original configuration.

2.2 In the current configuration

The space expression or eulerian:

$$\mathbf{e} = \mathbf{e}^{e} + \mathbf{e}^{p} \tag{6}$$

Where
$$\mathbf{e} = \frac{1}{2}(\mathbf{I} - \mathbf{b}^{-1}) \tag{7}$$

is the Almansy strain tensor, and

$$\mathbf{b} = \mathbf{F}\ \mathbf{F}^{\mathrm{T}} \tag{8}$$

is the Left Cauchy-Green tensor.

The “constitutive” Cauchy stress tensor comes given for:

$$\boldsymbol{\sigma}' = \frac{1}{J}\phi_*\ \mathbf{S}' \tag{9}$$

Where, ϕ_* is the push-forward transformations therefore it will be:

$$\boldsymbol{\sigma}' = \frac{1}{J}\mathbf{F}\,\mathbf{S}\,\mathbf{F}^{\mathrm{T}} \tag{10}$$

$$\dot{\boldsymbol{\sigma}}' = \frac{1}{J}\phi_*\ \dot{\mathbf{S}}' \tag{11}$$

The rate of deformation tensor:

$$\mathbf{d} = \mathrm{L}_{\mathrm{v}}(\mathbf{e}) = \frac{1}{2}\phi_*\ \dot{\mathbf{E}} \tag{12}$$

$$\mathbf{d} = \mathbf{d}^{\mathrm{e}} + \mathbf{d}^{\mathrm{p}} \tag{13}$$

In eq. (13) $\mathbf{d}^{\mathrm{e}}$, $\mathbf{d}^{\mathrm{p}}$ are the elastic part and plastic part of the rate of deformation tensor, respectively

$$\mathbf{d}^{\mathrm{p}} = \dot{\lambda}\,\frac{\partial g}{\partial \boldsymbol{\sigma}} \tag{14}$$

$$\mathbf{d}^{\mathrm{p}} = \dot{\lambda}\ \mathbf{m}^{\sigma} \tag{15}$$

where $\mathbf{m}^{\sigma} = \dfrac{\partial g}{\partial \boldsymbol{\sigma}}$ is the gradient of the plastic potential function.

The material is defined as hipoelastic, where the components of the rate stress tensor are lineal functions of the components of the rate of deformation tensor.

$$\dot{\boldsymbol{\sigma}}' = \mathbf{a}^{\mathrm{e}} : (\mathbf{d} - \mathbf{d}^{\mathrm{p}}) \tag{16}$$

$\mathbf{a}^{\mathrm{e}}$, is the tangent elasticity module in the current configuration obtained way push-forward transformation

$$\mathbf{a}^{\mathrm{e}} = J\ \phi_*\ \mathbf{D} \tag{17}$$

Following Weihe (1990), the flow rule can be formulated in terms of the space of sub-differentials representing a fan of admissible normal at each corner of the composite failure surface:

$$\partial f_{\lambda}\left\{\boldsymbol{\sigma}', \dot{\mathrm{s}}, \kappa, \mathbf{d}\right\} = \left\{\mathrm{a}\,|\,\left(\boldsymbol{\sigma}' - \boldsymbol{\sigma}'_{\mathbf{o}}\right) : \mathrm{a} \geq 0\,,\quad \forall\, \boldsymbol{\sigma}'_{\mathrm{o}} \in \mathrm{B}_{\lambda}\left\{\kappa, \mathbf{d}, \dot{\mathrm{s}}\right\}\right\} \tag{18}$$

The non-associated flow rule-based general constitutive equations for partially saturated soils can, therefore, be expressed as:

$$\dot{\boldsymbol{\sigma}}' = \mathbf{a}^{e} : (\mathbf{d} - \mathbf{d}^{p}) \tag{19}$$

$$(\boldsymbol{\sigma}' - \boldsymbol{\sigma}'_o) : \mathbf{A} : \mathbf{d}^{p} \geq 0 \,, \forall\, \boldsymbol{\sigma}'_o \in B(\mathbf{d}, \dot{s}, \kappa) \tag{20}$$

$$\dot{\kappa} = \mathrm{h}\left(\mathbf{d}^{p}\right) \tag{21}$$

The variation form of the non-associated flow rule can be reformulated using its rate form and the Kuhn-Tucker conditions, an identical form to the one for small deformation theory:

$$\dot{\lambda} \geq 0 \quad f(\boldsymbol{\sigma}', \kappa, s) \leq 0 \quad f\, \dot{\lambda} = 0 \tag{22}$$

3 CONSTITUTIVE STRESS

3.1 In the original configuration

Throughout this paper, we shall adopt Bishop´s (1959) stress tensor decomposition based on constitutive stresses.The partially saturated soils are generally described in terms of the "constitutive" second Piola-Kirchhoff stress tensor, and following Borja et al.(1997) comes given for:

$$\dot{\mathbf{S}}' = \dot{\mathbf{S}} - \dot{\mathbf{p}}_{\mathbf{w}}\, \mathbf{C}^{-1} = \dot{\mathbf{S}}_{\mathbf{n}} + \dot{s}\, \mathbf{C}^{-1} \tag{23}$$

$$s = (\, p_a - p_w\,) \tag{24}$$

$$\mathbf{S}_n = \mathbf{S} - \mathrm{p}_a\, \mathbf{C}^{-1} \tag{25}$$

thereby s is the matrix soil suction, $\mathbf{S}_{\mathbf{n}}$ the second Piola-Kirchhoff net stress tensor, $\mathbf{S}$ the second Piola-Kirchhoff total stress tensor , $\mathbf{C}$ symmetric right Cauchy-Green tensor, p_a , p_w the pore air and the pore water pressure, respectively.

3.2 The consistency condition

In the case of partially saturated soils the consistency condition during plastic loading, in the original configuration, comes given for:

$$\dot{f} = \frac{\partial f}{\partial \mathbf{S}'} : \dot{\mathbf{S}}' + \frac{\partial f}{\partial s}\, \dot{s} + \frac{\partial f}{\partial \kappa}\, \dot{\kappa} = 0 \tag{26}$$

Where f is the yield function defined as function of the three invariant of stress. The model used is MRS-Lade proposed by Sture et al.(1989), and extended to unsaturated soil by Schiava (2001).

$$\dot{f} = \mathbf{n}_{\mathbf{i}}^{\mathrm{S}} : \dot{\mathbf{S}}' + \mathrm{n}_{\mathrm{i}}^{\mathrm{s}}\, \dot{s} + \mathrm{r}_{\mathrm{i}}\, \dot{\kappa} = 0, \quad \mathrm{i} = 1,2,....\mathrm{u}' \tag{27}$$

$$\mathbf{n}^{\mathrm{S}} = \frac{\partial f}{\partial \mathbf{S}'} \tag{28}$$

$$\mathrm{r}_{\mathrm{i}} = \frac{\partial \mathrm{f}_{\mathrm{i}}}{\partial \kappa_{\mathrm{i}}} \tag{29}$$

$$\dot{\kappa} = h\left(\dot{\mathbf{E}}^{\mathbf{p}}\right) \tag{30}$$

As compared to the consistency condition of classical elastoplastic models, eq (27) has an additional term $n_i^s\,\dot{s}$ related with the evolution of the suction and the gradient of the yield surface with respect to the suction.

Now, the constitutive equation:

$$\dot{\mathbf{S}}' = \mathbf{D} : \dot{\mathbf{E}}^{e} = \mathbf{D} : \left(\dot{\mathbf{E}} - \dot{\mathbf{E}}^{\mathbf{p}}\right) \tag{31}$$

$$\dot{\mathbf{E}}^{\mathbf{p}} = \dot{\lambda}\left(\frac{\partial g}{\partial \dot{\mathbf{S}}'}\right) \tag{32}$$

$$\mathbf{m}^{\mathbf{S}'} = \frac{\partial g}{\partial \mathbf{S}'} \tag{33}$$

Is the gradient de yield potential function respect to “constitutive” second Piola-Kirchhoff stress tensor. For non associative yield rule is: $\mathbf{m}^{\mathbf{S}'} = \mathbf{A}^{-1} : \mathbf{n}^{\mathbf{S}'}$. For associative flow rule $\mathbf{A} = \mathbf{I}$. Where $\mathbf{I}$ is unit fourth order tensor

The flow rule is:

$$\dot{\mathbf{E}}'^{\mathbf{p}} = \sum_{i}^{U} \dot{\lambda}_i \mathbf{m}_i^{\mathbf{S}'} \tag{34}$$

$$\dot{\lambda}_i \geq 0 \tag{35}$$

$$f_i \dot{\lambda}_i = 0 \tag{36}$$

Replacing in the eq. (26), the expression of the plastic multiplier $\dot{\lambda}$ can be obtained as:

$$\dot{\lambda}_i = \frac{\mathbf{n}_i^{\mathbf{S}'} : \mathbf{D} : \dot{\mathbf{E}} + \mathbf{n}_i^{s} : \mathbf{I}\,\dot{s}}{\mathbf{n}_i^{\mathbf{S}'} : \mathbf{D} : \mathbf{m}_i^{\mathbf{S}'} - r\,h(\mathbf{m}_i^{\mathbf{S}'})} \tag{37}$$

Substituting the expression of second constitutive P-K tensor:

$$\dot{\mathbf{S}}' = \mathbf{D} : \dot{\mathbf{E}} - \frac{\mathbf{D} : \mathbf{m}_i^{\mathbf{S}'} \otimes \mathbf{n}_i^{\mathbf{S}'} : \mathbf{D}}{\mathbf{n}_i^{\mathbf{S}'} : \mathbf{D} : \mathbf{m}_i^{\mathbf{S}'} + H_i} : \dot{\mathbf{E}} - \frac{\mathbf{D} : \mathbf{m}_i^{\mathbf{S}'} \otimes n_i^{s}\, \mathbf{I}}{\mathbf{n}_i^{\mathbf{S}'} : \mathbf{D} : \mathbf{m}_i^{\mathbf{S}'} + H_i} : \mathbf{I}\,\dot{s} \tag{38}$$

We obtain the evolution of the total second P-K stress tensor

$$\dot{\mathbf{S}} = \dot{\mathbf{S}}' - \dot{p}_w \mathbf{C}^{-1} \begin{cases} \textit{Caso general} & \dot{\mathbf{S}} = \mathbf{D}_{\mathbf{ep}} : \dot{\mathbf{E}} + \left(-\mathbf{D}_{\mathbf{s}}\right) : \mathbf{I}\,\dot{s} - p_w\,\mathbf{C}^{-1} \\ Si\ \dot{p}_a = 0 \rightarrow & \dot{\mathbf{S}} = \mathbf{D}_{\mathbf{ep}} : \dot{\mathbf{E}} - \left(-\mathbf{D}_{\mathbf{s}} + \mathbf{C}^{-1}\right) : \mathbf{I}\,\dot{s} \\ Si\ \dot{p}_a = \dot{p}_w = 0 \rightarrow \dot{\mathbf{S}} = \mathbf{D}_{\mathbf{ep}} : \dot{\mathbf{E}} & \end{cases} \tag{39}$$

3.3 In the current configuration

The stress state is described for partially saturated soils as:

$$\dot{\boldsymbol{\sigma}}' = \dot{\boldsymbol{\sigma}} - \mathbf{I}\,\frac{\dot{p}_W}{J} = \dot{\boldsymbol{\sigma}}_n + \frac{\dot{s}}{J}\,\mathbf{I} \tag{40}$$

thereby $\boldsymbol{\sigma}'$ is the "constitutive" Cauchy stress tensor and $\mathrm{J} = \det(\mathbf{F})$ is the Jacobian of the solid phase motion.

In case of partially saturated soil, the consistent condition take the form:

$$\dot{\mathrm{f}} = \frac{\partial \mathrm{f}}{\partial \boldsymbol{\sigma}'} : \dot{\boldsymbol{\sigma}}' + \frac{\partial \mathrm{f}}{\partial \mathrm{s}} \dot{\mathrm{s}} + \frac{\partial \mathrm{f}}{\partial \kappa} \dot{\kappa} = 0 \tag{41}$$

$$\dot{f} = \mathbf{n}_{\mathbf{i}}^{\sigma} : \dot{\boldsymbol{\sigma}}' + \mathrm{n}_{\mathrm{i}}^{\mathrm{s}} \, \dot{\mathrm{s}} + \mathrm{r}_{\mathrm{i}} \, \dot{\kappa} = 0, \quad \mathrm{i} = 1,2,....\mathrm{u} \tag{42}$$

The expression of the plastic multiplier λ can be obtained as:

$$\dot{\lambda}_i = \frac{\mathbf{n}_{\mathbf{i}}^{\sigma} : \mathbf{a}^{\mathrm{e}} : \mathbf{d} + \mathbf{n}_{\mathbf{i}}^{\mathrm{s}} : \mathbf{I} \, \dot{\mathrm{s}}}{\mathbf{n}_{\mathbf{i}}^{\sigma} : \mathbf{a}^{\mathrm{e}} : \mathbf{m}_{\mathbf{i}}^{\sigma} - \mathrm{r} \, \mathrm{h}(\mathbf{m}_{\mathbf{i}}^{\sigma})} \tag{43}$$

The "constitutive" Cauchy stress tensor comes given for:

$$\dot{\boldsymbol{\sigma}}' = \mathbf{a}^{\mathrm{e}} : \mathbf{d} - \frac{\mathbf{a}^{\mathrm{e}} : \mathbf{m}_{\mathbf{i}}^{\sigma} \otimes \mathbf{n}_{\mathbf{i}}^{\sigma} : \mathbf{a}^{\mathrm{e}}}{\mathbf{n}_{\mathbf{i}}^{\sigma} : \mathbf{a}^{\mathrm{e}} : \mathbf{m}_{\mathbf{i}}^{\sigma} + \mathrm{H}_{\mathrm{i}}} : \mathbf{d} - \frac{\mathbf{a}^{\mathrm{e}} : \mathbf{m}_{\mathbf{i}}^{\sigma} \otimes \mathrm{n}_{\mathrm{i}}^{\mathrm{s}} \, \mathbf{I}}{\mathbf{n}_{\mathbf{i}}^{\sigma} : \mathbf{a}^{\mathrm{e}} : \mathbf{m}_{\mathbf{i}}^{\sigma} + \mathrm{H}_{\mathrm{i}}} : \dot{\mathrm{s}} \, \mathbf{I} \tag{44}$$

$$\dot{\boldsymbol{\sigma}}' = \mathbf{a}_{\mathbf{ep}} : \mathbf{d} + \mathbf{a}_{\mathrm{s}} : \dot{\mathrm{s}} \, \mathbf{I} = \mathbf{a}_{\mathbf{ep}} : \mathbf{d} + \mathbf{a}_{\mathrm{s}} : \mathbf{I}(\dot{\mathrm{p}}_{\mathrm{a}} - \dot{\mathrm{p}}_{\mathrm{w}}) \tag{45}$$

$$\mathbf{a}_{\mathbf{ep}} = \mathbf{a}^{\mathrm{e}} - \sum_{\mathbf{i=1}}^{\mathbf{U}} \left[\frac{\mathbf{a}^{\mathrm{e}} : \mathbf{m}^{\sigma} \otimes \mathbf{n}^{\sigma} : \mathbf{a}^{\mathrm{e}}}{\mathbf{n}^{\sigma} : \mathbf{a}^{\mathrm{e}} : \mathbf{m}^{\sigma} + \mathrm{H}} \right]_i \tag{46}$$

$$\mathbf{a}_{\mathrm{s}} = -\sum_{\mathbf{i=1}}^{\mathbf{U}} \left[\frac{\mathbf{a}^{\mathrm{e}} : \mathbf{m}^{\sigma} \otimes \mathbf{n}^{\mathrm{s}} \, \mathbf{I}}{\mathbf{n}^{\sigma} : \mathbf{a}^{\mathrm{e}} : \mathbf{m}^{\sigma} + \mathrm{H}_{\mathrm{i}}} \right] \tag{47}$$

We obtain the evolution of the total Cauchy stress tensor:

$$\dot{\boldsymbol{\sigma}} = \dot{\boldsymbol{\sigma}}' + \mathbf{I} \frac{\dot{\mathrm{s}}}{\mathrm{J}} = \left\{ \begin{array}{ll} General\ case \rightarrow & \mathbf{a}_{\mathbf{ep}} : \mathbf{d} + [-\mathbf{I} + \mathbf{a}_{\mathrm{s}}] : \frac{\dot{\mathrm{s}}}{\mathrm{J}} \mathbf{I} \\ \dot{p}_a = 0 \quad Case \rightarrow & \mathbf{a}_{\mathbf{ep}} : \mathbf{d} - [-\mathbf{I} + \mathbf{a}_{\mathrm{s}}] : \frac{\dot{\mathrm{p}}_{\mathrm{w}}}{\mathrm{J}} \mathbf{I} \\ \dot{p}_a = \dot{p}_w = 0 \, Case \rightarrow & \dot{\boldsymbol{\sigma}} = \dot{\boldsymbol{\sigma}}' = \mathbf{a}_{\mathbf{ep}} : \mathbf{d} \end{array} \right\} \tag{48}$$

4 YIELD CONDITION

The yield condition, is defined in terms of the first invariant of the effective stress tensor p, of the second and third invariant of the deviatoric stress tensor q and θ, respectively, and of the hardening/softening variables in the cone region κ_{cone}. Defining the effective pressure in terms of the net mean stress p_n and the suction s the generic shape of the cone takes the form:

$$f_{cone}\{p_n,q,\theta,s,\kappa_{cone}.\} = f\{q,\theta,s\} - \eta_{cone}\{\kappa_{cone}\}(p+s-p_c) = 0 \quad (49)$$

$$f\{q,\theta,s\} = q\left(1+\frac{q}{q_a}\right)^m g\{\theta\} \quad (50)$$

$$p = -\frac{I_{1n}}{3} \qquad q = \sqrt{3J_{2D}} \qquad \cos\ 3\theta = \frac{3\sqrt{3}}{2}\frac{J_{3D}}{\sqrt{(J_{2D})^3}} \quad (51)$$

Thereby I_{I_n} is the first invariant of the net stress tensor and J_{2D} and J_{3D} the second and third invariants of the deviatoric stress tensor, respectively. Finally, $g(\theta)$ is the Willam and Warnke factor which assures a continuous and smooth variation of the shear strength in the deviatoric plane as long as the so-called eccentricity parameter *e* fulfills the condition $\frac{1}{2} < e \le 1$. In the present formulation the plastic potential function in the cone regime is defined in the form:

$$g_{cone}\{p_n,q,\theta,s,\kappa_{cono}.\} = f\{q,\theta,s\} - n\ \eta_{cone}\{\kappa_{cone}\}(p+s-p_c) = 0 \quad (52)$$

where *n* is a scalar parameter. For a more elaborate description of the model, see Schiava and Etse (2006).

5 NUMERICAL SIMULATION

The model predictions of the failure response behaviour of unsaturated soils are analyzed for the uniaxial compression test, implementing the model in program of finite elements.

In the numeric simulation of axial symmetric test, the structure was model by means of a homogeneous quadrilaterals element of four nodes, applying vertical displacements in the top nodes. In the Figure 1 the answer is observed that is obtained with the extended of MRS-Lade model for small deformations and the model implemented for finite deformations. It is deduced that the influence on the level stress increases in finite deformations for increment suction values.

6 CONCLUSIONS

An elastoplastic constitutive model for unsaturated soils is proposed in finite deformations regime, it is considered the decomposition additive of the rate of deformation tensor and defined the material as hipoelastic where the components of the rate stress tensor are lineal functions of the components of the rate of deformation tensor. The simulation of the axial compression test showed the influence on the level stress increases in finite deformations for the increment suction values.

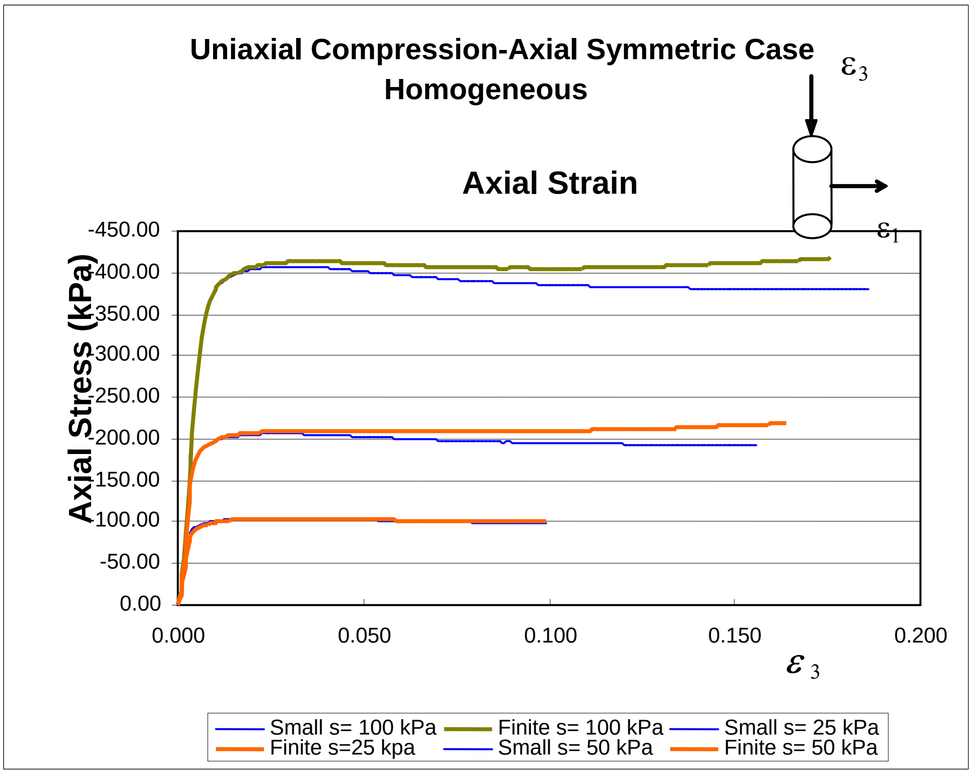

Figure 1. Uniaxial compression case

REFERENCES

Bishop A.W. The Principle of Effective Stress. *Teknisk. Ukeblad* , 39: 859-863, 1959.

Borja R.I. and Tamagnini C. Numerical Implementation of a Mathematical Model for Finite Strain Elastoplastic Consolidation. *Computational Plasticity, Fundamentals and Applications,* D.R.J. Owen, E. Oñate and E. Hinton (Eds), 1631-1640. CIMNE, 1997.

Garcia Garino C. *Un modelo Numérico para el análisis de Sólidos elastoplásticos sometidos a Grandes deformaciones*. Tesis Doctoral. Universitat Politécnica de Catalunya. Escola Técnica Superior D´enginyers de Camins, Canals I Ports. 1993.

Meshke G., Liu H. and Mang H.A. Large strain finite element analysis of snow. *Journal of Engineering Mechanics*, ASCE, 123: 591-602, 1996.

Schiava R. *Modelación elastoplástica de suelos cohesivos friccionales parcialmente Saturados.* Tesis Magister, UNSE, Argentina, 2001.

Schiava R. and Etse G. Constitutive modelling and discontinuous bifurcation assessment in unsaturated soil. *Journal of Applied Mechanics*, 73 I.1:1039-1044, 2006.

Simo J.C., and Meshke G. A new class of algoritms for classical plasticity extended to finite strains. Application to geomaterials. *Computational Mechanics*, 11: 253-278, 1993.

Sture S., Runesson R. and Macari E. Analysis and calibration of a three invariant plasticity model for granular materials. *Ingenieur Archive,* 59: 253-266. 1989.

Weihe, Stefan; Thesis: *Implicit integration schemes for multi-surface yield criteria subjected to hardening/softening behavior.* University of Colorado, Boulder, 1990.