AN EXAMPLE SHOWING SEVERE DIFFICULTIES IN THE EVALUATION OF EIGENVALUES OF SIMPLE TRIDIAGONAL NON-SYMMETRIC MATRICES

Fernando G. Basombrío
Grupo de Mecánica Computacional, TEMADI - Centro Atómico Bariloche - CNEA
C.C. 439, 8400 SC de Bariloche, Río Negro, Argentina

RESUMEN

Las matrices no normales carecen de un conjunto completo de autovectores ortogonales. En consecuencia no existe garantía para evitar el colapso de al menos algunos autovectores de la base. Para tales casos dicha base puede volverse fuertemente no ortogonal, sin control. En consecuencia, los resultados al evaluar autopares pueden ser extremadamente sensitivos a pequeñas perturbaciones originadas, por ejemplo, en errores numéricos. El propósito de esta nota es proveer de un simple ejemplo de matrices tridiagonales con valores constantes a lo largo de cada diagonal, para las cuales los espectros calculados con el paquete ARPACK (método de Arnoldi) o la subrutina EIGZF de IMSL, son completamente irrealistas. La principal conclusión es que para este tipo de problemas, ningún cálculo es posible sin una muy alta precisión. En el caso de nuestro ejemplo, el paquete debería utilizar unos 50 dígitos. Es decir, más de tres veces la doble precisión usual.

ABSTRACT

Non-normal matrices do not possess a complete set of orthogonal eigenvectors. Consequently no guaranty exists against the collapse of at least some of the eigenvectors in the basis. For such cases the eigenvectors basis may become strongly non-orthogonal, without control. Thus, the results of evaluating eigenpairs may be extremely sensitive to small perturbations originated, for example, by numerical errors. The aim of this short note is to provide a simple and detailed example of tridiagonal matrices with constant entries along each diagonal, for which the spectra calculated by ARPACK package (Arnoldi's approach) or by IMSL's routine EIGZF are completely irrealistic. For such kind of problems, no calculation is possible without a very high machine precision. This is our main conclusion. In the paticular example discussed here, the codes would need about 50 digits i.e., more than three times the usual double precision.

DESCRIPTION OF THE EXAMPLE

Peculiarities associated to non-normal matrices are well known (see [1-3], among others), specially the numerical difficulties in calculating eigenpairs. Nevertheless no sufficient attention seems to be paid to these issues, for example, by practitioners interested in solving advection-diffusion problems by finite elements or finite difference techniques. Also in the current literature such issues are rather absent. Such kind of matrices do not possess a complete set of orthogonal eigenvectors. Consequently, no guarantee exists against the collapse of at least some of the eigenvectors on the basis. This originates a deficiency of linear independence, because the eigenvectors basis may become strongly non-orthogonal, without control. Thus, the results of evaluating eigenpairs may be extremely sensitive to small perturbations originated, for example, by numerical errors. The aim of this short note is to provide a simple and detailed example of tridiagonal matrices with constant entries along each diagonal, for which the spectra calculated by ARPACK package (Arnoldi's approach) or by IMSL's routine EIGZF become completely irrealistic. We show that for such kind of problems, no calculation is possible without a very high machine precision. For the example here discussed, the packages would need to work with about 50 digits, i.e. more than three times the usual double precision.

The example comes from a discretization by linear finite elements of the one dimensional linear advection-diffusion problem,

$$u_t + c u_x - \nu u_{xx} = 0, \quad u(0,t) = u(l,t) = 0, \quad u(x,0) = u_0(x). \tag{1}$$

The full discretized problem is completely defined by two non-dimensional parameters: Courant's number $\mathcal{C} = \frac{c\Delta t}{\Delta x}$ and local Péclét number $p_e = \frac{c\Delta x}{\nu}$, where Δx and Δt are the space and time steps. Performing first the space discretization of (1), as usual we get a system of J ordinary differential equations for the J nodal unknowns $U_j(t),\ j = 1, 2, .., J$,

$$M\dot{\mathbf{U}} = A\mathbf{U} \tag{2}$$

where $\mathbf{U}(t) = \{U_1(t), .., U_J(t)\}$, M is the mass (tridiagonal) matrix and A is the advection-diffusion (tridiagonal and non-symmetric) matrix, respectively with entries ,

$$\begin{aligned} M_{j,j-1} &= \frac{1}{6}, & A_{j,j-1} &= \mathcal{C}(\frac{1}{2} + \frac{1}{p_e}) \\ M_{j,j} &= \frac{2}{3}, & A_{j,j} &= -2\frac{\mathcal{C}}{p_e} \\ M_{j,j+1} &= \frac{1}{6}, & A_{j,j+1} &= \mathcal{C}(-\frac{1}{2} + \frac{1}{p_e}) \end{aligned} \tag{3}$$

For further convinience both matrices were written in non-dimensional form, i.e., respectively multiplied by $\frac{1}{\Delta x}$ and $\frac{\Delta t}{\Delta x}$.

The eigenvalues z_s obtained from the generalized eigenvalue problem,

$$Av^s = z_s M v^s, \qquad s = 1, 2, .., J \tag{4}$$

(v^s is the s-eigenvector), are of fundamental importance for the analysis of outstanding properties of the algorithm. We point out that as the matrices M and A are non dimensional, the eigenvalues $z_s = \lambda_s \Delta t$ are non-dimensional too. Here λ_s designates the s-eigenvalue of the original dimensioned problem.

The generalized eigenvalues z_s in (4) can be obtained with the exact (implicit) formula,

$$z_s = \frac{A_d + 2A_u\sqrt{\frac{A_l - z_s M_l}{A_u - z_s M_u}}cos(\frac{J+1-s}{J+1}\pi)}{M_d + 2M_u\sqrt{\frac{A_l - z_s M_l}{A_u - z_s M_u}}cos(\frac{J+1-s}{J+1}\pi)} \tag{5}$$

where $A_l,\ A_d,\ A_u$ and $M_l,\ M_d,\ M_u$ are respectively the lower diagonal, diagonal and upper diagonal entries of the matrices A and M. The component j of the corresponding (normalized) eigenvector v^s is given by the expression,

$$v_j^s = \left(\frac{A_l - z_s M_l}{A_u - z_s M_u}\right)^{\frac{j-J}{2}} \frac{sin(\frac{J+1-s}{J+1}j\pi)}{sin(\frac{J+1-s}{J+1}J\pi)} \tag{6}$$

These formulae are obtained by simply adapting the usual procedure applied to current eigenvalue problems for tridiagonal matrices (see for example, [4] pp. 113-115).

RESULTS

Our example deals with the particular case for which the two involved parameters take the (prudent) values $\mathcal{C} = 3$ and $p_e = \frac{19}{10}$. Of course, for near values one arrives to similar results. The order of the matrices is $J = 89$. The exact entries of A and M are given by (3). We get,

$$A_l = \frac{117}{38}, \qquad A_d = -\frac{60}{19}, \qquad A_u = \frac{3}{38}. \tag{7}$$

Using expressions (5) and (6) we obtain the exact (real) eigenvalues, shown in TABLE I.

TABLE I: Exact eigenvalues z_s obtained with expression (5)
(Truncated)

-17.3787.,-17.3296.,-17.2483.,-17.1354.,-16.9919.,-16.8191.,-16.6185.,-16.3917.,-16.1404.,
-15.8666.,-15.5725.,-15.2600.,-14.9314.,-14.5887.,-14.2341.,-13.8697.,-13.4976.,-13.1196.,
-12.7375.,-12.3532.,-11.9682.,-11.5840.,-11.2020.,-10.8235.,-10.4494.,-10.0808., -9.7187..,
-9.3636..,-9.0164.., -8.6775.., -8.3474.., -8.0264..,-7.7149.., -7.4131..,-7.1211.., -6.8389..,
-6.5667..,-6.3044.., -6.0519.., -5.8092.., -5.5762..,-5.3526.., -5.1383..,-4.9331.., -4.7368..,
-4.5491..,-4.3699.., -4.1989.., -4.0357.., -3.8802..,-3.7321.., -3.5912..,-3.4572.., -3.3298..,
-3.2087..,-3.0938.., -2.9848.., -2.8815.., -2.7835..,-2.6908.., -2.6031..,-2.5202.., -2.4418..,
-2.3678..,-2.2981.., -2.2324.., -2.1706.., -2.1124..,-2.0579.., -2.0068..,-1.9589.., -1.9143..,
-1.8727..,-1.8340.., -1.7982.., -1.7651.., -1.7346..,-1.7068.., -1.6814..,-1.6585.., -1.6379..,
-1.6196..,-1.6036.., -1.5898.., -1.5782.., -1.5688..,-1.5615.., -1.5563..,-1.5531..

To solve (5), Newton's method was used. Calculations were made by means of the processor *Mathematica* [5], using an arithmetic of 50 digits. Eigenvalues in Table I (with all their digits) were calculated with absolute residuals ranging between 10^{-50} and 8.83×10^{-46}. The exact last eigenvector v^{89} is shown in Table II. It is worth observing the huge difference between their components, of more than 44 orders of magnitude. The last eigenvector offers the worst situation. The eigenvector v^{89} in Table II and the corresponding eigenvalue $z_{89} = -1.5531..$ verify equation (4) with residuals less than 10^{-50}.

TABLE II: Exact eigenvector v^{89} corresponding to $z_{89} = -1.5531..$, obtained with expression (6)
(Truncated)

$1.6959.. \cdot 10^{-44}, 1.0655.. \cdot 10^{-43}, 5.0189.. \cdot 10^{-43}, 2.1005.. \cdot 10^{-42}, 8.2381.. \cdot 10^{-42}, 3.1005.. \cdot 10^{-41},$
$1.1340.. \cdot 10^{-40}, 4.0614.. \cdot 10^{-40}, 1.4312.. \cdot 10^{-39}, 4.9794.. \cdot 10^{-39}, 1.7143.. \cdot 10^{-38}, 5.8509.. \cdot 10^{-38},$
$1.9822.. \cdot 10^{-37}, 6.6728.. \cdot 10^{-37}, 2.2338.. \cdot 10^{-36}, 7.4420.. \cdot 10^{-36}, 2.4685.. \cdot 10^{-35}, 8.1562.. \cdot 10^{-35},$
$2.6853.. \cdot 10^{-34}, 8.8130.. \cdot 10^{-34}, 2.8837.. \cdot 10^{-33}, 9.4105.. \cdot 10^{-33}, 3.0631.. \cdot 10^{-32}, 9.9472.. \cdot 10^{-32},$
$3.2231.. \cdot 10^{-31}, 1.0421.. \cdot 10^{-30}, 3.3633.. \cdot 10^{-30}, 1.0833.. \cdot 10^{-29}, 3.4835.. \cdot 10^{-29}, 1.1181.. \cdot 10^{-28},$
$3.5835.. \cdot 10^{-28}, 1.1466.. \cdot 10^{-27}, 3.6635.. \cdot 10^{-27}, 1.1687.. \cdot 10^{-26}, 3.7234.. \cdot 10^{-26}, 1.1845.. \cdot 10^{-25},$
$3.7634.. \cdot 10^{-25}, 1.1940.. \cdot 10^{-24}, 3.7838.. \cdot 10^{-24}, 1.1974.. \cdot 10^{-23}, 3.7849.. \cdot 10^{-23}, 1.1948.. \cdot 10^{-22},$
$3.7673.. \cdot 10^{-22}, 1.1863.. \cdot 10^{-21}, 3.7315.. \cdot 10^{-21}, 1.1722.. \cdot 10^{-20}, 3.6780.. \cdot 10^{-20}, 1.1525.. \cdot 10^{-19},$
$3.6075.. \cdot 10^{-19}, 1.1277.. \cdot 10^{-18}, 3.5208.. \cdot 10^{-18}, 1.0978.. \cdot 10^{-17}, 3.4187.. \cdot 10^{-17}, 1.0632.. \cdot 10^{-16},$
$3.3021.. \cdot 10^{-16}, 1.0241.. \cdot 10^{-15}, 3.1719.. \cdot 10^{-15}, 9.8096.. \cdot 10^{-15}, 3.0291.. \cdot 10^{-14}, 9.3392.. \cdot 10^{-14},$
$2.8747.. \cdot 10^{-13}, 8.8336.. \cdot 10^{-13}, 2.7096.. \cdot 10^{-12}, 8.2963.. \cdot 10^{-12}, 2.5351.. \cdot 10^{-11}, 7.7306.. \cdot 10^{-11},$
$2.3521.. \cdot 10^{-10}, 7.1399.. \cdot 10^{-10}, 2.1618.. \cdot 10^{-09}, 6.5279.. \cdot 10^{-09}, 1.9653.. \cdot 10^{-08}, 5.8981.. \cdot 10^{-08},$
$1.7638.. \cdot 10^{-07}, 5.2540.. \cdot 10^{-07}, 1.5582.. \cdot 10^{-06}, 4.5991.. \cdot 10^{-06}, 1.3499.. \cdot 10^{-05}, 3.9370.. \cdot 10^{-05},$
$1.1397.. \cdot 10^{-04}, 3.2711.. \cdot 10^{-04}, 9.2900.. \cdot 10^{-04}, 2.6047.. \cdot 10^{-03}, 7.1861.. \cdot 10^{-03}, 1.9413.. \cdot 10^{-02},$
$5.0965.. \cdot 10^{-02}, 1.2839.. \cdot 10^{-01}, 3.0313.. \cdot 10^{-01}, 6.3587.. \cdot 10^{-01}, 1.0000.. \cdot 10^{-00}$

For comparison purposes, the same problem was simultaneously solved with programs AUTOV2 and AUTOV3, respectively based on IMSL routine EIGZF and ARPACK package. Here we give details only for results obtained with the second programm. The eigenvalues are shown in Table III. Coincidences with the exact values in Table I are only acceptable for the first sixteen eigenvalues, where the number of accurate digits ranges between 6 and 3. Higher eigenvalues calculated with AUTOV3 are complex, with real parts differing from the correct ones. Fig. 1 depictes the exact spectrum together with the other two obtained numerically. It can be observed that the erroneous imaginary parts are far from being negligible. They introduce spurious oscillatory behaviours for the algorithm. It is also observed that the (false) predictions of both programs are rather similar, and suspiciously comparable to Fig. 12.1 in Ref. [6] p. 93, referred to Wilkinson's example of ill-conditioning for polynomial rootfinding. (Including the existence of separated points, as those appearing in our Fig.1

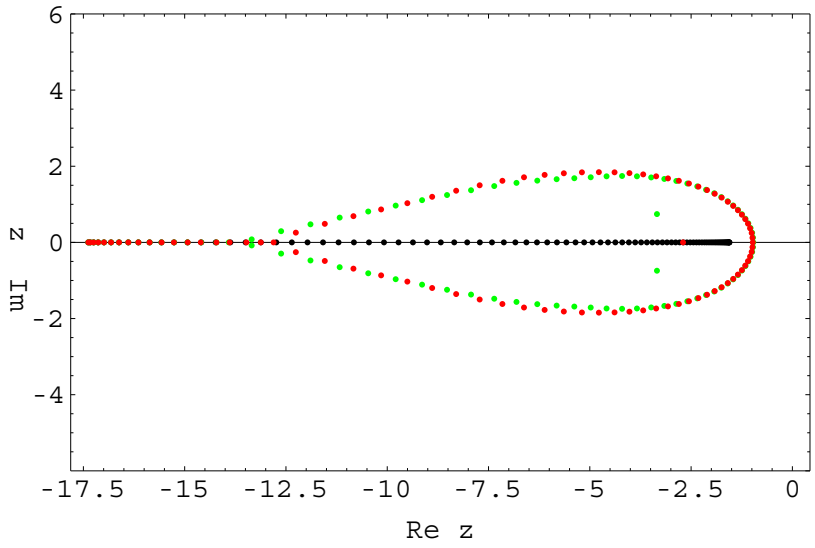

Figure 1. Comparison of the three spectra. Green (lightgray): ARPACK package. Red (stark gray): Subroutine EIGZF (IMSL). Black: Exact values. The three spectra coincide at the left (strictly real tail).

Finally, Table IV contains the (practically normalized) eigenvector v^{49} obtained with AUTOV3, corresponding to $z_{49} = -4.2027.. \pm 1.7453..i$, the conjugate eigenvalues of maximal imaginary part. This time the components differ in only 17 orders of magnitude instead of 44, as was the case for the exact eigenvector v^{89} in Table II. The 89^{th} eigenvector calculated with AUTOV3 (not shown here), also presents similar orders. Again, big differences appear between exact and numerical values. An immediate interpretation is possible: at least the first 55 components of the exact eigenvector are double precision zeroes, and thus will never be perceived by usual codes. The information contained in the exact eigenvector, is lost. Additionally, it becomes clear that by numerical errors, matrix-vector multiplication with 16 digits makes no sense in these cases, and so on with all remaining operations of the algorithm.

TABLE III: Eigenvalues z_s obtained with ARPACK package
(Truncated)

$-17.378.. + 0.0000..i, -17.329.. + 0.0000..i, -17.248.. + 0.0000..i, -17.135.. + 0.0000..i,$
$-16.992.. + 0.0000..i, -16.818.. + 0.0000..i, -16.618.. + 0.0000..i, -16.391.. + 0.0000..i,$
$-16.140.. + 0.0000..i, -15.866.. + 0.0000..i, -15.573.. + 0.0000..i, -15.259.. + 0.0000..i,$
$-14.929.. + 0.0000..i, -14.594.. + 0.0000..i, -14.216.. + 0.0000..i, -13.909.. + 0.0000..i,$
$-13.344.. + 0.0793..i, -13.344.. - 0.0793..i, -12.620.. + 0.2929..i, -12.620.. - 0.2929..i,$
$-11.894.. + 0.4755..i, -11.894.. - 0.4755..i, -11.173.. + 0.6493..i, -11.173.. - 0.6493..i,$
$-10.469.. + 0.8128..i, -10.469.. - 0.8128..i, -9.7899.. + 0.9653..i, -9.7899.. - 0.9653..i,$
$-9.1418.. + 1.1084..i, -9.1418.. - 1.1084..i, -8.5236.. + 1.2442..i, -8.5236.. - 1.2442..i,$
$-7.9299.. + 1.3701..i, -7.9299.. - 1.3701..i, -7.3583.. + 1.4786..i, -7.3583.. - 1.4786..i,$
$-6.8119.. + 1.5635..i, -6.8119.. - 1.5635..i, -6.2964.. + 1.6228..i, -6.2964.. - 1.6228..i,$
$-5.8185.. + 1.6593..i, -5.8185.. - 1.6593..i, -5.3824.. + 1.6837..i, -5.3824.. - 1.6837..i,$
$-4.9777.. + 1.7103..i, -4.9777.. - 1.7103..i, -4.5855.. + 1.7354..i, -4.5855.. - 1.7354..i,$
$-4.2027.. + 1.7453..i, -4.2027.. - 1.7453..i, -3.8348.. + 1.7357..i, -3.8348.. - 1.7357..i,$
$-3.4863.. + 1.7082..i, -3.4863.. - 1.7082..i, -3.3421.. + 0.7433..i, -3.3421.. - 0.7433..i,$
$-3.1614.. + 1.6661..i, -3.1614.. - 1.6661..i, -2.8634.. + 1.6118..i, -2.8634.. - 1.6118..i,$
$-2.5930.. + 1.5460..i, -2.5930.. - 1.5460..i, -2.3488.. + 1.4698..i, -2.3488.. - 1.4698..i,$
$-2.1284.. + 1.3842..i, -2.1284.. - 1.3842..i, -1.9304.. + 1.2902..i, -1.9304.. - 1.2902..i,$
$-1.7537.. + 1.1886..i, -1.7537.. - 1.1886..i, -1.5981.. + 1.0809..i, -1.5981.. - 1.0809..i,$
$-1.4627.. + 0.9693..i, -1.4627.. - 0.9693..i, -1.3458.. + 0.8555..i, -1.3458.. - 0.8555..i,$
$-1.2451.. + 0.7402..i, -1.2451.. - 0.7402..i, -1.1592.. + 0.6226..i, -1.1592.. - 0.6226..i,$
$-1.0882.. + 0.5017..i, -1.0882.. - 0.5017..i, -1.0332.. + 0.3779..i, -1.0332.. - 0.3779..i,$
$-0.9945.. + 0.2522..i, -0.9945.. - 0.2522..i, -0.9718.. + 0.1261..i, -0.9718.. - 0.1261..i,$

TABLE IV: Eigenvector v^{49} corresponding to $z_{49} = -4.2027.. \pm 1.7453..i$,
(maximal imaginary part) obtained with ARPACK package
(Truncated. Not normalized)

$+3.1308..\cdot 10^{-17} - 3.2192..\cdot 10^{-17} i$, $-1.0436..\cdot 10^{-16} - 3.0298..\cdot 10^{-17} i$, $-2.4786..\cdot 10^{-17} - 3.7873..\cdot 10^{-17} i$,
$+0.0000..\cdot 10^{-00} - 9.0896..\cdot 10^{-17} i$, $-1.4610..\cdot 10^{-16} - 7.5747..\cdot 10^{-18} i$, $4.1745..\cdot 10^{-17} + 8.3321..\cdot 10^{-17} i$,
$+1.0436..\cdot 10^{-16} + 6.7415..\cdot 10^{-16} i$, $+1.1479..\cdot 10^{-15} + 2.2724..\cdot 10^{-16} i$, $+8.3490..\cdot 10^{-17} - 8.6351..\cdot 10^{-16} i$,
$-1.2314..\cdot 10^{-15} - 1.3217..\cdot 10^{-15} i$, $-2.1133..\cdot 10^{-15} + 1.3483..\cdot 10^{-15} i$, $+1.5028..\cdot 10^{-15} + 3.6282..\cdot 10^{-15} i$,
$+6.2147..\cdot 10^{-15} - 8.7109..\cdot 10^{-16} i$, $-1.2262..\cdot 10^{-16} - 9.3472..\cdot 10^{-15} i$, $-1.4255..\cdot 10^{-14} - 2.7666..\cdot 10^{-15} i$,
$-6.9922..\cdot 10^{-15} + 2.0327..\cdot 10^{-14} i$, $+2.8261..\cdot 10^{-14} + 1.6520..\cdot 10^{-14} i$, $+3.2409..\cdot 10^{-14} - 3.8487..\cdot 10^{-14} i$,
$-4.9989..\cdot 10^{-14} - 5.7776..\cdot 10^{-14} i$, $-9.9854..\cdot 10^{-14} + 5.9794..\cdot 10^{-14} i$, $+6.5497..\cdot 10^{-14} + 1.6462..\cdot 10^{-13} i$,
$+2.6405..\cdot 10^{-13} - 5.6431..\cdot 10^{-14} i$, $-1.8221..\cdot 10^{-14} - 4.1205..\cdot 10^{-13} i$, $-6.2559..\cdot 10^{-13} - 7.6262..\cdot 10^{-14} i$,
$-2.7255..\cdot 10^{-13} + 9.2216..\cdot 10^{-13} i$, $+1.3196..\cdot 10^{-12} + 6.4295..\cdot 10^{-13} i$, $+1.3007..\cdot 10^{-12} - 1.8236..\cdot 10^{-12} i$,
$-2.4167..\cdot 10^{-12} - 2.4195..\cdot 10^{-12} i$, $-4.2532..\cdot 10^{-12} + 3.0281..\cdot 10^{-12} i$, $+3.4863..\cdot 10^{-12} + 7.1669..\cdot 10^{-12} i$,
$+1.1668..\cdot 10^{-11} - 3.4394..\cdot 10^{-12} i$, $-2.2333..\cdot 10^{-12} - 1.8434..\cdot 10^{-11} i$, $-2.8315..\cdot 10^{-11} - 1.2901..\cdot 10^{-12} i$,
$-9.0891..\cdot 10^{-12} + 4.2302..\cdot 10^{-11} i$, $+6.1391..\cdot 10^{-11} + 2.4357..\cdot 10^{-11} i$, $+5.2161..\cdot 10^{-11} - 8.6275..\cdot 10^{-11} i$,
$-1.1672..\cdot 10^{-10} - 1.0029..\cdot 10^{-10} i$, $-1.8044..\cdot 10^{-10} + 1.5040..\cdot 10^{-10} i$, $+1.8088..\cdot 10^{-10} + 3.0961..\cdot 10^{-10} i$,
$+5.1175..\cdot 10^{-10} - 1.9418..\cdot 10^{-10} i$, $-1.6319..\cdot 10^{-10} - 8.1942..\cdot 10^{-10} i$, $-1.2747..\cdot 10^{-09} + 3.8901..\cdot 10^{-11} i$,
$-2.6313..\cdot 10^{-10} + 1.9289..\cdot 10^{-09} i$, $+2.8375..\cdot 10^{-09} + 8.8290..\cdot 10^{-10} i$, $+2.0452..\cdot 10^{-09} - 4.0489..\cdot 10^{-09} i$,
$-5.5793..\cdot 10^{-09} - 4.1007..\cdot 10^{-09} i$, $-7.5814..\cdot 10^{-09} + 7.3645..\cdot 10^{-09} i$, $+9.1737..\cdot 10^{-09} + 1.3271..\cdot 10^{-08} i$,
$+2.2295..\cdot 10^{-08} - 1.0461..\cdot 10^{-08} i$, $-1.0122..\cdot 10^{-08} - 3.6204..\cdot 10^{-08} i$, $-5.7058..\cdot 10^{-08} + 6.1030..\cdot 10^{-09} i$,
$-5.2110..\cdot 10^{-09} + 8.7438..\cdot 10^{-08} i$, $+1.3031..\cdot 10^{-07} + 2.9910..\cdot 10^{-08} i$, $+7.7915..\cdot 10^{-08} - 1.8864..\cdot 10^{-07} i$,
$-2.6432..\cdot 10^{-07} - 1.6490..\cdot 10^{-07} i$, $-3.1495..\cdot 10^{-07} + 3.5631..\cdot 10^{-07} i$, $+4.5693..\cdot 10^{-07} + 5.6406..\cdot 10^{-07} i$,
$+9.6448..\cdot 10^{-07} - 5.4555..\cdot 10^{-07} i$, $-5.7791..\cdot 10^{-07} - 1.5896..\cdot 10^{-06} i$, $-2.5389..\cdot 10^{-06} + 4.6886..\cdot 10^{-07} i$,
$+6.5123..\cdot 10^{-08} + 3.9406..\cdot 10^{-06} i$, $+5.9488..\cdot 10^{-06} + 8.9639..\cdot 10^{-07} i$, $+2.8507..\cdot 10^{-06} - 8.7295..\cdot 10^{-06} i$,
$-1.2422..\cdot 10^{-05} - 6.4945..\cdot 10^{-06} i$, $-1.2912..\cdot 10^{-05} + 1.7062..\cdot 10^{-05} i$, $+2.2428..\cdot 10^{-05} + 2.3744..\cdot 10^{-05} i$,
$+4.1405..\cdot 10^{-05} - 2.7772..\cdot 10^{-05} i$, $-3.1361..\cdot 10^{-05} - 6.9342..\cdot 10^{-05} i$, $-1.1230..\cdot 10^{-05} + 2.9714..\cdot 10^{-05} i$,
$+1.6392..\cdot 10^{-05} + 1.7656..\cdot 10^{-04} i$, $+2.6995..\cdot 10^{-04} + 1.9880..\cdot 10^{-05} i$, $+9.8048..\cdot 10^{-05} - 4.0138..\cdot 10^{-04} i$,
$-5.7952..\cdot 10^{-04} - 2.4890..\cdot 10^{-04} i$, $-5.2093..\cdot 10^{-04} + 8.0962..\cdot 10^{-04} i$, $+1.0873..\cdot 10^{-03} + 9.8864..\cdot 10^{-04} i$,
$+1.7624..\cdot 10^{-03} - 1.3875..\cdot 10^{-03} i$, $-1.6439..\cdot 10^{-03} - 3.0042..\cdot 10^{-03} i$, $-4.9352..\cdot 10^{-03} + 1.7174..\cdot 10^{-03} i$,
$+1.3370..\cdot 10^{-03} + 7.8700..\cdot 10^{-03} i$, $+1.2166..\cdot 10^{-02} - 3.8304..\cdot 10^{-05} i$, $+3.0529..\cdot 10^{-03} - 1.8378..\cdot 10^{-02} i$,
$-2.6752..\cdot 10^{-02} - 9.0956..\cdot 10^{-03} i$, $-2.0903..\cdot 10^{-02} + 3.8362..\cdot 10^{-02} i$, $+5.1497..\cdot 10^{-02} + 3.9657..\cdot 10^{-02} i$,
$+7.7460..\cdot 10^{-02} - 6.9918..\cdot 10^{-02} i$, $-8.0218..\cdot 10^{-02} - 1.1931..\cdot 10^{-01} i$, $-2.4604..\cdot 10^{-01} + 1.0185..\cdot 10^{-01} i$,
$+7.3756..\cdot 10^{-02} + 2.5535..\cdot 10^{-01} i$, $+8.2641..\cdot 10^{-01} - 5.0539..\cdot 10^{-02} i$

CONCLUSIONS

The example here discussed clearly shows how difficult may become the numerical evaluation of eigenpairs, even for simple advection-diffusion linear one-dimensional problems of very small size, leading to non-normal tridiagonal matrices. The problem is severely ill-conditioned, due to the loss of linear independence of the eigenvectors basis. For this class of problems, no calculations with usual codes are possible without a very high arithmetic precision, perhaps no less than 50 digits according to our example. Otherwise, specialized algorithms should be developed. We hope that this example would serve as a warning against careless use of double precision codes to solve non-symmetric eigenvalue problems.

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