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NEW DEVELOPMENTS IN ELASTIC DEGRADATION AND DAMAGE: ANISOTROPIC FORMULATIONS AND EVOLUTION LAWS IN PSEUDO-LOG SPACE

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ABSTRACT

Anisotropic damage modeling still poses a number of open challanges. One of the most important is how to formulate the evolution laws in a way that is simple and makes physical sense. The theory tells us that loading function and damage rule have to be defined in the space of conjugate forces to the primary damage variable. Choosing the 2nd-order integrity tensor (or any of the usual related tensors) as such variable, the resulting conjugate force lacks physical meaning and proposing evolution laws becomes a difficult task. A (2nd-order) pseudo-log rate of damage is proposed which remedies this problem and exhibits a number of additional advantages. A first simple model is developed based on these ideas, which exhibits very promising features.

RESUMEN

La modelización del daño anisótropo todavía presenta un buen número de cuestiones abiertas. Uno de las más importantes es como formular las leyes de evolución de una forma simple y que tenga sentido físico. La teoría nos dice que la función de carga y la regla de flujo para la variable daño deben definirse en el espacio de las fuerzas termodinámicas conjugadas a la variable de daño primaria. Eligiendo como tal el tensor de integridad de segundo orden (o cualquiera de los tensores relacionados usuales), la fuerza conjugada resultante no tiene un sentido físico claro, y proponer leyes de evolución se convierte en una tarea difícil. En este artículo se define una tasa de variación cuasi-logarítmica de la variable daño, que remedia este problema y presenta diversas ventajas adicionales. A partir de este concepto se desarrolla un primer modelo constitutivo concreto que ofrece resultados muy prometedores.

INTRODUCTION

Anisotropic degradation and damage entails considerably complexity, with a number of aspects not completely solved at present [1]. In previous papers, the author and coworkers contributed with the proposal of a unified theoretical framework for elastic degradation and damage [2], with the analysis of spurious energy dissipation of stiffness recovery schemes [3], and with the study of the constitutive localization properties of scalar damage models, based on the spectral analysis of the so-called 'acoustic tensor' [4]. Most of these results were summarized in the previous MECOM conference [5].

The problem of formulating evolution laws for anisotropic damage in a simple, consistent and understandable way has been undertaken in a recent publication [6], the main results of which are summarized in the following.

THEORETICAL FRAMEWORK FOR ELASTIC DEGRADATION AND DAMAGE

A general theoretical framework to formulate elastic degradation and damage in small strains, in a way that is similar to classical elasto-plasticity was proposed in recent years [2], and only the essential equations are summarized in this section.

In the simplest setting of purely elastic degradation and damage, it is assumed that unloading always leads to the origin with some secant stiffness/compliance, and reloading follows the same path until the envelope is reached again and nonlinear behavior resumes (Fig. 1).

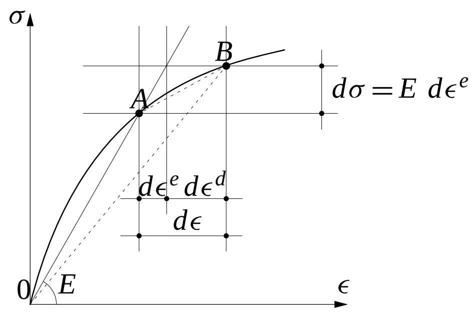

Fig. 1. Elastic-Degrading behavior and decomposition of the strain increments

Total values of stresses and strains at any time are related by the secant expressions

$$\boldsymbol{\sigma} = \mathbf{E} : \boldsymbol{\epsilon} \qquad ; \qquad \boldsymbol{\epsilon} = \mathbf{C} : \boldsymbol{\sigma} \tag{1a,b}$$

where $\mathbf{E}$ and $\mathbf{C}$ are the fourth-order stiffness and compliance tensors, assumed with major symmetry, which are inverse to each other, i.e. $\mathbf{E} : \mathbf{C} = \mathbf{C} : \mathbf{E} = \mathbf{I}_4^{\text{sym}}$ (fourth-order symmetric indentity tensor, defined as $\mathbf{I}_4^{\text{sym}} = (\mathbf{I}\otimes\mathbf{I} + \mathbf{I}\bar{\otimes}\mathbf{I})/2$ where $\mathbf{I}$=second-order indentity tensor with Cartesian components $I_{ij} = \delta_{ij}$ Kronecker delta, and products $\mathbf{A} = \mathbf{b}\underline{\otimes}\mathbf{c}$ and $\mathbf{A} = \mathbf{b}\bar{\otimes}\mathbf{c}$ correspond to the Cartesian component expressions $A_{ijkl} = b_{ik}c_{jl}$ and $A_{ijkl} = a_{il}b_{jk}$ respectively).

It is also assumed that stiffness and compliance are functions of a damage variable $\boldsymbol{\mathcal{D}}$, which may be scalar, vectorial or tensorial. The elastic energy per unit volume at any stage of the damage process, u, may be expressed as

$$u = \frac{1}{2}\,\boldsymbol{\epsilon} : \mathbf{E}(\boldsymbol{\mathcal{D}}) : \boldsymbol{\epsilon} = \frac{1}{2}\,\boldsymbol{\sigma} : \mathbf{C}(\boldsymbol{\mathcal{D}}) : \boldsymbol{\sigma} \tag{2a,b}$$

For isothermal conditions, one may differentiate to obtain the equations of incremental energy balance, dissipation $\dot{d}$, and conjugate forces $-\boldsymbol{\mathcal{Y}}$:

$$\dot{u} = \boldsymbol{\sigma} : \dot{\boldsymbol{\epsilon}} - \dot{d} \quad , \qquad \dot{d} = (-\boldsymbol{\mathcal{Y}}) \star \dot{\boldsymbol{\mathcal{D}}} \quad , \qquad -\boldsymbol{\mathcal{Y}} = \frac{\partial u}{\partial \boldsymbol{\mathcal{D}}} = \frac{1}{2}[\boldsymbol{\sigma} \otimes \boldsymbol{\sigma}] :: \frac{\partial \mathbf{C}}{\partial \boldsymbol{\mathcal{D}}} \tag{3a,b,c,d}$$

where the symbol $\star$ means full contraction of all indices of the damage variable.

The conjugate forces $-\boldsymbol{\mathcal{Y}}$ constitute the space in which the loading surface $F(-\boldsymbol{\mathcal{Y}}, \mathbf{p}) = 0$ and the ‘flow rule’ for damage (or *damage rule*) $\boldsymbol{\mathcal{M}}$ must be defined, in order to achieve a fully consistent elastic-damage formulation. In this space, the damage evolution and the normal to the surface are defined as:

$$\dot{\boldsymbol{\mathcal{D}}} = \dot{\lambda}\, \boldsymbol{\mathcal{M}} \qquad , \qquad \boldsymbol{\mathcal{N}} = \left.\frac{\partial F}{\partial(-\boldsymbol{\mathcal{Y}})}\right|_{\lambda} \tag{4a,b}$$

In the theoretical framework proposed [2], these thermodynamic concepts are perfectly compatible with plasticity-like concepts and expressions in stress (or strain) space, which make the formulation more intuitive. A first step is to consider the intermediate (fourth-order) space of forces $-\mathbf{Y}$ conjugate to the compliance rate $\dot{\mathbf{C}}$, in which we rephrase the dissipation, loading function $F(-\mathbf{Y},\mathbf{p})$, and define fourth-order *compliance rule* $\mathbf{M}$ and normal to the surface $\mathbf{N}$:

$$\dot{d} = \frac{1}{2}\boldsymbol{\sigma}:\dot{\mathbf{C}}:\boldsymbol{\sigma} = (-\mathbf{Y})::\dot{\mathbf{C}} \quad ; \quad -\mathbf{Y} = \frac{\partial u}{\partial \mathbf{C}} = \frac{1}{2}\boldsymbol{\sigma}\otimes\boldsymbol{\sigma} = (-\boldsymbol{\mathcal{Y}})\star\frac{\partial \mathbf{C}}{\partial \boldsymbol{\mathcal{D}}} \quad ; \quad \mathbf{N} = \left.\frac{\partial F}{\partial(-\mathbf{Y})}\right|_{\lambda} = \boldsymbol{\mathcal{N}}\star\frac{\partial \mathbf{C}}{\partial \boldsymbol{\mathcal{D}}}$$

$$\dot{\mathbf{C}} = \dot{\lambda}\,\mathbf{M} \quad ; \quad \mathbf{M} = \boldsymbol{\mathcal{M}}\star\frac{\partial \mathbf{C}}{\partial \boldsymbol{\mathcal{D}}} \tag{5a-i}$$

These quantities may be finally related to the usual stress-space in which loading function is given as $F(\boldsymbol{\sigma},\mathbf{p})$, with normal to the surface $\mathbf{n}$, degradaing strains $\dot{\boldsymbol{\epsilon}}^d$ (see Fig. 1) and flow rule $\mathbf{m}$:

$$\mathbf{n} = \left.\frac{\partial F}{\partial \boldsymbol{\sigma}}\right|_{\lambda} = \mathbf{N}:\boldsymbol{\sigma}$$
$$\dot{\boldsymbol{\epsilon}}^d = \dot{\lambda}\,\mathbf{m} \quad ; \quad \mathbf{m} = \mathbf{M}:\boldsymbol{\sigma} \tag{6a-d}$$

From these concepts, one may consider rate equations identical to the ones traditional in elasto-plasticity, and obtain the well known expressions of the inelastic multiplier $\dot{\lambda}$ and tangential stiffness in which the only difference is the secant stiffness $\mathbf{E}$ instead of the initial one

$$\dot{\lambda} = \frac{1}{\bar{H}}\,\mathbf{n}:\mathbf{E}:\dot{\boldsymbol{\epsilon}} \tag{7}$$

$$\dot{\boldsymbol{\sigma}} = \mathbf{E}^{\tan}:\dot{\boldsymbol{\epsilon}} \quad ; \quad \mathbf{E}^{\tan} = \mathbf{E} - \frac{1}{\bar{H}}\,\mathbf{E}:\mathbf{m}\otimes\mathbf{n}:\mathbf{E} \tag{8a,b}$$

and the hardening parameters in strain and stress space are defined in the usual way:

$$\bar{H} = \left.\frac{\partial F}{\partial \lambda}\right|_{\epsilon} = H + \mathbf{n}:\mathbf{E}:\mathbf{m} \quad , \quad H = \left.\frac{\partial F}{\partial \lambda}\right|_{\sigma} \tag{9a,b}$$

Similar to plasticity, F and $\mathbf{m}$ are restricted in such a way that the denominator $\bar{H} = H + \mathbf{n}:\mathbf{E}:\mathbf{m}$ remains always positive. The model is called associated *in the stress space* (traditional definition) when $\mathbf{m}$ is proportional to $\mathbf{n}$ and consequently the tangent stiffness exhibits major symmetry. If $\mathbf{m}$ is derived from a potential Q, associativity may be alternatively stated as $Q = F$. Other definitions of associativity may be established at compliance level if $\mathbf{M}$ is parallel to $\mathbf{N}$, which implies the former, or in damage space if $\boldsymbol{\mathcal{M}}$ is parallel to $\boldsymbol{\mathcal{N}}$, which implies all of them. The latter may also be called full associativity [2].

BASIC ISOTROPIC DAMAGE

Using the theoretical framework described in Sec. 2, it is possible to formulate a variety of damage models depending mainly on the nature and choice of damage variables $\boldsymbol{\mathcal{D}}$, and the dependency of stiffness or compliance on those variables, $\mathbf{E} = \mathbf{E}(\boldsymbol{\mathcal{D}})$ or $\mathbf{C} = \mathbf{C}(\boldsymbol{\mathcal{D}})$. The simplest models are those in which the initial stiffness (and therefore also the compliance) is isotropic, and its degraded counterpart also maintains isotropy. In particular, the traditional "$(1-D)$" scalar damage model is that one in which all the components of the stiffness tensor are reduced with the same coefficient $(1-D)$, where D is a damage variable varying from 0 to 1. In [2], a strain-based formulation of this type was derived in the general framework presented, and it was shown that a number of models available in the literature [7, 8, 9] were included as particular cases. Here, the same formulation is rewritten in stress space, with more convenient choices of inelastic multiplier and damage variable which makes expressions look simpler and allows us to introduce the concept of logarithmic scalar damage.

First, consider the general form of the isotropic stiffness and compliance tensors:

$$\mathbf{E} = \Lambda\,\mathbf{I}\otimes\mathbf{I} + G\big(\mathbf{I}\overline{\otimes}\mathbf{I} + \mathbf{I}\underline{\overline{\otimes}}\mathbf{I}\big) \quad , \quad \mathbf{C} = \frac{-\nu}{E}\mathbf{I}\otimes\mathbf{I} + \frac{1+\nu}{2E}\big(\mathbf{I}\overline{\otimes}\mathbf{I} + \mathbf{I}\underline{\overline{\otimes}}\mathbf{I}\big) \tag{10a,b}$$

where Λ and G are the Lamé constants, linked to the Young modulus E and Poisson ratio ν by the classical relations

$$\Lambda = \frac{\nu E}{(1+\nu)(1-2\nu)} \quad , \qquad G = \frac{E}{2(1+\nu)} \tag{11a,b}$$

In the "$(1-D)$" scalar damage model, the following well know expressions are assumed for the secant stiffness and its inverse compliance:

$$\mathbf{E} = (1-D)\,\mathbf{E}^0 \qquad ; \qquad \mathbf{C} = \frac{1}{1-D}\,\mathbf{C}^0 \tag{12a,b}$$

where $\mathbf{E}^0$ and $\mathbf{C}^0$ are the initial stiffness and compliance tensors given by (10a,b) with initial values of elastic constants Λ^0, G^0 or E^0, ν^0. i.e.

$$\mathbf{E}^0 = \Lambda^0\,\mathbf{I}\otimes\mathbf{I} + G^0\big(\mathbf{I}\underline{\otimes}\mathbf{I}+\mathbf{I}\bar{\otimes}\mathbf{I}\big) \quad , \quad \mathbf{C}^0 = \frac{-\nu^0}{E^0}\mathbf{I}\otimes\mathbf{I} + \frac{1+\nu^0}{2E^0}\big(\mathbf{I}\underline{\otimes}\mathbf{I}+\mathbf{I}\bar{\otimes}\mathbf{I}\big) \tag{13a,b}$$

Differentiating (12b) yields

$$\dot{\mathbf{C}} = \frac{\dot{D}}{(1-D)^2}\,\mathbf{C}^0 \tag{14}$$

A new logarithmic scalar damage variable L is defined as the primary damage variable $\boldsymbol{\mathcal{D}}$

$$\boldsymbol{\mathcal{D}} = \text{scalar} = L = \ln\frac{1}{1-D} \qquad ; \qquad D = 1 - e^{-L} \tag{15a,b,c}$$

While the conventional damage variable D varies between 0 and 1, the logarithmic damage L varies between 0 and ∞. Having introduced L, we can rewrite (12) as

$$\mathbf{E} = e^{-L}\,\mathbf{E}^0 \qquad ; \qquad \mathbf{C} = e^{L}\,\mathbf{C}^0 \tag{16a,b}$$

This expression of $\mathbf{C}$ may be differentiated to obtain an alternative form to (14):

$$\dot{\mathbf{C}} = \dot{L}\,e^{L}\mathbf{C}^0 = \dot{L}\,\mathbf{C} \tag{17a,b}$$

The partial derivative $\partial\mathbf{C}/\partial\boldsymbol{\mathcal{D}}$ may be immediately calculated, and $\dot{L}$ itself may be used as the inelastic multiplier of the formulation:

$$\frac{\partial\mathbf{C}}{\partial\boldsymbol{\mathcal{D}}} = \frac{\partial\mathbf{C}}{\partial L} = \mathbf{C} \qquad ; \qquad \dot{\lambda} = \dot{L} = \frac{\dot{D}}{1-D} \tag{18a,b,c,d}$$

This leads to the identification of the "m" terms of the general theory, which take the convenient simple form of the current value of compliance and strain:

$$\boldsymbol{\mathcal{M}} = scalar = 1 \quad ; \quad \mathbf{M} = \mathbf{C} \quad ; \quad \mathbf{m} = \mathbf{C} : \boldsymbol{\sigma} = \boldsymbol{\epsilon} \tag{19a-d}$$

The dissipation equation leads to the force $-\boldsymbol{\mathcal{Y}}$=scalar=$-\mathcal{Y}$, conjugate to the logarithmic damage L, which turns out to be equal to the current (secant) elastic energy:

$$\dot{d} = \frac{1}{2}\,\boldsymbol{\sigma} : \mathbf{C} : \boldsymbol{\sigma}\,\dot{L} = (-\mathcal{Y})\,\dot{L} \quad ; \quad -\mathcal{Y} = \frac{1}{2}\,\boldsymbol{\sigma}:\mathbf{C}:\boldsymbol{\sigma} = u \tag{20a,b,c,d}$$

In order to achieve an associated formulation, the loading surface is written in terms of the conjugate force $-\mathcal{Y} = u$ and the damage state L (equivalent to D), in the format

$$F = u - r(L) = 0 \tag{21}$$

This actually represents a general form of $F(u,L)$ since, from any other expression $F'(u,L)=0$, one can always isolate $u=r(L)$ and rewrite as above (in particular, this definition includes other functions usually found in the literature such as those written in terms of the strain- or stress-based undamaged energies $\bar{u}^0=\boldsymbol{\epsilon}:\mathbf{E}^0:\boldsymbol{\epsilon}/2=u/(1-D)$, or $u^0=\boldsymbol{\sigma}:\mathbf{C}^0:\boldsymbol{\sigma}/2=(1-D)u$).

From F, the various gradients of the loading function at constant damage may be obtained:

$$\boldsymbol{\mathcal{N}} = scalar = \frac{\partial F}{\partial(-\mathcal{Y})} = 1 \quad ; \quad \mathbf{N} = \frac{\partial F}{\partial(-\mathbf{Y})} = \mathbf{C} \quad ; \quad \mathbf{n} = \frac{\partial F}{\partial \boldsymbol{\sigma}} = \mathbf{C} : \boldsymbol{\sigma} = \boldsymbol{\epsilon} \tag{22a,b,c}$$

Note that the three gradients $\mathcal{N}$, $\mathbf{N}$ and $\mathbf{n}$ are equal to the corresponding rules $\mathcal{M}$, $\mathbf{M}$ and $\mathbf{m}$ in the theory, which means associativity at all levels. In general, associativity depends on the particular choice of F such that its gradients are parallel to the damage rule. In the case of scalar damage, however, because both damage rule and gradient of F are scalars, it is sufficient that $\mathcal{N}$ exists and it will automatically be parallel to $\mathcal{M}$. Therefore, the only condition for full associativity is that F be expressed in terms of the conjugate force, i.e. in this case, of u (a more detailed discussion on the various levels of associativity in damage models and related considerations may be found in [2]).

The hardening/softening modulus $H=-\partial F/\partial\lambda$ at constant stress, is also obtained from (21) as

$$H = \frac{\partial r}{\partial L} - u \tag{23}$$

Finally, with $\mathbf{m}$, $\mathbf{n}$ and H, the expression for the tangent stiffness is obtained:

$$\mathbf{E}^{\tan} = e^{-L}\,\mathbf{E}^0 - \frac{1}{H}\,\boldsymbol{\sigma}\otimes\boldsymbol{\sigma} \qquad ; \qquad \bar{H} = \frac{\partial r}{\partial L} + u \tag{24a,b}$$

As described, the model has only the hardening/softening function $r(L)$ (or, equivalently, $r(D)$) to be defined. This function may be identified from a single stress-strain curve from experiments, for instance from a uniaxial test. Once it has been chosen, however, all other features of the model are automatically fixed.

If further degrees of freedom are needed in the model in order to fit additional experimental data without abandoning the domain of isotropic degradation, the model would have to be modified. In order to focus on the main objective of evolution laws based on the pseudo-log damage rate, this will not be pursued in this paper. However, a simple extension along this line has already been advanced in [10] and is developed in more detail and inserted in the general context of an ‘extended’ anisotropic degradation in [11].

ANISOTROPIC SECANT STIFFNESS USING SECOND ORDER DAMAGE TENSORS

Damage variables

Disregarding vectors due to theoretical and practical shortcomings [12, 13], a second-order symmetric tensor seems to be the simplest way to represent anisotropic damage with reasonable generality. Also, similar to strain or stress, the second-order symmetric damage tensor can be decomposed spectrally and represented graphically in a convenient way. All those advantages were recognized by several authors who proposed either the direct generalization of D to a *second-order damage tensor* $\bar{\mathbf{D}}$ which varies between 0 and $\mathbf{I}$ as damage progresses [14, 15, 16], or the use of an *integrity tensor* $\bar{\boldsymbol{\phi}}=\mathbf{I}-\bar{\mathbf{D}}$ which has exactly the opposite variation [17, 18] (note that, for consistency with the general theory developed in [2, 19], we use notation with overbar for variables in strain space and without overbar for their counterparts in stress space). These two tensors share principal axes and their principal values vary between 0 and 1, and are related according to $D_{(i)}=1-\bar{\phi}_{(i)}$.

Actually, one can think of a number of second-order tensors to characterize damage, all with the same principal axes and simple relations between their principal values; the choice of which one to use is mainly a matter of convenience. Additionally to the integrity tensor $\bar{\boldsymbol{\phi}}$, here we introduce its square root $\bar{\mathbf{w}}$ (which also varies from $\mathbf{I}$ to 0) and their inverses $\boldsymbol{\phi}$ and $\mathbf{w}$ (which vary from $\mathbf{I}$ to ∞). These tensors and their principal values satisfy the following relations:

$$\bar{\boldsymbol{\phi}} = \bar{\mathbf{w}}\cdot\bar{\mathbf{w}} \quad , \quad \boldsymbol{\phi} = \mathbf{w}\cdot\mathbf{w} \quad , \quad \bar{\boldsymbol{\phi}}\cdot\boldsymbol{\phi} = \boldsymbol{\phi}\cdot\bar{\boldsymbol{\phi}} = \mathbf{I} \quad , \quad \bar{\mathbf{w}}\cdot\mathbf{w} = \mathbf{w}\cdot\bar{\mathbf{w}} = \mathbf{I} \tag{25a,b,c,d}$$

$$\bar{\phi}_{(i)} = \bar{w}_{(i)}^2 \quad , \quad \phi_{(i)} = w_{(i)}^2 \quad , \quad \bar{\phi}_{(i)} = \frac{1}{\phi_{(i)}} \quad , \quad \bar{w}_{(i)} = \frac{1}{w_{(i)}} \qquad \text{(26a,b,c,d)}$$

In the case of isotropic degradation, all these tensors reduce to the volumetric form:

$$\bar{\boldsymbol{\phi}} = \bar{\phi}\, \mathbf{I} \quad , \quad \bar{\mathbf{w}} = \bar{w}\, \mathbf{I} \quad , \quad \boldsymbol{\phi} = \phi\, \mathbf{I} \quad , \quad \mathbf{w} = w\, \mathbf{I} \qquad \text{(27a,b,c,d)}$$

Due to the *energy equivalence approach* which will be introduced next, equivalence of these variables to the scalar D used in the previous section will involve a square root

$$\bar{\phi} = \bar{w}^2 = \frac{1}{\phi} = \frac{1}{w^2} = \sqrt{1-D} \qquad \text{(28a,b,c,d)}$$

The structure of expressions (27) suggest a product-type decomposition also in the general case of anisotropic damage:

$$\bar{\boldsymbol{\phi}} = \bar{\phi}\, \bar{\boldsymbol{\psi}} \quad , \quad \bar{\mathbf{w}} = \bar{w}\, \bar{\mathbf{v}} \quad , \quad \boldsymbol{\phi} = \phi\, \boldsymbol{\psi} \quad , \quad \mathbf{w} = w\, \mathbf{v} \qquad \text{(29a,b,c,d)}$$

where the scalars ϕ, $\bar{\phi}$, w and $\bar{w}$ satisfy previous relations (28a-c), and also have the meaning of the 1/3 power of the determinant of their tensor counterparts

$$\bar{\phi} = (det\, \bar{\boldsymbol{\phi}})^{1/3} \quad , \quad \bar{w} = (det\, \bar{\mathbf{w}})^{1/3} \quad , \quad \phi = (det\, \boldsymbol{\phi})^{1/3} \quad , \quad w = (det\, \mathbf{w})^{1/3} \qquad \text{(30a,b,c,d)}$$

Tensors $\boldsymbol{\psi}$ and $\bar{\boldsymbol{\psi}}$ are isochoric (with unit determinant) and inverse to each other, and so are their square root tensors $\mathbf{v}$ and $\bar{\mathbf{v}}$:

$$\bar{\boldsymbol{\psi}} = \bar{\mathbf{v}} \cdot \bar{\mathbf{v}} \quad , \quad \boldsymbol{\psi} = \mathbf{v} \cdot \mathbf{v} \quad , \quad \bar{\boldsymbol{\psi}} \cdot \boldsymbol{\psi} = \boldsymbol{\psi} \cdot \bar{\boldsymbol{\psi}} = \mathbf{I} \quad , \quad \bar{\mathbf{v}} \cdot \mathbf{v} = \mathbf{v} \cdot \bar{\mathbf{v}} = \mathbf{I} \qquad \text{(31a,b,c,d)}$$

Note that, in all these product-type decompositions of the damage tensors, the "product-volumetric" part (determinant to power 1/3) may be interpreted as representing the isotropic part of the damage, while the isochoric part (with unit determinant) would represent its anisotropic part. This separation of effects will be important in subsequent sections.

Effective stress and strain, energy equivalence

While in isotropic degradation the effect of the scalar damage variables on the stiffness or compliance may be easily established (12), in the anisotropic case with second-order damage tensors the task becomes considerably more complicated, and it is convenient to introduce first some additional concepts. Degradation may be understood as the average effect of distributed microcracks. *Effective stress* $\boldsymbol{\sigma}^{\text{eff}}$ and *effective strain* $\boldsymbol{\epsilon}^{\text{eff}}$ are then defined as the stress and strain to which the material between microcracks is subjected. In this context, the relation between effective stress and effective strain describes the behavior of the undamaged material skeleton, which in this case is assumed to be linear elastic, i.e.

$$\boldsymbol{\sigma}^{\text{eff}} = \mathbf{E}^0 : \boldsymbol{\epsilon}^{\text{eff}} \qquad ; \qquad \boldsymbol{\epsilon}^{\text{eff}} = \mathbf{C}^0 : \boldsymbol{\sigma}^{\text{eff}} \qquad \text{(32a,b)}$$

On the other hand, damage variables must relate the effective quantities to their *nominal* or apparent counterparts, which are the ones measured externally and must satisfy equilibrium and compatibility at structural level. In the literature, the relation between nominal and effective quantities has been established mainly in three ways: strain equivalence, stress equivalence and energy equivalence. In strain equivalence [20], effective and nominal strains are assumed equal and stresses differ, while stress equivalence is the opposite. These assumptions may be interpreted microscopically in terms of parallel or series arrangements of elements which fail progressively during the degradation process. In spite of strain equivalence being the most widely used, these two approaches exhibit the significant theoretical shortcoming of producing non-symmetrical secant stiffness and compliance tensors, which introduces loss of energy conservation in the unloading-reloading regime.

In contrast, *energy equivalence* produces symmetric secant stiffness and compliance tensors. It is assumed [15] that the elastic energy stored in terms of effective quantities with undamaged stiffness and in terms of nominal quantities with secant stiffness must be the same (this actually requires the undamaged behavior to be linear elastic; see a more general derivation based on the Principle of Virtual Work [21]). As a result, neither effective strain nor effective stress coincide with their nominal counterparts. Rather, assuming that the relations are linear, they must be given by the same fourth-order "damage-effect" tensor $\bar{\boldsymbol{\alpha}}$, or its inverse $\boldsymbol{\alpha}$ (i.e. $\boldsymbol{\alpha}:\bar{\boldsymbol{\alpha}}=\bar{\boldsymbol{\alpha}}:\boldsymbol{\alpha}=\mathbf{I}_4^{\text{sym}}$) in the following reciprocal form:

$$\boldsymbol{\sigma}=\bar{\boldsymbol{\alpha}}:\boldsymbol{\sigma}^{\text{eff}} \quad , \quad \boldsymbol{\sigma}^{\text{eff}}=\boldsymbol{\alpha}:\boldsymbol{\sigma} \quad , \quad \boldsymbol{\epsilon}=\boldsymbol{\alpha}^T:\boldsymbol{\epsilon}^{\text{eff}} \quad , \quad \boldsymbol{\epsilon}^{\text{eff}}=\bar{\boldsymbol{\alpha}}^T:\boldsymbol{\epsilon} \qquad \text{(33a,b,c,d)}$$

where superscript $(.)^T$ stands for transposed in the major sense, (i.e. in Cartesian components $\alpha^T_{ijkl}=\alpha_{klij}$).

Combining equations (33) with (32), one recovers the secant relations (1a,b) where

$$\mathbf{E}=\bar{\boldsymbol{\alpha}}:\mathbf{E}^0:\bar{\boldsymbol{\alpha}}^T \quad , \quad \mathbf{C}=\boldsymbol{\alpha}^T:\mathbf{C}^0:\boldsymbol{\alpha} \qquad \text{(34a,b)}$$

Symmetrized nominal-effective relations and resulting secant stiffness/compliance

Trying to establish relation (33a) in terms of a second-order damage tensor as a direct generalization of the one-dimensional relation $\sigma=\bar{\phi}\sigma^{\text{eff}}$ where $\bar{\phi}$ is an effective area reduction, one has $\boldsymbol{\sigma}=\bar{\boldsymbol{\phi}}\cdot\boldsymbol{\sigma}^{\text{eff}}$, where symmetry cannot be ensured for $\boldsymbol{\sigma}$ even if $\boldsymbol{\sigma}^{\text{eff}}$ and $\bar{\boldsymbol{\phi}}$ are symmetric. This suggests that some form of symmetrization should be applied. Both "sum-type" and "product-type" symmetrizations have been considered in the literature [15], which lead to two different forms of the damage-effect tensor $\bar{\boldsymbol{\alpha}}$. Among the two, the product-type seems to be the most convenient due to theoretical and practical advantages, the main one being that both stiffness-based and compliance-based versions of the theory turn out to be completely equivalent, which does not happen with the sum-type [6].

This form of symmetrization leads to the following relations between nominal and effective quantities (where the symmetry of $\mathbf{w}$ and its inverse has been taken into account)

$$\boldsymbol{\sigma}^{\textbf{eff}}=\mathbf{w}\cdot\boldsymbol{\sigma}\cdot\mathbf{w} \quad , \quad \boldsymbol{\epsilon}^{\textbf{eff}}=\bar{\mathbf{w}}\cdot\boldsymbol{\epsilon}\cdot\bar{\mathbf{w}} \quad , \quad \boldsymbol{\sigma}=\bar{\mathbf{w}}\cdot\boldsymbol{\sigma}^{\textbf{eff}}\cdot\bar{\mathbf{w}} \quad , \quad \boldsymbol{\epsilon}=\mathbf{w}\cdot\boldsymbol{\epsilon}^{\textbf{eff}}\cdot\mathbf{w} \qquad \text{(35a,b,c,d)}$$

which corresponds to the damage-effect tensors

$$\bar{\boldsymbol{\alpha}}=\frac{1}{2}\,(\bar{\mathbf{w}}\otimes\bar{\mathbf{w}}+\bar{\mathbf{w}}\bar{\otimes}\bar{\mathbf{w}}) \quad ; \quad \boldsymbol{\alpha}=\frac{1}{2}\,(\mathbf{w}\otimes\mathbf{w}+\mathbf{w}\bar{\otimes}\mathbf{w}) \qquad \text{(36a,b)}$$

In a 6×6 matrix representation and selecting the reference system in the principal axes of damage, tensor $\bar{\boldsymbol{\alpha}}$ exhibits a diagonal form with diagonal components $\bar{w}_{(i)}\bar{w}_{(j)}$ (remember that $\bar{w}^2_{(i)}=\bar{\phi}_{(i)}$), which is a scheme found often in the literature on anisotropic damage [15, 22, 23, 24]:

$$\bar{\boldsymbol{\alpha}}=\begin{bmatrix} \bar{\phi}_{(1)} & & & & & \\ & \bar{\phi}_{(2)} & & & & \\ & & \bar{\phi}_{(3)} & & & \\ & & & \bar{w}_{(1)}\bar{w}_{(2)} & & \\ & & & & \bar{w}_{(2)}\bar{w}_{(3)} & \\ & & & & & \bar{w}_{(3)}\bar{w}_{(1)} \end{bmatrix} \qquad \text{(37)}$$

Replacing expressions (35) into (32), one obtains the following equations (better expressed in Cartesian component form):

$$E_{ijkl}=\bar{w}_{ip}\bar{w}_{jq}\bar{w}_{kr}\bar{w}_{ls}E^0_{pqrs} \quad ; \quad C_{ijkl}=w_{ip}w_{jq}w_{kr}w_{ls}C^0_{pqrs} \qquad \text{(38a,b)}$$

Further replacing the isotropic elastic stiffness and compliance tensors and making the appropriate products and substitutions, one finally obtains

$$\mathbf{E}=\Lambda^0\,\bar{\boldsymbol{\phi}}\otimes\bar{\boldsymbol{\phi}}+G^0\left(\bar{\boldsymbol{\phi}}\otimes\bar{\boldsymbol{\phi}}+\bar{\boldsymbol{\phi}}\,\bar{\otimes}\,\bar{\boldsymbol{\phi}}\right) \;; \quad \mathbf{C}=-\frac{\nu^0}{E^0}\,\boldsymbol{\phi}\otimes\boldsymbol{\phi}+\frac{1+\nu^0}{2E^0}\left(\boldsymbol{\phi}\otimes\boldsymbol{\phi}+\boldsymbol{\phi}\,\bar{\otimes}\,\boldsymbol{\phi}\right) \qquad \text{(39a,b)}$$

which obviously can be equally rewritten in terms of any other pair of elastic constants, obtaining in any case expressions analogous to the isotropic ones (10), in which all second-order unit tensors $\mathbf{I}$ (or Kronecker deltas δ_{ij} in Cartesian components) have been replaced by $\bar{\boldsymbol{\phi}}$ (stiffness) or $\boldsymbol{\phi}$ (compliance). Note that the expression for $\mathbf{E}$ (39a) actually corresponds to the model proposed by Valanis (1990), although in that case it was derived directly from a specific form of the elastic potential, rather than using effective stress and effective strain concepts.

It is also useful to represent the secant compliance **C** obtained, in a 6×6 matrix form, selecting as the reference system the principal axes of damage. This matrix may be compared to the traditional compliance matrix for orthotropic elasticity:

$$\mathbf{C}=\begin{bmatrix} \phi_1^2\frac{1}{E} & \phi_1\phi_2\frac{-\nu}{E} & \phi_1\phi_3\frac{-\nu}{E} & & & \\ \phi_1\phi_2\frac{-\nu}{E} & \phi_2^2\frac{1}{E} & \phi_2\phi_3\frac{-\nu}{E} & & & \\ \phi_1\phi_3\frac{-\nu}{E} & \phi_2\phi_3\frac{-\nu}{E} & \phi_3^2\frac{1}{E} & & & \\ & & & \phi_1\phi_2\frac{1+\nu}{E} & & \\ & & & & \phi_2\phi_3\frac{1+\nu}{E} & \\ & & & & & \phi_1\phi_3\frac{1+\nu}{E} \end{bmatrix} \ , \quad \mathbf{C}^{\mathrm{orth}}=\begin{bmatrix} \frac{1}{E_1} & \frac{-\nu_{12}}{E_2} & \frac{-\nu_{13}}{E_3} & & & \\ \frac{-\nu_{21}}{E_1} & \frac{1}{E_2} & \frac{-\nu_{23}}{E_3} & & & \\ \frac{-\nu_{31}}{E_1} & \frac{-\nu_{32}}{E_2} & \frac{1}{E_3} & & & \\ & & & \frac{1}{G_{12}} & & \\ & & & & \frac{1}{G_{23}} & \\ & & & & & \frac{1}{G_{31}} \end{bmatrix} \qquad \text{(40a,b)}$$

obtaining the following equivalences:

$$E_1=\bar{\phi}_1^2 E \ , \quad E_2=\bar{\phi}_2^2 E \ , \quad E_3=\bar{\phi}_3^2 E \ , \quad G_{12}=\bar{\phi}_1\bar{\phi}_2\frac{E}{2(1+\nu)} \ , \quad G_{23}=\bar{\phi}_2\bar{\phi}_3\frac{E}{2(1+\nu)} \ , \quad G_{31}=\bar{\phi}_3\bar{\phi}_1\frac{E}{2(1+\nu)}$$
$$\nu_{12}=\frac{\bar{\phi}_1}{\bar{\phi}_2}\nu \ , \quad \nu_{13}=\frac{\bar{\phi}_1}{\bar{\phi}_3}\nu \ , \quad \nu_{21}=\frac{\bar{\phi}_2}{\bar{\phi}_1}\nu \ , \quad \nu_{23}=\frac{\bar{\phi}_2}{\bar{\phi}_3}\nu \ , \quad \nu_{31}=\frac{\bar{\phi}_3}{\bar{\phi}_1}\nu \ , \quad \nu_{32}=\frac{\bar{\phi}_3}{\bar{\phi}_2}\nu \qquad \text{(41a-l)}$$

In these relations, the 9 independent orthotropic elastic constants (the 12 in previous equations are subject to the three symmetry constraints $\nu_{12}/E_2=\nu_{21}/E_1$, etc.), are generated from 5 independent parameters: E, ν plus the three principal values of damage ϕ_i. Therefore, this secant stiffness corresponds only to a restricted form of orthotropy, which will lead to what we call *basic formulation* of anisotropic damage. For the particular case of isotropic damage, the 'basic' formulation collapses into the "$1-D$" model, which is also known to be a restricted form of isotropic damage in which only E degrades while ν remains constant.

Nevertheless, in spite of its limited character, the 'basic' formulation of anisotropic damage seems the most appropriate framework to introduce the concepts of pseudo-log rate of damage and related evolution laws, as done in the following sections. A more general respresentation of anisotropy requires to consider additional independent damage variables or parameters. A first step along this line is the 'extended' anisotropic formulation proposed recently [11], in which the number of independent parameters is increased to 6, encompassing new types of degradation such as purely deviatoric or von-Mises damage.

PSEUDO-LOG RATE OF DAMAGE AND CONJUGATE FORCE

Next now is to select what damage variable will play the role of $\boldsymbol{\mathcal{D}}$ in the general theory of Sect. 2, and to calculate the corresponding conjugate force $-\boldsymbol{\mathcal{Y}}$. As the first step in this process, **C** given in (39b) is differentiated, which is better expressed in Cartesian components:

$$\dot{C}_{ijkl}=\frac{\partial C_{ijkl}}{\partial \phi_{pq}}\dot{\phi}_{pq} \quad ; \qquad \frac{\partial C_{ijkl}}{\partial \phi_{pq}}=\frac{-\nu^0}{2E^0}\Big[(\delta_{ip}\delta_{jq}+\delta_{iq}\delta_{jp})\phi_{kl}+\phi_{ij}(\delta_{kp}\delta_{lq}+\delta_{kq}\delta_{lp})\Big]+$$
$$+\frac{1+\nu^0}{4E^0}\Big[(\delta_{ip}\delta_{kq}+\delta_{iq}\delta_{kp})\phi_{jl}+\phi_{ik}(\delta_{jp}\delta_{lq}+\delta_{jq}\delta_{lp})+(\delta_{ip}\delta_{lq}+\delta_{iq}\delta_{lp})\phi_{jk}+\phi_{il}(\delta_{jp}\delta_{kq}+\delta_{jq}\delta_{kp})\Big] \qquad \text{(42a,b)}$$

Replacing now this expression into (5a), the dissipation rate $\dot{d}$ is obtained as:

$$\dot{d}=\left[\frac{-\nu^0}{E^0}\,(\boldsymbol{\sigma}:\boldsymbol{\phi})\,\boldsymbol{\sigma}+\frac{1+\nu^0}{E^0}\,\boldsymbol{\sigma}\cdot\boldsymbol{\phi}\cdot\boldsymbol{\sigma}\right]:\dot{\boldsymbol{\phi}} \qquad (43)$$

If the inverse integrity tensor $\boldsymbol{\phi}$ itself is taken as the primary damage variable, the term between brackets may be immediately identified as the corresponding conjugate force. This force, analogous to what was obtained in [9] in terms of stiffness and strains, has no clear physical meaning, which makes it difficult to propose and interpret loading functions and damage rules. From (43), however, one can realize that $\boldsymbol{\sigma} : \boldsymbol{\phi} = tr\left(\boldsymbol{\sigma}^{\text{eff}}\right)$, which motivates the search for an expression of the conjugate force in terms of effective quantities exclusively. This is possible and conveniently achieved by changing the damage variable involved in the dissipation equation (43), from the rate of the inverse integrity, $\dot{\boldsymbol{\phi}}$, to the rate of a *pseudo-logarithmic damage* tensor, $\dot{\mathbf{L}}$, defined as:

$$\dot{\mathbf{L}} = -2\,\mathbf{w} \cdot \dot{\bar{\boldsymbol{\phi}}} \cdot \mathbf{w} = 2\,\bar{\mathbf{w}} \cdot \dot{\boldsymbol{\phi}} \cdot \bar{\mathbf{w}} \qquad \text{or} \qquad \dot{\boldsymbol{\phi}} = \frac{1}{2}\,\mathbf{w} \cdot \dot{\mathbf{L}} \cdot \mathbf{w} \quad , \quad \dot{\bar{\boldsymbol{\phi}}} = -\frac{1}{2}\,\bar{\mathbf{w}} \cdot \dot{\mathbf{L}} \cdot \bar{\mathbf{w}} \tag{44a,b,c}$$

If the principal axes of damage remain constant, the new tensor coincides with the logarithm of the square inverse integrity tensor, i.e. $\mathbf{L} = \ln \boldsymbol{\phi}^{\mathbf{2}}$ (logarithm of a tensor defined as a tensor function, i.e. with same principal axes and logarithm of the principal values). Otherwise, this equivalence is not valid for $\mathbf{L}$, although it is for its volumetric part $L = tr(\mathbf{L})/3$. The lack of a general relation between total values of $\mathbf{L}$ and $\boldsymbol{\phi}$ does not actually represent a practical difficulty because the pseudo-log damage is only used in rate form due to its properties of exhibiting a convenient conjugate force. Once the damage rule in terms of $\dot{\mathbf{L}}$ is established, the rate $\dot{\boldsymbol{\phi}}$ may be always evaluated with (44b) and the integration process needed in the numerical implementation of the model can be always carried out directly in terms of $\boldsymbol{\phi}$, which is the variable that enters directly the expressions of secant compliance or stiffness.

If (44b) is replaced into (43), and the two factors $\mathbf{w}$ go into the brackets, we obtain

$$\dot{d} = \left[\frac{-\nu^0}{E^0} (tr\,\boldsymbol{\sigma}^{\text{eff}}) \boldsymbol{\sigma}^{\text{eff}} + \frac{1+\nu^0}{E^0} \boldsymbol{\sigma}^{\text{eff}} \cdot \boldsymbol{\sigma}^{\text{eff}} \right] : \dot{\mathbf{L}} \tag{45}$$

The term between brackets may be indentified with the new conjugate force, which, using linear elastic relations (32) and (13), may be rewritten in the simple form:

$$-\boldsymbol{\mathcal{Y}} = \frac{1}{2}\, \boldsymbol{\sigma}^{\mathbf{eff}} \cdot \boldsymbol{\epsilon}^{\mathbf{eff}} \tag{46}$$

Because $\boldsymbol{\sigma}^{\mathbf{eff}}$ and $\boldsymbol{\epsilon}^{\mathbf{eff}}$ remain coaxial, $-\boldsymbol{\mathcal{Y}}$ also shares the same principal axes. The principal values are $-\mathcal{Y}_{(i)} = \sigma^{\text{eff}}_{(i)} \epsilon^{\text{eff}}_{(i)}/2$, and the first invariant $tr(-\boldsymbol{\mathcal{Y}}) = \boldsymbol{\sigma}^{\mathbf{eff}} : \boldsymbol{\epsilon}^{\mathbf{eff}}/2 = u$ (current elastic energy).

Another property which is very important with regard to the formulation of evolution laws, is that the product-type decomposition of $\dot{\boldsymbol{\phi}}$ becomes a sum-type decomposition for $\dot{\mathbf{L}}$ [6]. Due to that, *the volumetric part of the damage rule only generates increments of isotropic damage*, while *the deviatoric part is solely responsible for anisotropic degradation*. This property makes it possible to establish restrictions on the "damage rule" for a physically admissible evolution of $\boldsymbol{\phi}$ (see next section).

LOADING FUNCTION AND DAMAGE RULE

The loading function F is defined in terms of the conjugate forces $-\boldsymbol{\mathcal{Y}}$ and of the previous history. Here we consider the following type of expression:

$$F = f(-\boldsymbol{\mathcal{Y}}) - r(history) \tag{47}$$

The simplest choice for f is in terms of the invariants of $-\boldsymbol{\mathcal{Y}}$. This actually does not contradict the anisotropic nature of the model because this conjugate force tensor involves effective stress and effective strain, and if these are replaced using (35a,b), the square root integrity tensor comes into the picture resulting in an anisotropic loading function in terms of nominal stress or strain. Thus, it makes sense to consider the space of principal values of the conjugate force $-\mathcal{Y}_{(1)}$, $-\mathcal{Y}_{(2)}$, $-\mathcal{Y}_{(3)}$. In that space, one may represent concepts such as p-axis, deviatoric planes, loading surface $F = 0$ and damage rule, analogous to what is customary in the principal stress space in the context of plasticity theory (Fig. 2).

The choice of pseudo-log damage rate and the space defined by its conjugate force bring about a number of interesting advantages. As mentioned before, it turns out that the *volumetric part* of the pseudo-log flow rule, represented in the

$-\mathcal{Y}_{(1)}, -\mathcal{Y}_{(2)}, -\mathcal{Y}_{(3)}$ space by its component parallel to the p-axis, causes only increments of *isotropic degradation*. On the other hand, the *deviatoric part* of the pseudo-log damage rule (i.e. its component on the deviatoric plane) causes only increments of *anisotropic* degradation. In this way, we have a very simple and understandable separation of effects that may be very useful for the development of specific models. For instance, it is trivial to verify that the traditional "$(1-D)$" associated scalar damage model is recovered with a loading surface parallel to the π-plane.

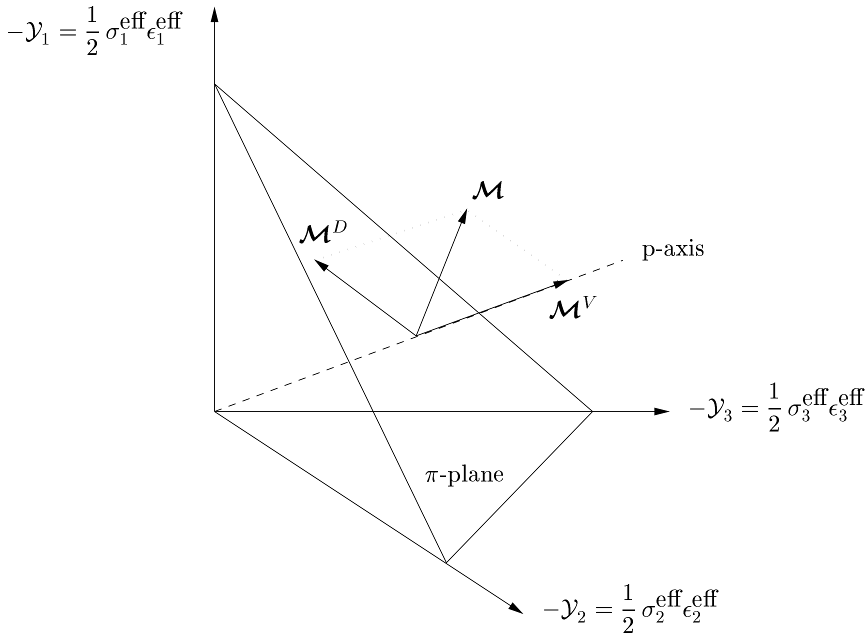

Fig. 2. Space of principal values of the conjugate force $-\mathcal{Y}_1, -\mathcal{Y}_2, -\mathcal{Y}_3$

The condition that the dissipation (45) must be always positive, leads to the conditions that the loading surface must be convex in the $-\boldsymbol{\mathcal{Y}}$ space, and must include its origin, analogous to similar arguments classical in elasto-plasticity [25, 26]. Additional constraints to the pseudo-log damage rule may be derived from its own definition as the rate of a damage measure. The inverse integrity $\boldsymbol{\phi}$ has been defined as a tensor which evolves between $\mathbf{I}$ and ∞ as damage progresses. If s_i denotes the Cartesian components of a generic unit vector ($s_i s_i = 1$), the projection $\mathbf{s}\cdot\boldsymbol{\phi}\cdot\mathbf{s}$ may be interpreted as a geometric measure of the damage on a plane with normal oriented with that direction (i.e. inverse of a stress-carrying area fraction). Due to the irreversible nature of damage (no healing is considered in this study), it seems reasonable to assume that the damage on any given plane should always increase or remain constant, but never decrease. This means that, for any orientation $\mathbf{s}$, we must have $\mathbf{s}\cdot\dot{\boldsymbol{\phi}}\cdot\mathbf{s} \geq 0$, i.e. the integrity rate $\dot{\boldsymbol{\phi}}$ must be positive semi-definite (all its eigenvalues be positive or zero). By replacing (44b) in the previous equation, one obtains:

$$\frac{1}{2}\,\mathbf{s}'\cdot\dot{\mathbf{L}}\cdot\mathbf{s}' = \frac{\dot{\lambda}}{2}\,\mathbf{s}'\cdot\boldsymbol{\mathcal{M}}\cdot\mathbf{s}' \geq 0 \quad ; \qquad \mathbf{s}' = \mathbf{w}\cdot\mathbf{s} \tag{48}$$

Because the square root integrity tensor $\mathbf{w}$ is non-singular, $\mathbf{s}'$ may also take any arbitrary orientation. Since, additionally, the inelastic multiplier $\dot{\lambda}$ must be non-negative, this means that the pseudo-log damage rule $\boldsymbol{\mathcal{M}}$ must also be positive semi-definite, i.e. that its principal values must satisfy $\mathcal{M}_{(1)} \geq 0$, $\mathcal{M}_{(2)} \geq 0$ and $\mathcal{M}_{(3)} \geq 0$. In terms of a geometric representation in Fig. 2, this implies that the vector representing the damage rule should be part of the positive-positive-positive octant, which is a severe restriction if compared with traditional flow rules in stress space. For instance, associated models with surfaces similar to von Mises or Drucker-Prager (in which the normal may have negative component on one of the axes) are not allowed here. On the other hand, a surface similar to Rankine in the $-\mathcal{Y}_{(1)}, -\mathcal{Y}_{(2)}, -\mathcal{Y}_{(3)}$ space would sit in the limit of the stated restriction, with only one positive principal value of $\boldsymbol{\mathcal{M}}$ at a time, while the other two are zero. This model, that we will call 'pseudo-Rankine', actually exhibits very appealing properties and is developed in detail and illustrated with some application examples in the next section.

EXAMPLE MODEL: GENERALIZED PSEUDO-RANKINE

For the first, simple, associated, anisotropic damage model developed in this framework, F is defined according to (47), using for f an algebraic expression taken from the literature [27], and for r an exponential decay function of the volumetric component of the logarithmic damage tensor, L:

$$f = \left(\left(-\mathcal{Y}_{(1)}\right)^{b+1} + \left(-\mathcal{Y}_{(2)}\right)^{b+1} + \left(-\mathcal{Y}_{(3)}\right)^{b+1} \right)^{\frac{1}{b+1}} \quad , \quad r = \frac{\sigma^2_{\text{peak}}}{2E^0} \exp\left(- \frac{3\sigma^2_{\text{peak}}}{2E^0 g_f} L \right) \qquad \text{(49a,b,c)}$$

The surface $F=0$ takes different shapes in $-\mathcal{Y}_{(1)}, -\mathcal{Y}_{(2)}, -\mathcal{Y}_{(3)}$ space depending on parameter b. For $b=0$, it is a π-plane and the model collapses into isotropic damage. For $b \to \infty$, the surface approaches a Rankine-type criterion and the model exhibits maximum anisotropic character. The cross-section of such a surface with the $-\mathcal{Y}_{(1)}$,$-\mathcal{Y}_{(2)}$ plane, is represented in Fig. 3 for different values of b.

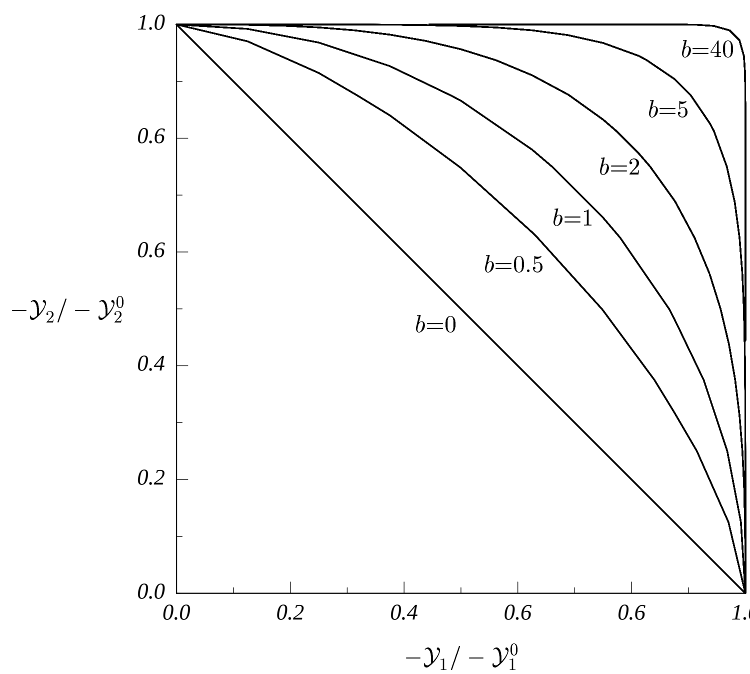

Fig. 3. Two-dimensional cross-section of the loading surface, with coordinate plane $-\mathcal{Y}_1$, $-\mathcal{Y}_2$, for various values of b.

The coefficients of the exponential resistance function (49b) are simple expressions of the tensile strength σ^{peak} and the fracture energy per unit volume g_f (area enclosed under the uniaxial stress-strain diagram). To obtain these expressions, first a generic exponential function $r = r_0 \exp(-kL)$ is assumed and then r_0 and k are identified from the analytical solution of the pure tension case (which, remarkably, is available for any value of $b \geq 0$). The uniaxial stress solution exhibits a postpeak power-law decay with exponent $-(g_f + r_0)/(g_f - r_0)$ [11].

To verify its capabilities under complex loading, the model has been implemented and used to solve Willam's test [28], which is becoming a typical benchmark for anisotropic cracking and damage formulations. This test consists of two load steps in plane stress. First, uniaxial loading is applied until the peak. Second, strain increments are applied to all in-plane degrees of freedom in the proportion $[\epsilon_{xx}, \epsilon_{yy}, \epsilon_{xy}] = [1, 1.5, 1]$. This represents increments of tensile strain in all directions, accompanied by a rotation of the principal axes which slows down progressively with a final asymptotic value of 52.02°. Fixed parameter values are $E^0 = 10^7$ MPa, $\nu^0 = 0.2$, $\sigma_{\text{peak}} = 10^4$ kPa and $g_f = 15$ kPa (i.e. three times the elastic energy at peak). The analysis is repeated for various values of parameter b. Some of the salient results for the extreme cases of $b=0$ (isotropic damage) and $b=40$ (very close to pure pseudo-Rankine, maximum anisotropy), are shown in Fig. 4.

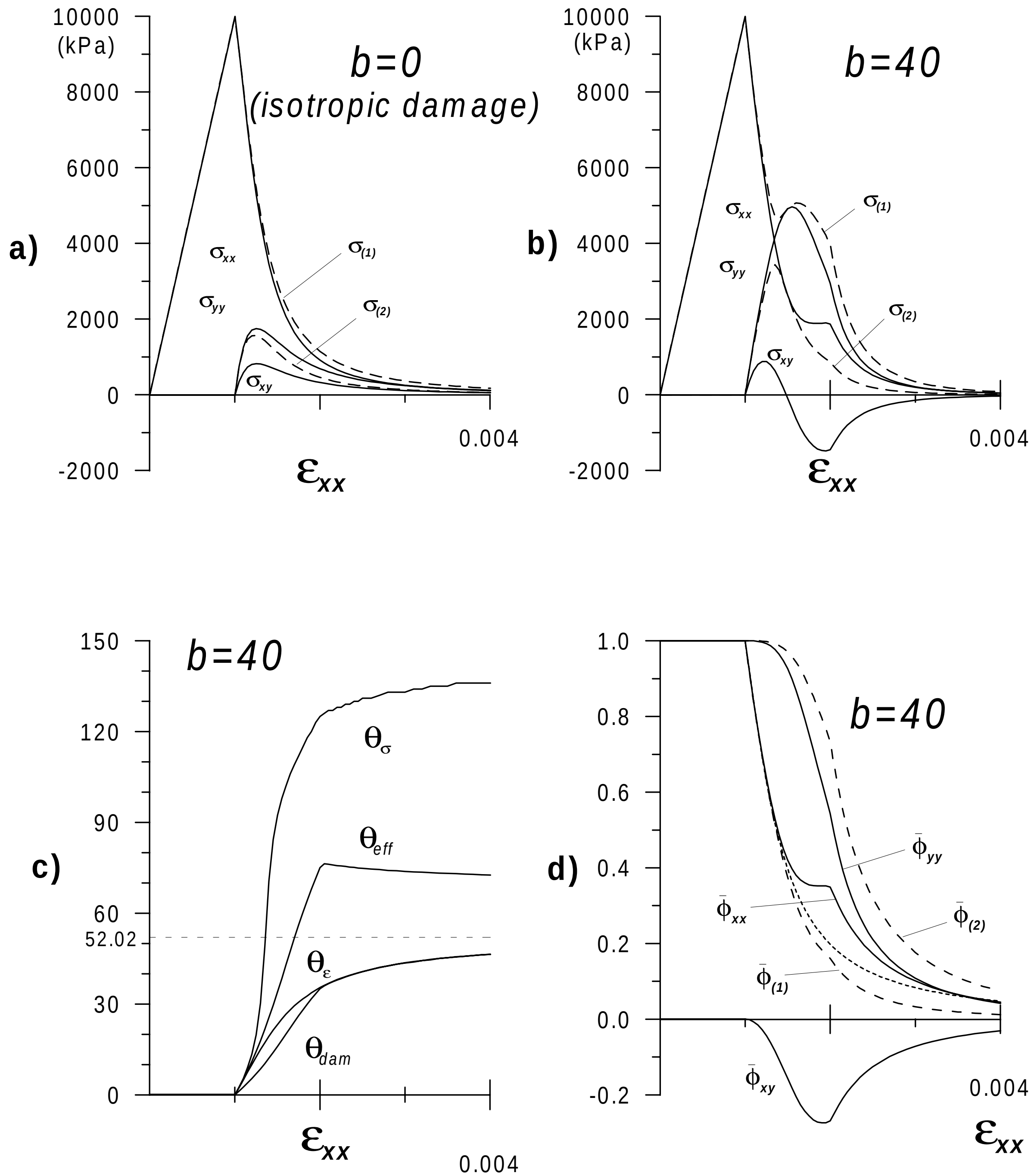

Fig. 4. Results of Willam's test. Evolution of: a) stress components for isotropic damage (b=0); b) same for highly anisotropic damage (b=40); c) angles of the first principal directions of prescribed strain, stress, damage and effective stress with x axis, for b=40; and d) damage components for b=40.

Evolution of stress components varies considerably from isotropic damage in Fig. 4a, to $b = 40$ (highly anisotropic) in Fig. 4b. Differences include a second peak of $\sigma_{(1)}$, a plateau followed by an abrupt drop in σ_{xx}, and a sign inversion of σ_{xy}. In Fig. 4.c. the angles formed by the first principal direction of prescribed strain and calculated nominal stress, effective stress and damage, are represented. The main principal direction of damage slightly underrotates the prescribed strain, while effective stress/strain, and especially nominal stress, largely overrotate it. Damage components for $b=40$ (Fig. 4d) also evolve quite differently from the isotropic case (dotted line), as one would expect for a highly anisotropic response . Additional results of these calculations may be found in [6].

CONCLUDING REMARKS

The theoretical framework for elastic degradation and damage proposed by the author some years ago has been revisited and developed further for specific types of damage variables. For isotropic degradacion, a convenient logarithmic damage variable has been introduced that leads to very simple expressions. For anisotropic damage, the format of a 'basic formulation' has been cast into the framework proposed. The new concept of pseudo-log rate of damage has been introduced, which exhibits convenient properties, leads to a simple and elegant format of conjugate forces and allows a physical interpretation of damage rules. Implementation, even in a tentative simple form, has offered meaningful results for a complex case such as Willam's test. All this configures a very promising scene for further developments and practical applications of the new formulation proposed.

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