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ERROR ANALYSIS OF THE STABILIZED BOUNDARY PENALTY METHOD

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Abstract. The Stabilized Boundary Penalty Method (SBMPM) enforces Dirichlet boundary conditions though a penalty function related to the mesh-size. We derive a priori error estimates for this method, and we prove that they always give, at least theoretically, an optimal rate of convergence. We also derive an a posteriori error estimate and we propose an adaptive loop. Numerical examples show that SBPM is highly flexible, produces accurate results and it is a very efficient adaptive method.

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1 INTRODUCTION

Let Ω be a bounded domain in $\mathbb{R}^d$, $d \geq 2$, with Lipschitz boundary $\partial\Omega$. We consider the model problem with homogeneous Dirichlet boundary conditions

$$\begin{aligned} -\Delta u = f \qquad & \text{in } \Omega, \\ u = 0 \qquad & \text{on } \partial\Omega, \end{aligned} \tag{1}$$

with exact solution $u_{\mathscr{L}}$. The boundary penalty method (BPM) for approximating solution $u_{\mathscr{L}}$ has been used for a while. The basic idea of BPM is to impose Dirichlet boundary conditions weakly by using Robin type boundary condition with a penalty positive parameter $\phi \in \mathbb{R}$. Given a finite element space $\mathbb{V}_{\mathscr{T}}$, the weak formulation using BMP now reads: find $U_{\mathscr{T},\phi} \in \mathbb{V}_{\mathscr{T}}$ such that

$$\int_\Omega \nabla U_{\mathscr{T},\phi} \cdot V + \int_{\partial\Omega} \phi\, U_{\mathscr{T},\phi} V = \int_\Omega f\, V, \qquad \forall V \in \mathbb{V}_{\mathscr{T}}. \tag{2}$$

We immediately note that this method is not consistent in the sense that $u_{\mathscr{L}}$ does not solve equation (1).

The continuous version of (2) is:

Find $u_\phi \in H^1(\Omega)$ such that

$$\int_\Omega \nabla u_\phi \cdot v + \int_{\partial\Omega} \phi\, u_\phi\, v = \int_\Omega f\, v, \qquad \forall v \in H^1(\Omega). \tag{3}$$

The error analysis of

$$||u_{\mathscr{L}} - U_{\mathscr{T},\phi}||_1$$

could be separated into the problems:

1. Estimate
$$||u_{\mathscr{L}} - u_\phi||_1.$$
2. Estimate
$$||u_\phi - U_{\mathscr{T},\phi}||_1.$$

There are various reasons for studying BMP. One of then is that it allows us to impose Dirichlet boundary conditions even with more general finite element spaces where this task is expensive. Other authors choose this method in their work on non-matching grids or problems with interior sub-domain interfaces. A basic issue in numerical applications is the error analysis of the method. Some of the first works on this subject are (Babuška, 1970, 1973a) and (Barrett and Elliott, 1986) among others. In all these works ϕ is a real parameter. On the other hand, in the two papers (Barboza and Hughes, 1991, 1992), an stabilization technique was proposed for the Lagrange multiplier theory of Babŭska (Babuška, 1973b). Furthermore, in the classical method by Nitsche, other authors have introduced stabilizing or penalization functions ϕ, which are strongly related to the mesh-size function of the mesh $\mathscr{T}$ (Juntunen and Stenberg, 2005; Stenberg, 1995). These stabilization technique has been thoroughly studied in (Pitkäranta, 1973). Among other things, he showed that the stability and error analysis are more easily performed using mesh dependent norms.

This paper aims at error analysis of the BPM with penalization functions. In particular, the stabilized BPM.

Given $\alpha > 0$, we consider functions $\phi \in L^\infty_\alpha(\partial\Omega) := \{\phi \in L^\infty(\partial\Omega) \,|\, [\phi] := \inf_{x\in\partial\Omega} \phi(x) \geq \alpha\}$.

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Our first main result is an a priori error estimate for the continuous problem:
Main 1:

$$||u_{\mathscr{L}} - u_\phi||_1 \preccurlyeq ||f||_0 \, [\phi]^{-1/2}.$$

Under the assumption $u_{\mathscr{L}} \in H^{1+r}(\Omega), r > 0$, and $||u_{\mathscr{L}}||_{1+r} \preccurlyeq ||f||_0$, our second result is an a priori error estimate for the discrete approximation:
Main 2:

$$||u_\phi - U_{\mathscr{T},\phi}||_1 \preccurlyeq (h^r + [\phi]^{-1/2}) \, ||f||_0,$$

where h is, as usual, the maximum of the mesh-size.

Next, we introduce the stabilized boundary penalty method (SBPM), which is BPM with penalty functions of the form $\phi \approx h^{-k}$, where h is the mesh-size function and k is a positive number. We define an a posteriori error estimator $\eta_{\mathscr{T}}(U_{\mathscr{T},\phi})$ and the adaptive loop. It can be proved that the adaptive process converges, that is

$$\eta_{\mathscr{T}_n}(U_{\mathscr{T}_n,\phi_n}) \to 0.$$

But we shall not deal with this rather technical result here. It will be proved in a forthcoming paper.

An a posteriori error estimator for BMP ($\phi \in \mathbb{R}$) was defined in (Eriksson et al., 2004). There, the adaptive process consists of two stages: refinement and adjustment of parameter ϕ. In SBPM, the penalty function adjusts automatically in triangles that are refined.

The *hp-Fem* spaces $\mathscr{F}_{\mathscr{X}}^{(m,r)}$ are next introduced for its use in numerical experiments (cf. (Zuppa, 2008; Zuppa et al., 2007)) . Spaces $\mathscr{F}_{\mathscr{X}}^{(m,r)}$ are well suited for experiments because:

- They always produce singular stiffness-matrix as well as other Generalized Finite Element methods.
- Imposing Dirichlet boundary conditions is not cheap like in traditional FEM. Then, the use of a method like SBPM or Nitsche method is mandatory. In (Zuppa et al., 2007) we have discussed the good behavior of $\mathscr{F}_{\mathscr{X}}^{(m,r)}$ under the Nitsche method, which is considered the most straightforward approach in this kind of problems. The numerical experiment confirms that SBPM is a very efficient adaptive method.

2 NOTATION AND PRELIMINARIES

We use standard notation for Sobolev spaces, norms, and seminorms. We denote by $\langle .,. \rangle$ the L^2-inner product. If no confusion seems likely, the duality pairing between a Hilbert space and its dual will be also denoted by $\langle .,. \rangle$. Here and throughout, C denotes a generic constant that can change in every situation. Sometimes, we hide the constant using instead the symbol $\preccurlyeq$.

Remark 1 *The choice of model (1) is made for ease of presentation, since similar results seem to hold for the mixed non-homogenous boundary value problem*

$$\begin{aligned} \mathscr{L}u := -div(A\nabla u) + cu = f \qquad & in\ \Omega, \\ u = g \qquad on\ \partial\Omega_D, \qquad & g \in H^{1/2}(\Gamma_D), \\ A\frac{\partial u}{\partial n} = h \qquad on\ \partial\Omega_N, \qquad & h \in H^{-1/2}(\Gamma_N). \end{aligned} \tag{4}$$

where $\partial\Omega = \partial\Omega_D \cup \partial\Omega_N$ and $c \geq 0$. However, a main ingredient in our proofs is the possibility of solving the $\mathscr{L}$-harmonic Dirichlet problem with L^2 data (see Theorem 5).

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The bilinear form $\mathscr{B}: H^1(\Omega)\times H^1(\Omega)\to\mathbb{R}$ is defined to be

$$\mathscr{B}[u,v] := \int_\Omega \nabla u\cdot\nabla v, \qquad \forall u,v\in H^1(\Omega).$$

2.1 Harmonic functions

A coercive scalar inner product $[.,.]_\partial$ on $H^1(\Omega)$ is defined by

$$[u,v]_\partial := \mathscr{B}[u,v] + \int_{\partial\Omega} \Gamma u\,\Gamma v \qquad \forall u,v\in H^1(\Omega),$$

where $\Gamma: H^1(\Omega)\to H^{1/2}(\partial\Omega)$ is the trace operator. The associated norm is

$$||u||_\partial := ([u,u]_\partial)^{1/2} \qquad \forall u\in H^1(\Omega).$$

A function $u\in H^1(\Omega)$ is harmonic if

$$\mathscr{B}[u,v]=0 \qquad \forall v\in H^1_0(\Omega). \tag{5}$$

The subspace of $H^1(\Omega)$ of all harmonic functions will be denoted by $\mathscr{H}$.

Subspaces $H^1_0(\Omega)$ and $\mathscr{H}$ are ∂-orthonormal and

$$H^1(\Omega) = H^1_0(\Omega)\oplus_\partial \mathscr{H}. \tag{6}$$

The restriction of the trace map

$$\Gamma: \mathscr{H}\to H^{1/2}(\partial\Omega) \tag{7}$$

is a linear isomorphism with inverse $E:=\Gamma^{-1}$. Furthermore, for every $u\in\mathscr{H}$, $D_n u := \frac{\partial u}{\partial n}$ exists as an element of $H^{-1/2}(\partial\Omega)$ and

$$\mathscr{B}[u,v] = \langle D_n u, v\rangle\,, \qquad \forall u,v\in\mathscr{H}. \tag{8}$$

Notation 2 *For every $v\in H^1(\Omega)$, we write $u=u_0+\widehat{u}$, $u_0\in H^1_0(\Omega)$ and $\widehat{u}\in\mathscr{H}$, according to (6).*

3 THE PENALTY METHOD

Given $\alpha>0$, the space $L^\infty_\alpha(\partial\Omega)$ is defined to be

$$L^\infty_\alpha(\partial\Omega) := \{\phi\in L^\infty(\partial\Omega)\,|\,[\phi] := \inf_{x\in\partial\Omega}\phi(x)\geq\alpha\}.$$

For $\phi\in L^\infty_\alpha(\partial\Omega)$, we define the continuous symmetric bilinear form

$$\mathscr{B}_\phi[u,v] := \mathscr{B}[u,v] + \int_{\partial\Omega}\phi\,\Gamma u\,\Gamma v, \qquad \forall u,v\in H^1(\Omega).$$

We note that

$$\mathscr{B}_\phi[u,u] \geq \mathscr{B}[u,u] + \alpha\int_{\partial\Omega}(\Gamma u)^2 \geq C^2\,||u||_1^2, \qquad \forall u\in H^1(\Omega), C>0. \tag{9}$$

Therefore, the form $\mathscr{B}_\phi$ is $\phi-$uniformly coercive.

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It follows that the variational problem:

$$\mathscr{B}_\phi[u,v] = \langle f,v\rangle, \qquad \forall v \in H^1(\Omega), \tag{10}$$

admits a unique solution $u_\phi \in H^1(\Omega)$.

From (10) we observe that

$$\mathscr{B}_\phi[u_\phi,u_\phi] \le ||f||_0||u_\phi||_1,$$

and from (9), we get

$$\mathscr{B}_\phi[u_\phi,u_\phi] \preccurlyeq ||f||_0{}^2, \qquad \forall \phi \in L^\infty_\alpha(\partial\Omega). \tag{11}$$

In particular,

$$\int_{\partial\Omega} \phi\, (\Gamma u)^2 = ||\phi^{1/2}\Gamma u_\phi||^2_{0,\partial\Omega} \preccurlyeq ||f||_0^2, \qquad \forall \phi \in L^\infty_\alpha(\partial\Omega). \tag{12}$$

Remark 3 *It is clear that u_ϕ is the weak solution of the Robin boundary value problem*

$$-\Delta u = f \qquad \text{in } \Omega, \tag{13}$$

$$D_n u + \phi u = 0 \qquad \text{on } \partial\Omega. \tag{14}$$

The main idea of the penalty method is to use this perturbed variational problem with the hope that $u_\phi \to u_{\mathscr{L}}$ if $[\phi] \to \infty$.

Using the ∂-orthogonality we notice that (10) can be rewritten as

$$\mathscr{B}[u_{\phi,0},v_0] + \mathscr{B}[\widehat{u}_\phi,\widehat{v}] + \int_{\partial\Omega} \phi\, \Gamma\widehat{u}_\phi\, \Gamma\widehat{v} = \langle f,v_0\rangle + \langle f,\widehat{v}\rangle . \tag{15}$$

First at all, setting $\widehat{v} = 0$, we get

$$\mathscr{B}[u_{\phi,0},v] = \langle f,v\rangle, \qquad \forall v \in H^1_0(\Omega).$$

Corollary 4 *For all $\phi \in L^\infty_\alpha(\partial\Omega)$, we have $u_{\phi,0} = u_{\mathscr{L}}$.*

On the other hand, setting $v_0 = 0$ in (15), we obtain

$$\mathscr{B}[\widehat{u}_\phi,\widehat{v}] + \int_{\partial\Omega} \phi\, \Gamma\widehat{u}_\phi\, \Gamma\widehat{v} = \langle f,\widehat{v}\rangle, \qquad \forall \widehat{v} \in \mathscr{H}. \tag{16}$$

Now, recalling (8), we can rewrite this inequality as

$$\left\langle D_n\widehat{u}_\phi,\overline{v}\right\rangle_{1/2} + \left\langle \phi\, \Gamma\widehat{u}_\phi,\overline{v}\right\rangle = \langle f,E\overline{v}\rangle := F_\partial(\overline{v}), \qquad \forall \overline{v} \in H^{1/2}(\partial\Omega). \tag{17}$$

A difficult result of harmonic analysis (Dahlberg, 1977) gives that $F_\partial : H^{1/2}(\partial\Omega) \to \mathbb{R}$ can be extended in a appropriate way to a continuous application over $L^2(\partial\Omega)$:

Theorem 5 *The operator E can be extended to a continuous operator $\widetilde{E} : L^2(\partial\Omega) \to L^2(\Omega)$, such that $\widetilde{E}(\overline{v})$ is harmonic for every $\overline{v} \in L^2(\partial\Omega)$.*

As a consequence, we obtain that the application $\widetilde{F_\partial} : L^2(\partial\Omega) \to \mathbb{R}$ defined by $\widetilde{F_\partial}(\overline{v}) := \int_\Omega f\, \widetilde{E}(\overline{v})$ is continuous.

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Corollary 6 *There exists a constant $C = C(\Omega)$ such that*

$$|F_\partial(\bar{v})| \le C\,||f||_0\,||\bar{v}||_{0,\partial\Omega}, \qquad \forall \bar{v} \in H^{1/2}(\partial\Omega). \tag{18}$$

Theorem 7 *There exists a constant $C = C(\Omega)$ such that*

$$||u_{\mathscr{L}} - u_\phi||_1 = ||\widehat{u}_\phi||_1 \le C\,||f||_0^2\,[\phi]^{-1/2}, \qquad \forall \phi \in L^\infty_\alpha(\partial\Omega). \tag{19}$$

Proof. To see this, we insert $\widehat{u}_\phi$ in (16). Then, by (18)

$$||\widehat{u}_\phi||_1^2 \le \mathscr{B}[\widehat{u}_\phi, \widehat{u}_\phi] + \int_{\partial\Omega} \phi\,(\Gamma\widehat{u}_\phi)^2 \le C\,||f||_0\,||\Gamma\widehat{u}_\phi||_{0,\partial\Omega}. \tag{20}$$

For the other hand, using (12) we obtain

$$[\phi]\int_{\partial\Omega}(\Gamma\widehat{u}_\phi)^2 \le \int_{\partial\Omega}\phi\,(\Gamma\widehat{u}_\phi)^2 \preccurlyeq ||f||_0^2.$$

and

$$||\Gamma\widehat{u}_\phi||_{0,\partial\Omega} \preccurlyeq ||f||_0\,[\phi]^{-1/2}. \tag{21}$$

Putting together (20) and (21) the assertion follows. ■

4 DISCRETE APPROXIMATION AND A PRIORI ESTIMATES

We suppose from now on that $\overline{\Omega}$ is a polyhedral domain that is triangulated by a conforming triangulation $\mathscr{T} = \{T\}$. We assume also

Condition 8 *The exact solution satisfies*

- $u_{\mathscr{L}} \in H^{1+r}(\Omega)$, for some $r > 0$.
- $||u_{\mathscr{L}}||_{1+r} \preccurlyeq |||f|||$, where $|||\,.\,|||$ is some norm such that $||f||_0 \le |||f|||$.

Remark 9 *Constant r depends on Ω and the data ($r = 1$ if Ω is convex).*

Let $h_{\mathscr{T}}$ stand for the mesh-size of $\mathscr{T}$; namely $h_{\mathscr{T}} := \max_{T\in\mathscr{T}}\{h_T\}$, where $h_T := |T|^{1/d}$ and $|T|$ is the Lebesgue measure of T. We define also the mesh-size at the boundary by

$$h_{\mathscr{T},\partial} := \max_{T\in\mathscr{T},\, T\cap\partial\Omega\neq\varnothing}\{h_T\}.$$

We suppose all meshes considered here satisfy a uniform shape-regularity condition.

Now, let $\mathbb{V}_{\mathscr{T}} \subset H^1(\Omega)$ be a finite dimensional space of functions that will be used for approximating problem (10). The requirement for $\mathbb{V}_{\mathscr{T}}$ is the existence of a Scott-Zhang interpolation operator (Scott and Zhang):

Condition 10 *There exists a continuous linear map $\mathscr{P} : H^1(\Omega) \to \mathbb{V}_{\mathscr{T}}$ which satisfies:*

- *There exists a constant C depending only on Ω and the mesh-quality of $\mathscr{T}$ such that*

$$||u - \mathscr{P}u||_s \le C h_{\mathscr{T}}^{1+r-s}\,|u|_{1+r}, \qquad \forall u \in H^1(\Omega),\, 0 \le s < 1 + r.$$

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- *$\mathscr{P}$ preserves homogeneous boundary values. That is, $\mathscr{P}u \in \mathbb{V}_{0,\mathscr{T}} := \mathbb{V}_{\mathscr{T}} \cap H_0^1(\Omega)$ if $u \in H_0^1(\Omega)$.*

We also assume, as it is usual, the conditions:

- $\mathbb{V}_{\mathscr{T}} \subset C(\overline{\Omega})$
- For all $V \in \mathbb{V}_{\mathscr{T}}, V\,|\,T$ is a polynomial of degree $\leq p$, for some fixed positive integer p.

Remark 11 *$\mathbb{V}_{\mathscr{T}}$ can be a standard finite element space or the hp-fem spaces $\mathscr{F}_{\mathscr{X}}^{m,r}$ defined in (Zuppa, 2008), for example, for which the use of a penalty method is more suitable These spaces will be used in the numerical experiments bellow.*

Given $\phi \in L^\infty_\alpha(\partial\Omega)$, the discrete formulation of the penalty method is:
Find $U_\phi \in \mathbb{V}_{\mathscr{T}}$ such that

$$\mathscr{B}_\phi[U_\phi, V] = \langle f, V\rangle, \qquad \forall V \in \mathbb{V}_{\mathscr{T}}. \tag{22}$$

By a standard coercivity argument the solution U_ϕ exists and is unique.
The main objective of this section is to estimate the error

$$||\,u_{\mathscr{L}} - U_\phi\,||_1 .$$

First of all, we write

$$||\,u_{\mathscr{L}} - U_\phi\,||_1 \leq ||\,u_{\mathscr{L}} - u_\phi\,||_1 + ||\,u_\phi - U_\phi\,||_1.$$

We notice that the first term of the sum is already estimated. By (19) we have

$$||\,u_{\mathscr{L}} - u_\phi\,||_1 \leq C\,||\,f\,||_0^2\,[\phi]^{-1/2} \leq C\,|||\,f\,|||^2\,[\phi]^{-1/2}.$$

Now, we shall proceed to estimate the second term . Using orthogonality of the error and Cea's lemma, we obtain

$$\begin{aligned} \mathscr{B}_\phi[u_\phi - U_\phi, u_\phi - U_\phi] &\leq C \inf_{V\in\mathbb{V}_{\mathscr{T}}} \mathscr{B}_\phi[u_\phi - V, u_\phi - V] \\ &\leq C \inf_{V\in\mathbb{V}_{0,\mathscr{T}}} \mathscr{B}_\phi[u_\phi - V_0, u_\phi - V_0]. \end{aligned}$$

Therefore, setting $V_0 = \mathscr{P}u_{\mathscr{L}}$, writing $u_\phi = u_{\mathscr{L}} + \widehat{u}_\phi$, and using the property that

$$\mathscr{B}[\widehat{u}_\phi, V] = 0, \qquad \forall V \in H_0^1(\Omega),$$

we have

$$\begin{aligned} \mathscr{B}_\phi[u_\phi - \mathscr{P}u_{\mathscr{L}}, u_\phi - \mathscr{P}u_{\mathscr{L}}] &= \mathscr{B}[u_{\mathscr{L}} - \mathscr{P}u_{\mathscr{L}}, u_{\mathscr{L}} - \mathscr{P}u_{\mathscr{L}}] \\ &\quad + \mathscr{B}_\phi[\widehat{u}_\phi, \widehat{u}_\phi]. \end{aligned}$$

Using now (8), (10), and the equivalence of norms, we can estimate the first term as follows:

$$\begin{aligned} \mathscr{B}[u_{\mathscr{L}} - \mathscr{P}u_{\mathscr{L}}, u_{\mathscr{L}} - \mathscr{P}u_{\mathscr{L}}] &\leq C h_{\mathscr{T}}^{2r}\,||\,u_{\mathscr{L}}\,||_{1+r}^2 \\ &\leq C h_{\mathscr{T}}^{2r}\,|||\,f\,|||^2. \end{aligned}$$

For the other hand, we know already that

$$\mathscr{B}_\phi[\widehat{u}_\phi, \widehat{u}_\phi] \leq C\,|||\,f\,|||^4\ [\phi]^{-1}.$$

Putting together all these estimates, we have obtained

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Theorem 12 *Assuming the regularity condition (8), there holds*

$$||u_{\mathscr{L}} - U_\phi||_1 \preccurlyeq (h^r_{\mathscr{T}} \,|||f|||+ [\phi]^{-1/2}\, ||f||^2_0), \qquad \forall \phi \in L^\infty_\alpha(\partial\Omega). \tag{23}$$

The hidden constant depends only on the mesh-quality and other constants related to equivalence of the involved norms.

5 THE STABILIZED BOUNDARY PENALTY METHOD (SBPM)

A useful particular class of weight functions is related to the stabilized penalty method. Let $\mathscr{R}$ be the union of the internal sides of the triangulation. That is,

$$\mathscr{R} := (\cup_{T\in\mathscr{T}} \partial T) \cap \Omega .$$

We define the mesh size function $H_T \in L^\infty(\overline{\Omega})$ by

$$H_T(x) := \begin{cases} 0, & x \in \mathscr{R}, \\ h_T & T \text{ is unique triangle such that } x \in T. \end{cases}$$

Remark 13 *Note that* $h_\partial^{-1} := H_T^{-1} \,|\, \partial\Omega$ *belongs to* $L^\infty_\alpha(\partial\Omega)$, *where* $\alpha = h_{\mathscr{T},\partial}$.

Given $\beta, k > 0$, we define

$$\phi = \phi(\mathscr{T}, \beta, k) := \beta^{-2} h_\partial^{-k} . \tag{24}$$

For this class of functions we have

$$[\phi]^{-1} \le \beta^2 h^k_{\mathscr{T},\partial} , \tag{25}$$

and estimate (23) can be written

$$||u_{\mathscr{L}} - U_\phi||_1 \preccurlyeq (h^r_{\mathscr{T}} \,|||f|||+ \beta\, h^{k/2}_{\mathscr{T},\partial}\, ||f||^2_0) . \tag{26}$$

Remark 14 *The last estimate shows that the stabilized penalty method gives, at least theoretically, the right rate of convergence as long as* $k \approx 2r$. *From the computational point of view, however, we notice that the contribution to the stiffness matrix of* $\mathscr{B}[U,V]$ *in a triangle* T *is proportional to* h_T^{d-2}, *while the contribution of* $\int_{T\cap\partial\Omega} UV$ *is of order* h_T^{d-1}. *Then, the choice* $k = 1$ *gives the same order of contribution in both terms. Numerical experiments show that* $k = 1$ *is the optimal choice, whereas* $k > 1$ *can degrade the condition number of the stiffness-matrix and the quality of the results. Furthermore, SBMP outperforms the rather pessimistic rate of convergence of (26) if* $k = 1$.

6 AN A POSTERIORI ERROR ESTIMATOR AND ADAPTIVITY

In this section we introduce an error estimator for the error $e := u_{\mathscr{L}} - U_\phi$.

Notation 15 *For the sake of notational simplicity, we shall omit the symbol of the domain of integration when the domain is all* Ω *or* $\partial\Omega$. *Furthermore, we shall omit the symbol of the trace operator* Γ *for functions on the boundary. The context should clarify any confusion.*

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The procedure for obtaining the residual error estimator is standard, and we shall omit some details. Here, we shall only deal with penalization functions of the form $\phi \approx h_{\mathscr{T}}^{-k}$.

For all $v \in H^1(\Omega)$, we write

$$\mathscr{B}_\phi[e,v] = \mathscr{B}_\phi[u_{\mathscr{L}},v] + \langle D_n u_{\mathscr{L}}, \Gamma v\rangle - \mathscr{B}_\phi[U_\phi,v] = f\,v + \langle D_n u_{\mathscr{L}}, v\rangle - \mathscr{B}_\phi[U_\phi,v].$$

Now,

$$\begin{aligned}
\mathscr{B}_\phi[U_\phi,v] &= \mathscr{B}[U_\phi,v] + \phi\, U_\phi\, v \\
&= \sum_{T\in\mathscr{T}} \left[\int_T \nabla U_\phi \cdot \nabla v + \int_{T\cap\partial\Omega} \phi\, U_\phi\, v\right] \\
&= \sum_{T\in\mathscr{T}} \left[-\int_T \Delta U_\phi\, v + \int_{\partial T} D_n U_\phi\, v + \int_{T\cap\partial\Omega} \phi\, U_\phi\, v\right].
\end{aligned}$$

Separating the internal sides from those in the boundary, and introducing the standard jump operator J, we get

$$\begin{aligned}
\mathscr{B}_\phi[e,v] &= \sum_{T\in\mathscr{T}} \int_T [\Delta U_\phi + f]\, v + \left\langle D_n u_{\mathscr{L}} - D_n U_\phi, c\right\rangle \\
&\quad + \sum_{T\in\mathscr{T}} \int_{\partial T\cap\mathscr{R}} J(\nabla U_\phi)\, v - \sum_{T\in\mathscr{T}} \int_{\partial T\cap\partial\Omega} \phi\, U_\phi\, v.
\end{aligned}$$

Summing up the relation

$$\mathscr{B}_\phi[e,\mathscr{P}v] = 0,$$

we obtain

$$\begin{aligned}
\mathscr{B}_\phi[e,v] &= \sum_{T\in\mathscr{T}} \int_T [\Delta U_\phi + f]\,[v - \mathscr{P}v] + \left\langle D_n u_{\mathscr{L}} - D_n U_\phi, v - \mathscr{P}v\right\rangle \\
&\quad + \sum_{T\in\mathscr{T}} \int_{\partial T\cap\mathscr{R}} J(D_n U_\phi)\,[v - \mathscr{P}v] - \sum_{T\in\mathscr{T}} \int_{\partial T\cap\partial\Omega} \phi\, U_\phi\,[v - \mathscr{P}v].
\end{aligned}$$

Then,

$$\begin{aligned}
\mathscr{B}_\phi[e,v] &\preccurlyeq \sum_{T\in\mathscr{T}} ||\,\Delta U_\phi + f\,||_{0,T}\, ||v - \mathscr{P}v||_{0,T} \\
&\quad + \sum_{T\in\mathscr{T}} ||J(D_n U_\phi)\,||_{0,\partial T\cap\mathscr{R}}\, ||v - \mathscr{P}v||_{0,\partial T\cap\mathscr{R}} \\
&\quad + \sum_{T\in\mathscr{T}} h_T^{-k}\, ||\,U_\phi\,||_{0,\partial T\cap\partial\Omega}||\,v - \mathscr{P}v||_{0,\partial T\cap\partial\Omega}.
\end{aligned}$$

We have omitted the term $\left\langle D_n u_{\mathscr{L}} - D_n U_\phi, v - \mathscr{P}v\right\rangle$ because, by (**??**), it can be absorbed in the other terms.

Now, it is well known that the following interpolation estimate for the Scott-Zhang operator holds for all $v \in H^1(\Omega)$

$$\sum_{T\in\mathscr{T}} h_T^{-2}||v - \mathscr{P}v||_{0,T}^2 + h_T^{-1}||\,v - \mathscr{P}v||_{0,\partial T}^2 \preccurlyeq |\,v\,|_1,$$

where the hidden constant depends only on the mesh-quality of $\mathscr{T}$.

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Using this estimate, the fact that $|\,.\,|_1 \preccurlyeq (\mathscr{B}_\phi[.,.])^{1/2}$, other standard inequalities, and setting $v = e$, we finally obtain

$$\begin{aligned}\mathscr{B}_\phi[e,e] \quad \leq \quad & \sum_{T\in\mathscr{T}} h_T^2\,||\,\Delta U_\phi + f\,||_{0,T}^2 \\ & + \sum_{T\in\mathscr{T}} h_T\,||J(D_n U_\phi)\,||_{0,\partial T\cap\mathscr{R}}^2 \\ & + \sum_{T\in\mathscr{T}} h_T^{-k+1}\,||\,U_\phi\,||_{0,\partial T\cap\partial\Omega}^2.\end{aligned}$$

For each $T \in \mathscr{T}$, we define the local estimator by

$$\begin{aligned}\eta_{\mathscr{T}}^2(U_\phi,T) \quad : \quad & = [h_T^2\,||\,\Delta U_\phi + f\,||_{0,T}^2 + h_T\,||J(D_n U_\phi)\,||_{0,\partial T\cap\mathscr{R}}^2 \\ & + h_T^{-k+1}\,||\,U_\phi\,||_{0,\partial T\cap\partial\Omega}^2],\end{aligned} \tag{27}$$

and the total error estimator by

$$\eta_{\mathscr{T}}(U_\phi) := \left(\sum_{T\in\mathscr{T}} \eta_{\mathscr{T}}^2(U_\phi,T)\right)^{1/2}. \tag{28}$$

We have proved then:

Theorem 16 *There holds*

$$\mathscr{B}_\phi[u_{\mathscr{L}} - U_\phi, u_{\mathscr{L}} - U_\phi] \preccurlyeq \eta_{\mathscr{T}}^2(U_\phi).$$

The hidden constant depends only on the mesh-quality and several constants appearing in the equivalence of norms.

Adaptivity methods are now widely used in numerical simulation of partial differential equations to achieve better accuracy with minimum degrees of freedom. Recently, several convergence and optimality results have been obtained for adaptive FEM on elliptic PDEs, which justify the advantage of local refinement over uniform refinement of the triangulations.

Given an initial mesh $\mathscr{T}_0$ with associated finite element space $\mathbb{V}_0$, a typical loop of adaptivity for stationary problems through local refinement involves

$$\textbf{SOLVE} \rightarrow \textbf{ESTIMATE} \rightarrow \textbf{MARK} \rightarrow \textbf{REFINE},$$

and produces a sequence of successive meshes $\mathscr{T}_0 \geq ... \geq \mathscr{T}_n \geq \mathscr{T}_{n-1} \geq ...$ and finite elements spaces $\mathbb{V}_n := \mathbb{V}_{\mathscr{T}_n}$.

Module SOLVE:

Given β, k fixed, for $n \geq 1$, we apply the stabilized penalty method to solve

$$\mathscr{B}_{\phi_n}[U_{\phi_n}, V] = \langle f, V\rangle, \qquad \forall V \in \mathbb{V}_n,$$

where

$$\phi_n = \beta^{-2} h_{\partial,n}^{-k}.$$

Module ESTIMATE:

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We calculate the error estimator

$$\eta_{\mathscr{T}_n}(U_{\phi_n}).$$

Module MARK:

Elements where the local error indicator (27) are high must be selected for refinement. A standard methodology relies on Dörfler marking: given a parameter $\theta \in (0,1]$, the set of marked elements $\mathscr{M}_n \subset \mathscr{T}_n$ satisfies the property

$$\eta_{\mathscr{T}_n}(U_{\phi_n},\mathscr{M}_n) \geq \theta\, \eta_{\mathscr{T}_n}(U_{\phi_n}).$$

In general, we assume:

Condition 17 *$\mathscr{M}_n$ contains at least an element T^* where $\eta_{\mathscr{T}_n}(U_{\phi_n},T^*) = \max_{T\in\mathscr{T}_n}\{\eta_{\mathscr{T}_n}(U_{\phi_n},T)\}$.*

Module REFINE:

We suppose that a function REFINE is at our disposal which implements iterative or recursive bisection. In our 2-dimensional numerical experiments we have used the standard newest vertex bisection.

Remark 18 *Observe that in this adaptive procedure, the penalization function ϕ adjust automatically. This is one of the differences with the method in (Eriksson et al., 2004).*

It can be shown that the sequence of functions $H_{\mathscr{T}_n}$ converge in the L^∞-norm to a function H_∞ that can be nonzero. Generally speaking, adaptivity theories rely on the issue of showing the convergence of the adaptive iteration, that is:

$$\eta_{\mathscr{T}_n}(U_{\phi_n}) \to 0,$$

instead the convergence $H_{\mathscr{T}_n} \to 0$ (see (Morin et al., 2007)).

It can be proved that there holds:

Assuming the mark condition 17, the adaptive stabilized penalty method converge. That is,

$$\eta_{\mathscr{T}_n}(U_{\phi_n}) \to 0,$$

and, as a consequence,

$$||u_{\mathscr{L}} - U_{\phi_n}||_1 \to 0.$$

The rather technical arguments of the proof of this result, as well as other studies on the rate of convergence of the adaptive process will be presented in a forthcoming paper.

7 HP-FEM

In this section we describe the *hp-fem* function spaces which will be used in our numerical experiments. Instead of classical FEM, the imposition of Dirichlet boundary conditions could be expensive to compute. Therefore, BPM or Nitsche methods are more appropriate for dealing with these problems (Zuppa et al., 2007; Zuppa, 2008).

hp-fem is a particular version of *hp*-clouds where functions $\{\mathscr{W}_i\}$ were chosen as a standard FEM partition of unity of degree m. They were introduced in (Zuppa, 2008) were we can find a treatment of more general *hp-clouds*.

We define the standard finite element space

$$\mathbb{F}_m(\mathscr{T}) := \{V \in H^1(\Omega)\,|\,V_{|T} \in \mathbb{P}_m(T), T \in \mathscr{T}\},$$

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where $m \in \mathbb{N}$ is a fixed polynomial degree and $\mathbb{P}_m$ denotes the space of all polynomials of degree $\leq m$. The pair $(\mathscr{T}, \mathbb{F}_m(\mathscr{T}))$ is naturally associated with a set of nodes $\{x_{\mathscr{T},i}\}_{i=1;\ldots N}$ and the set of canonical shape-functions $\{\mathscr{W}_{\mathscr{T},i}\}_{i=1;\ldots N}$ of $\mathbb{F}_m(\mathscr{T})$ such that

$$\mathscr{W}_{\mathscr{T},i}(x_{\mathscr{T},j}) = \delta_{i,j},\, i,j = 1,\ldots,N.$$

$\mathbb{F}_m(\mathscr{T})$ has algebraic precision of degree m; that is, for every $P \in \mathbb{P}_m$ we have

$$P(x) = \sum_{i=1}^{N} P(x_{\mathscr{T},i})\, \mathscr{W}_{\mathscr{T},i}(x), \quad \forall x \in \overline{\Omega}.$$

For any node $x_{\mathscr{T},i}$ we denote

$$\omega_{\mathscr{T},i} := \bigcup_{x_{\mathscr{T},i} \in T} T,$$

the support of the shape function $\mathscr{W}_{\mathscr{T},i}$.

Given an integer $r \geq 1$, we define the *hp-FEM* space

$$\mathscr{F}^{(m,r)}_{\mathscr{X},\mathscr{T}} := \left\{ V \,|\, V = \sum_{i=1}^{N} T_i[x_{\mathscr{T},i}]\, \mathscr{W}_{\mathscr{T},i} \right\},$$

where each $T_i[x_i]$ is a Taylor polynomial of degree $\leq r$

$$T_i[x_{\mathscr{T},i}](x) := \sum_{|\alpha| \leq r} c_{i,\alpha}\, (x - x_{\mathscr{T},i})^{\alpha}.$$

The space $\mathscr{F}^{(m,r)}_{\mathscr{X},\mathscr{T}}$ reproduces polynomials of degree $m+r$ in the following sense.

Theorem 19 *There exist constants $c_\alpha = c(m,r,\alpha), 1 \leq |\alpha| \leq r$, such that for every $P \in \mathbb{P}_{m+r}$*

$$\sum_{i=1}^{N} \left(P(x_i + \sum_{1 \leq |\alpha| \leq r} c_\alpha D^\alpha P(x_i)(x - x_i)^\alpha \right) \mathscr{W}_i(x) = P(x), \quad \forall x \in \overline{\Omega}.$$

Remark 20 *Constants c_α can be effectively calculated (see (Zuppa, 2008)).*

Remark 21 *We observe that the modified Taylor polynomials expanded at nodes x_i contain polynomials of low degree that are reproduced by the class $\{\mathscr{W}_i\}$. It was observed by Duarte and Oden (Duarte and Oden, 1995) that this situation produces singular stiffness matrix because shape functions in the hp cloud space are not linearly independent. To overcome this drawback, they propose an enrichment that only uses monomials of degree between $m+1$ and $m+r$. The approximation space $\mathscr{F}^{(m,r)}_{\mathscr{D}}$ built with this approach is also considered in (Zuppa et al., 2007), but its numerical behavior in Galerkin schemes is poor. That is why Duarte-Oden also recommend the use of a $0-$reproducing partition of unity and low order Taylor polynomials.*

As we have remarked above, the standard Galerkin procedure using shape functions $\{\mathscr{W}_{i,\alpha}\}$ leads to a singular stiffness matrix. The bilinear associate form is yet coercive, and the solution is univocally determined. This is a common feature of most Generalized Finite Element methods. However, the use of direct solvers like subroutines MA27 and MA47 of Harwell Subroutine Library was successful even when the nullity of the stiffness matrix was large (cf. (Strouboulis et al., 2001)). It was also shown there that round-off errors do not play a significant role in solving the linear system, i.e., the round-off error is also the same as when a finite element linear system is solved. An iterative algorithm was also given. Therefore, there exist nowadays efficient solvers to deal with singular or near singular linear systems. In our numerical experiments at low r we have used the direct solvers of LAPACK without any additional precautions.

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8 NUMERICAL EXPERIMENTS

We shall consider the adaptive process for the weak formulation of a Poisson model problem

$$\begin{array}{ll} -\Delta u = f & \text{in } \Omega = (0,1)\times(0,1), \\ u = g & \text{on } \Gamma_D = [0,1]\times\{0\}, \\ u_n = h & \text{on } \Gamma_D = \partial\Omega\backslash.\,\Gamma_D, \end{array}$$

where Ω is a bounded, polyhedral domain in $\mathbb{R}^2$, all of which present some kind of singularity.

8.1 Models

8.1.1 Model 1

This is a Poisson equation with mixed boundary conditions and high local gradient

$$\begin{array}{ll} -\Delta u = f & \text{in } \Omega = (0,1)\times(0,1), \\ u = 0 & \text{on } \Gamma_D = [0,1]\times\{0\}, \\ u_n = 0 & \text{on } \Gamma_D = \partial\Omega\backslash.\,\Gamma_D, \end{array}$$

where f is chosen such that the exact solution is

$$\overline{u}(x,y) = 5x^2(1-x)^2(e^{10x^2}-1)y^2(1-y)^2(e^{10y^2}-1).$$

8.1.2 Model 2

We consider the Dirichlet problem

$$\begin{array}{rcll} -\Delta u & = & f & \text{in } \Omega = (0,1)\times(0,1), \\ u|\partial\Omega & = & \overline{u}, & \end{array}$$

with exact solution

$$\overline{u}(x,y) = \arctan\left(50\,y\,(x-0.5)\right).$$

The boundary Dirichlet condition presents a high tangential derivative which causes oscillatory behavior in *hp-clouds* methods.

8.1.3 Model 3

This is the classical L-shape problem

$$\begin{array}{rcll} -\Delta u & = & 0 & \text{in } \Omega, \\ u|\partial\Omega & = & \overline{u}, & \end{array}$$

in the domain $\Omega = (-1,1)^2\backslash[0,1)^2$ with exact solution

$$\overline{u}(r,\theta) = r^{\frac{2}{3}}\sin(\frac{2}{3}\theta).$$

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8.1.4 Model 4. A crack problem

We consider the Dirichlet problem

$$\begin{aligned} -\Delta u &= 1 \qquad \text{in } \Omega, \\ u|\partial\Omega &= \overline{u}, \end{aligned}$$

in the domain $\Omega = \{|x| + |y| < 1\} \backslash \{0 \leq x \leq 1, y = 0\}$. The exact solution is

$$\overline{u}(r,\theta) = r^{\frac{1}{2}} \sin\frac{\theta}{2} - \frac{1}{4} r^2.$$

8.2 Initial mesh

A really simple minimal mesh is given by hand. Figure 1 shows the mesh $\mathscr{T}_0$ for models 3 and 1, 2 respectively.

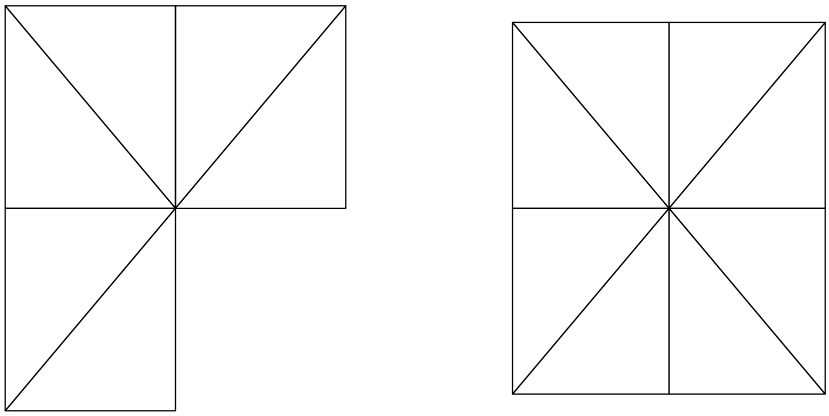

Figure 1: Initial meshes for models 3 and 1,2

8.3 Other numerical settings

- We summarize results for $\mathscr{F}^{1,1}_{\mathscr{X},\mathscr{T}}$ ($\mathscr{F}^{1,1}$), the standard FEM method with $\mathbb{F}_2$ $(fe2)$.
- A 3$-$point quadrature formula has been used at interior cells and a 2$-$point quadrature formula has been used at boundary cells.
- We shall only summarize results for the absolute energy error

  $$||\nabla e|| := ||\nabla(\overline{u} - u_{\mathscr{T}_n})||_0,$$

  calculated with a 7$-$point quadrature formula at interior cells, *vs Dof*.

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- The algebraic systems have been solved with the slash operator of MATLAB (©The MathWorks) which uses a direct LAPACK solver, without any additional precaution, since this scheme works well at low order Taylor polynomials.

- For numerical stability reasons, Taylor polynomials $T_i[x_{\mathscr{T},i}]$ are taken in the normalized form
$$T_i[x_{\mathscr{T},i}](x) := \sum_{|\alpha|\leq r} c_{i,\alpha} \left(\frac{1}{h_{\omega,i}^{|\alpha|}}\right) (x - x_{\mathscr{T},i})^{\alpha},$$
where $h_{\omega,i} := diameter(\omega_{\mathscr{T},i})$. Then, we assume that the standard shape functions are defined by
$$\mathscr{W}_{i,\alpha}(x) := \left(\frac{1}{h_{\omega,i}^{|\alpha|}}\right) (x - x_{\mathscr{T},i})^{\alpha}\, \mathscr{W}_i(x), \tag{29}$$
for $i = 1,\ldots,N$ and $0 \leq |\alpha| \leq r$.

- Finally, as we are dealing with the stabilized penalty method (SBMP), we have set $\beta = 0.01, k = 1$.

8.4 Results and conclusions of the experiments

Figures 2 and 3 summarize the convergence results for models 1-2 and models 3-4 respectively.

Several computational features are demonstrated in the figures:

- As expected, when the mesh does not capture the fine properties of the solution of singular problems, we observe the typical snaking behavior of *hp-clouds* methods: there is a bad-behavior plateau (sometimes, even worse than 1-FEM). With finer meshes , $\mathscr{F}^{1,1} + SPM$ has an optimal decay comparable to standard 2-FEM.

- SBPM handled of Dirichlet boundary conditions is very robust, the accuracy being quite insensitive to singular problems.

- SBPM and Nitsche's method, which is reputed as more efficient, give identical results (see (Zuppa et al., 2007) for a study of the same models under Nitsche's method).

Summarizing, the numerical experiments show that SPBM is highly flexible, produces accurate results, and is a very efficient adaptive method.

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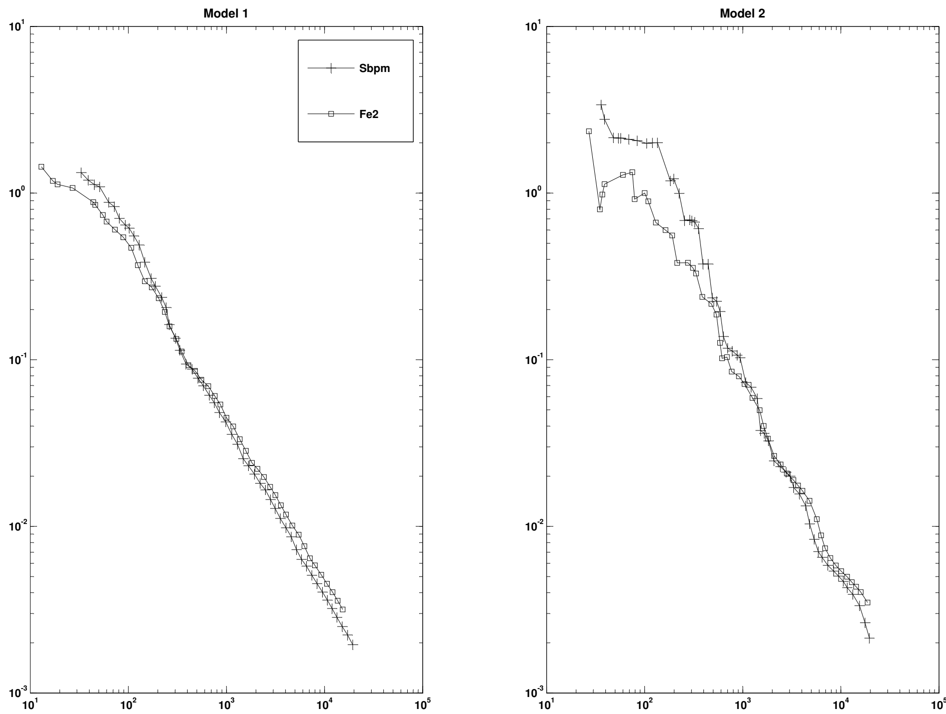

Figure 2: Energy error. Models 1 and 2

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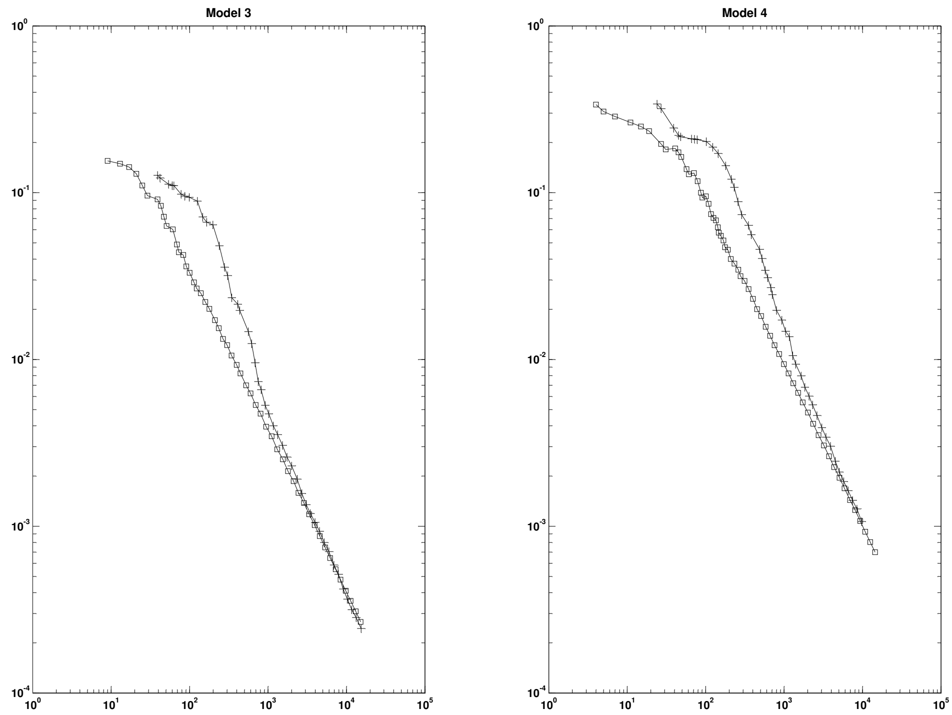

Figure 3: Energy error. Models 3 and 4

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