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UNSTEADY WALL SHEAR STRESS MODEL FOR PRESSURE TRANSIENT PROBLEMS IN LAMINAR AND TURBULENT PIPE FLOWS

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ABSTRACT

A transient shear stress model for the solution of water-hammer problems for laminar and turbulent flows in pipes is presented. The model is based on an expansion of the radial profile of the axial velocity in exponential and polynomial functions, and the solution of the resulting set of equations by the method of characteristics. The present model can be included with only minor modifications into any existing code based on the characteristics method.

The new model is tested against both experimental results and mathematical models and numerical simulations of other authors, showing good agreement.

INTRODUCTION

The usual way of solving a water-hammer problem is to represent the unsteady viscous stress on the pipe wall with a model similar to that used for steady flow. That is, it is assumed that the shear stress at the wall is proportional only to the local mean velocity; this is the so-called *quasi-steady model*.

However, due to the very presence of pressure waves traveling along the pipe, the instantaneous velocity profiles found during the transient are far from those of the steady state. The discrepancies turn to be more important next to the pipe wall, where the Richardson's effect [6] is present. Due to this, the quasi-steady model does a poor job in modeling the shear stress at the wall.

To overcome this difficulty, more complex models have been proposed in the past, e.g. those of Zielke [7] and Vardy and Hwang [5]. Both models, like the one we are proposing in the present work, rely on the method of characteristics for solving the resulting set of equations. The model of Zielke is based on the solution of a 1D problem in which the wall shear stress in transient laminar pipe flow is related to the instantaneous mean velocity and to weighted past velocity changes; therefore, this method requires that the history of all the previously calculated nodal velocities be kept in storage. On the other hand, Vardy and Hwang propose a quasi two-dimensional model by discretizing the domain in a set of concentric cylindrical annuli, and solving a water-hammer problem for each of them.

It the following, a new model for calculating the shear stress at the wall for transient laminar and turbulent pipe flow is proposed. This is an extension of the model for laminar flows proposed by the present authors in [1] and [2]. It is based on an expansion of the radial profile of the axial velocities into polynomial and exponential functions, and the solution of the resulting set of equations by the method of characteristics.

MATHEMATICAL MODEL

Motivation

In references [3] and [4], Prado and Marchegiani solve a transient laminar problem which is close to a water-hammer problem in a pipe. They consider an infinite pipe, and impose a given time-dependent pressure gradient along the pipe axis that resembles the pressure histories obtained by Zielke in [5]. They proceed by discretizing the Navier-Stokes equations in r and t via Finite Differences in a given section of the pipe, and present *snapshots* of the velocity profiles during the transient.

The shape of these profiles led the present authors to the idea that they may be represented by a polynomial expansion in r with reasonable accuracy. This is the key idea behind the laminar method presented in [1][2]. In this context, the traditional *quasi-steady* method is an expansion of the velocity profile in a polynomial of degree 2, and thus may be regarded as a special case of the new method (the reader is encouraged to look in [1] and [2] for the details).

In this paper, the method is extended to turbulent flows. In our numerical experiences, expanding the velocity profiles only into polynomials have proven not to be adequate for this type of flows. This is mainly due to the fact that the high degree polynomials needed to represent the extremely high gradients observed near the wall for high Reynolds numbers led to axial velocity profiles with strong oscillations in the radial direction (a known undesirable feature of polynomial expansions). As is shown in the rest of the paper, an answer to this problem seems to be the addition of an exponential term to the expansion.

Variables, parameters and non-dimensionalization

Once non-dimensionalized, the problem depends on the following four non-dimensional parameters: the initial Mach number, $Ma_0 = \overline{V}_0 / a$, the initial Reynolds number, $Re_0=\rho \overline{V}_0 2R/\mu$, the initial Froude number, $Fr_0=\overline{V}_0/(gR)^{1/2}$, and the non-dimensional pipe length, $\hat{L} = l / R$, where l is the pipe length, $\overline{V}_0$ is the initial average velocity, R is the pipe radius, ρ and μ are the density and viscosity of the fluid, a is the speed of sound in the pipe, and g is the gravity.

The non-dimensionalization of radial and axial positions, r and z, times, t, velocities, v, pressures, p, stresses, τ, and viscosities, μ, follow:

$$\hat{r} = \frac{r}{R}; \quad \hat{z} = \frac{z}{l}; \quad \hat{t} = \frac{t}{l/a}; \quad \hat{v} = \frac{v}{\overline{V}_0}; \quad \hat{p} = \frac{p}{\rho a \overline{V}_0}; \quad \hat{\tau} = \frac{\tau}{\rho a \overline{V}_0}; \quad \hat{\mu} = \frac{\mu}{\rho a R}$$

For clarity, the symbol ^ is dropped in the rest of the paper, and all variables and parameters are assumed to be non-dimensional.

Water-hammer governing equations

In the following, "mean" refers to time dependent magnitudes in which turbulence has been averaged out (by ensemble or time averaging) and "average" refers to spatial averages of mean magnitudes in the cross-sectional area, A.

Under the assumptions: a) horizontal rigid pipe of constant radius, b) purely axial flow, c) constant mean pressure across the pipe section, d) average flow velocity negligible with respect to the fluid speed of sound, and e) negligible viscous normal stresses, the averaged equations for the water-hammer phenomenon are reduced to the following hyperbolic system of two coupled partial differential equations for the average static pressure and velocity, $p(z,t)$ and $V(z,t)$:

$$\frac{\partial p}{\partial t} + \frac{\partial V}{\partial z} = 0, \qquad \frac{\partial V}{\partial t} + \frac{\partial p}{\partial z} = S, \tag{1}$$

The average velocity is defined by

$$V(z,t)=\int_A v(r,z,t)dA\Big/\int_A dA=\int_0^1 vrdr\Big/\int_0^1 rdr=2\int_0^1 v(r,z,t)rdr \tag{2}$$

where $v(r,z,t)$ denotes the mean axial velocity. The source term, S, comes from the integration of the friction term on the pipe section, as follows:

$$S(z,t)=L\int_A \frac{1}{r}\frac{\partial r\tau_z}{\partial r}dA\Big/\int_A dA=L\int_0^1 \frac{\partial r\tau_z}{\partial r}dr\Big/\int_0^1 r\,dr=2L\int_0^1 \frac{\partial r\tau_z}{\partial r}dr=-2L\tau_w(z,t) \tag{3}$$

where $\tau_z(r,z,t)$ is the mean axial shear stress, and $\tau_w(z,t)=-\tau_z(1,z,t)$ the mean wall shear stress.

Quasi-steady model for the wall stress

In the traditional *quasi-steady* model, τ_w is evaluated as

$$\tau_w=Ma_0 f\frac{|V|V}{8}, \tag{4}$$

where Darcy's and Weissbach's friction factor, f, is assumed to depend only on the average velocity, V (or more properly on the Reynolds number, *Re*), and thus is obtainable using Colebrook's formula (Moody's chart). Therefore, the source S is in this case only a function of V for a given pipe geometry and material and for a given fluid, that is,

$$S=S(V)=-LMa_0 f(V)\frac{|V|V}{4}. \tag{5}$$

Proposed transient model for the wall stress

In order to achieve a better representation of the time dependent mean shear stress at the wall, a more realistic representation of the mean velocity profile must be considered. If the flow is considered axisymmetric, the mean axial velocity profile, $v(r,z,t)$, may be aproximated by a polynomial on the radial coordinate, r, (see [1][2]), plus an exponential term, in the form

$$v(r,z,t)\cong\tilde{v}(r,z,t)=\sum_{j\in J}a_j(z,t)f_j(r) \tag{6}$$

where the expansion functions f_j are defined as:

$$f_1(r)=\left(1-e^{-(1-r^2)/d}\right)\Big/\left(1-e^{-1/d}\right) \quad ; \qquad f_j(r)=(1-r^j) \quad \forall j\geq 2 \tag{7}$$

The coefficients a_j and the parameter d will be defined below; for now, we consider d as a known positive value. The set J is a user selected small set of natural numbers, $J=\{j|\,j\in N\}$. Note that Eq.(6) verifies the non-slip condition at the wall and has null radial derivative at the centerline. Also, note that f_1 has been chosen so that $\lim_{d\to\infty} f_1(r)=f_2(r)=(1-r^2)$.

In order to calculate the coefficients a_j, we start by defining the following weight-averaged velocities:

$$V_i=V_i(z,t)=\int_A \tilde{v}(r,z,t)\,r^i\,dA\Big/\int_A r^i\,dA=\int_0^1 \tilde{v}r^{i+1}dr\Big/\int_0^1 r^{i+1}dr=(i+2)\int_0^1 \tilde{v}(r,z,t)r^{i+1}dr \tag{8}$$

where i = 0, 1, 2, ..., dim(J)-1. For i=0, $V_0(z,t)\cong V(z,t)$, so that the "order zero weight-averaged velocity" is "coincident" with the average velocity (within the precision of our expansion, Eq(6)).

Once the set J is defined, the coefficients $a_j(z,t)$ are determined (as functions of V_i) as follows. Eq.(6) is replaced into Eq.(8), giving

$$V_i = (i+2)\int_0^1 \tilde{v}(r,z,t)\, r^{i+1} d\bar{r} = \sum_{j\in J} G_{ij}\, a_j \tag{9}$$

where the matrix G_{ij} is defined by

$$G_{ij} = (i+2)\int_0^1 f_j(r)\, r^{i+1}\, dr = I_i \delta_{1j} + (1-\delta_{1j})\left(1 - \frac{i+2}{i+j+2}\right) \tag{10}$$

where δ_{ij} is the Kronecker delta, and I_i is defined recursively by:

$$I_0 = \left(1/\left(1-e^{-1/d}\right) - d\right);\; I_1 = (I_0 + d)\left(1 - \tfrac{3}{2}d + \tfrac{3}{4}d^{3/2} e^{-1/d}\sqrt{\pi}\, Erfi(d^{-1/2})\right),\; I_i = I_0 + d\left(1 - \tfrac{i+2}{2} I_{i-2}\right)\; \forall i \geq 2 \tag{11}$$

where $Erfi(x)$ is the imaginary error function. From Eq.(9), the coefficients a_j can be evaluated:

$$a_j = \sum_{i=0}^{\dim(J)-1} G_{ji}^{-1} V_i\,. \tag{12}$$

It should be noted that the index j is not a complete sequence of natural numbers, and therefore the matrix notation of Eqs.(9) to (12), and other equations below, are somehow an abuse of notation. Their meaning is, nevertheless, quite clear, although care must be taken when coding these equations.

Using Eqs.(6) and (12), the approximated mean velocity profile can be put in terms of the V_i as:

$$\tilde{v}(r,z,t) = \sum_{i=0}^{\dim(J)-1}\left[\sum_{j\in J} G_{ij}^{-1} f_j(r)\right] V_i(z,t)\,. \tag{13}$$

Using Eq. (13), the wall shear stress, τ_w, can be determined (for a Newtonian fluid) as,

$$\tau_w(z,t) = -\frac{2Ma_0}{Re_0}\left.\frac{\partial \tilde{v}}{\partial r}\right|_{r=1} = -\frac{2Ma_0}{Re_0}\sum_{i=0}^{\dim(J)-1}\left[\sum_{j\in J} G_{ji}^{-1} f_j'(1)\right] V_i(z,t) = \frac{2Ma_0}{Re_0}\sum_{i=0}^{\dim(J)-1}\left[\sum_{j\in J} G_{ji}^{-1}\left[2\delta_{1j} + j(1-\delta_{1j})\right]\right] V_i(z,t) \tag{14}$$

Consequently, *the present model considers that the shear stress at the wall, τ_w, is not only a function of the average velocity, V, but also a function of the weight-averaged velocities, V_i.*

Equations for the weight-averaged velocities

Our model needs not only solve for the average pressure and velocity, p and V, but also for the weight-averaged velocities, V_i. Under the assumptions (a) to (e), the axial momentum equation for the flow of a Newtonian fluid along a horizontal rigid pipe of constant radius is reduced to

$$\frac{\partial v}{\partial t} + \frac{\partial p}{\partial z} = \left[\frac{L}{r}\frac{\partial r\tau_z}{\partial r}\right] \tag{15}$$

The procedure for obtaining the additional equations is (see [2] for details):

i) Eq.(15) is multiplied by r^i, for $i = 0,1,2,\ldots,\dim(J)-1$;

ii) the resulting equation is then integrated on the cross sectional area, A, of the pipe;

iii) the derivatives are taken out of the integrals;

iv) everything is divided by $\int r^i dA$;

v) using the definition of V_i, Eq.(9), we finally get (in the form of Eq.(1)):

$$\frac{\partial V_i}{\partial t} + \frac{\partial p}{\partial z} = S_i\,, \tag{16}$$

where we introduce the weight-averaged sources

$$S_i = L\int_A \frac{\partial r\tau_z}{\partial r} r^{i-1} dA \Big/ \int_A r^j dA = L\int_0^1 \frac{\partial r\tau_z}{\partial r} r^j dr \Big/ \int_0^1 r^{i+1} dr = (i+2)L\int_0^1 \frac{\partial r\tau_z}{\partial r} r^i dr \tag{17}$$

for i=0,1,2,..,dim(J)-1. To evaluate Eq.(17), we make the following additional assumptions:

f) the transient is fast enough for the turbulence to remain the same along the process, and

g) the initial velocity condition, $v_o(r) = v(r,t=0)$, corresponds to developed flow.

We can then compute the initial axial shear-stress using the given initial wall shear stress, τ_{w0}, as:

$$\tau_z(r,t=0) = -\tau_{w0}\ r = \mu_{eff}(r)\, v'_0(r) \tag{18}$$

The (non-dimensional) effective viscosity, μ_{eff} (molecular+eddy), can then be evaluated as:

$$\mu_{eff}(r) = -\tau_{w0} r / v'_0(r) \cong -\tau_{w0} r / \tilde{v}'_0(r) = -\tau_{w0}\ r \Big/ \sum_{j\in J} a_{0j} f'_j(r) \tag{19}$$

where a_{0j} are the initial values of coefficients a_j, evaluated using Eq.(12) and the weight-averaged initial velocities, V_{0i}. These velocities, in turn, are evaluated (either analitically or numerically) with Eq.(8) and the approximation $\tilde{v}_0(r)$ to the initial profile $v_0(r)$.

After some manipulations, it can be shown that the sources can be expressed as functions of the weight-averaged velocities V_k, in the following way:

$$S_i(z,t) = \sum_{j\in J} H_{ij}\, a_j(z,t) = \sum_{k=0}^{\dim(J)-1} \left(\sum_{j\in J} H_{ij}\, G^{-1}_{jk} \right) V_k(z,t) \tag{20}$$

where the matrix H_{ij} is defined by:

$$H_{ij} = -(i+2)L\tau_{w0} \int_0^1 r^i \frac{\partial}{\partial r}\left(r^2 f'_j(r) \Big/ \sum_{l\in J} a_{0l} f'_l(r) \right) dr \tag{21}$$

Unlike the laminar case [2], H_{ij} is cumbersome to evaluate, and lacks of any evident analytical solution. It is, however, feasible, small ($dim(J)\times dim(J)$), and may be computed at t=0 and stored.

By combining Eq.(15) with Eqs.(1), and recalling that $V_0 \equiv V$, the following hyperbolic system of partial differential equations is obtained

$$\frac{\partial p}{\partial t} + \frac{\partial V_0}{\partial z} = 0, \tag{22.a}$$

$$\frac{\partial V_0}{\partial t} + \frac{\partial p}{\partial z} = S_0 \tag{22.b}$$

$$\frac{\partial V_i}{\partial t} + \frac{\partial p}{\partial z} = S_i \quad ; \quad i = 1,2,...,\dim(J)-1 \tag{22.c}$$

The non-dimensional characteristic slopes of Eqs.(22) in the (t,z) plane, $\zeta = dz/dt$, are

$$\zeta_{1,2} = \pm 1\ , \qquad \zeta_{3,\cdots,\dim(J)+1} = 0\,. \tag{23}$$

Substracting Eq.(22.b) from Eq.(22.c) to eliminate the pressure gradient term and rewriting Eqs.(22) as a system of ODEs along the characteristic lines defined in Eqs.(23), we finally get a hyperbolic system of $dim(J)$+1 equations in the unknowns $\{p, V_0, V_1, \ldots, V_{\dim(J)+1}\}$, as follows:

$$\frac{dV_0}{dt} + \frac{dp}{dt} = S_0 \qquad ; \frac{dz}{dt} = +1 \tag{24.a}$$

$$\frac{dV_0}{dt} - \frac{dp}{dt} = S_0 \qquad ; \frac{dz}{dt} = -1 \tag{24.b}$$

$$\frac{dV_i}{dt} - \frac{dV_0}{dt} = S_i - S_0 \quad ; i = 1, 2, ..., \dim(J)\text{-}1 \ ; \frac{dz}{dt} = 0 \tag{24.c}$$

Determining the non-dimensional parameter d

For a given set J, the parameter d depends only on the initial conditions, more precisely on the initial Reynolds number, Re_0; that is, $d=d_J(Re_0)$. It may be interpreted as a rough measure of the extension of the laminar sublayer. Its value is defined by asking the expansion of the initial profile, $\breve{v}(r,t=0)$, to exactly match the initial shear stress at the wall, τ_{w0}, and can be determined as follows:

Step 1: Compute the initial wall stress, τ_{w0}, with Colebrook's formula and Eq.(4).

Step 2: Find a mathematical expression, $\breve{v}_0(r)$, for the "real" initial velocity profile, $v_0(r)$. *There is no need for $\breve{v}_0(r)$ to accurately represent the velocity gradients near the wall, nor even to have null velocity at the wall itself; this approximated profile is only required to be a good approximation in the core of the flow. It may, for example, be a power law of the type $\breve{v}_0(\bar{r})=V_{max}(1-\bar{r})^{1/n}$.*

Step 3: Provide a *seed* value for d.

Step 4: Compute the shear stress at the wall, τ_w, using $\breve{v}_0(r)$ and Eqs.(8), (10) and (14).

Step 5: if the difference $|\tau_w - \tau_{w0}|$ is less that a specified tolerance, stop. If not, change the value of d and repeat from Step 4.

This algorythm may be applied completely *offline*, in the sense that a table d vs Re can be constructed for each set J, and stored for later use or for finding a suitable interpolation function $d_J(Re)$.

For laminar cases, this algorithm does no apply; in these cases d is set by hand to a big value (~20) so as to make $f_1(r) \sim 1-r^2$ (that is, the initial laminar velocity profile).

Numerical results

The system of equations (24) is solved via the Methods of Characteristics (for details see [2]). The model is tested against two experimental works: low Re laminar results for oil (Holmboe and Rouleau, referenced in [7]) and high Re laminar and low Re turbulent results for water (Bergant, Simpson and Vítkovský [8]). Both experiments deal with pressure transients originated by the sudden closure of the downstream valve in a tank-pipe-valve hydraulic system (Fig.(1)). In the following numerical tests, the set J has been chosen to be $J=\{1,2,...,9\}$.

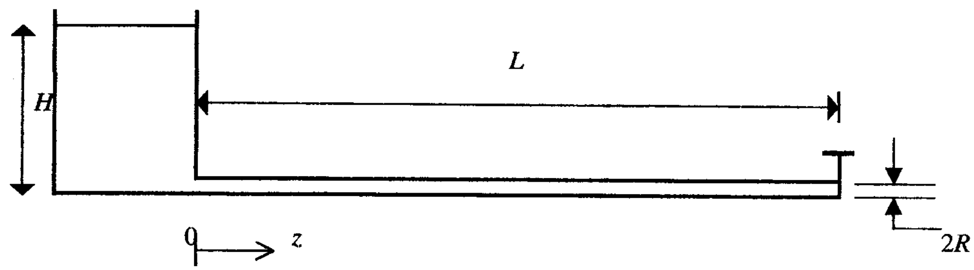

Figure 1: Geometric configuration.

In figure (2), pressure results using the proposed method are compared against measurements due to Holmboe and Rouleau and numerical simulations using the quasi-steady method. The improvement in the solution is evident.

In figure (3), the wall shear stress obtained with our method is compared against the stresses computed using the quasi-steady method and numerical results due to Vardy and Hwang [5], based on a finite-differences discretization of the radial direction into 24 concentric annuli. The similarity between our results and Vardy's, and the huge difference between these results and those obtained with the quasi-steady method, is clearly seen . For more details and results regarding this case, see [2].

The evolution of pressure at the valve and at the midpoint of the pipe for the experiment of Bergant et al is shown in figure (4). Results obtained with the quasi-steady method and experimental measurements are compared against the method proposed in this paper. Although the results are not as good as those obtained for Holmboe's case, the improvement is more than evident.

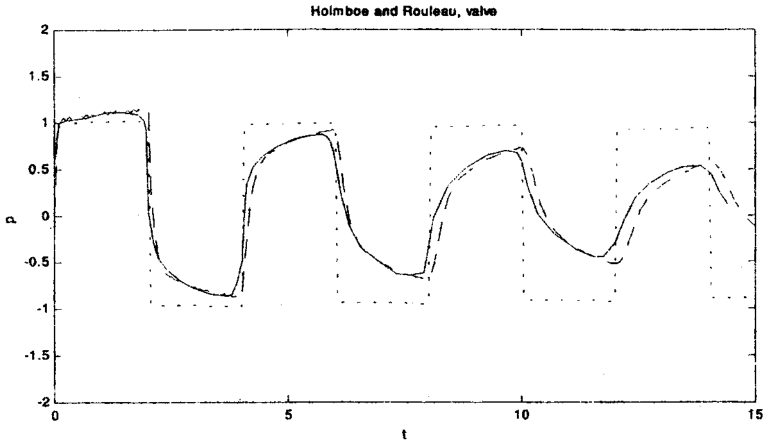

Figure 2: Case Holmboe and Rouleau. Pressure histories at $z=L$.
(—— experiment, ----- quasi-steady, — — present model $J=\{1,2,...,9\}$)

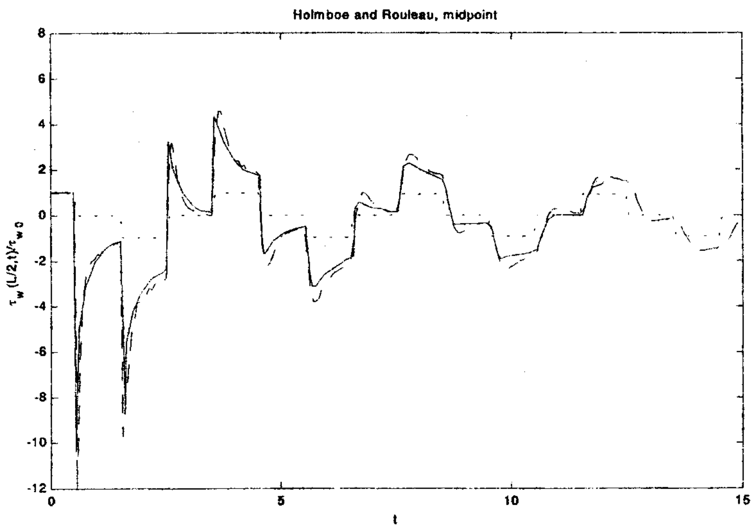

Figure 3: Case Holmboe and Rouleau. Wall shear stress at $z=L/2$.
(—— Vardy and Hwang [5], ----- quasi-steady, — — present model $J=\{1,2,...,9\}$)

Although not shown in the numerical results presented, two main drawbacks have been detected: a) high sensitivity of the results to the value of the parameter d , and b) extremely high values of coefficientes a_j in some cases, which are prone to lead to round-off numerical errors. These issues must be properly addressed before the method is considered well established.

Conclusions

A method for evaluating friction during pressure transients has been presented. The method is a generalization of that for laminar flows proposed by the present authors in [1,2], and provides a much

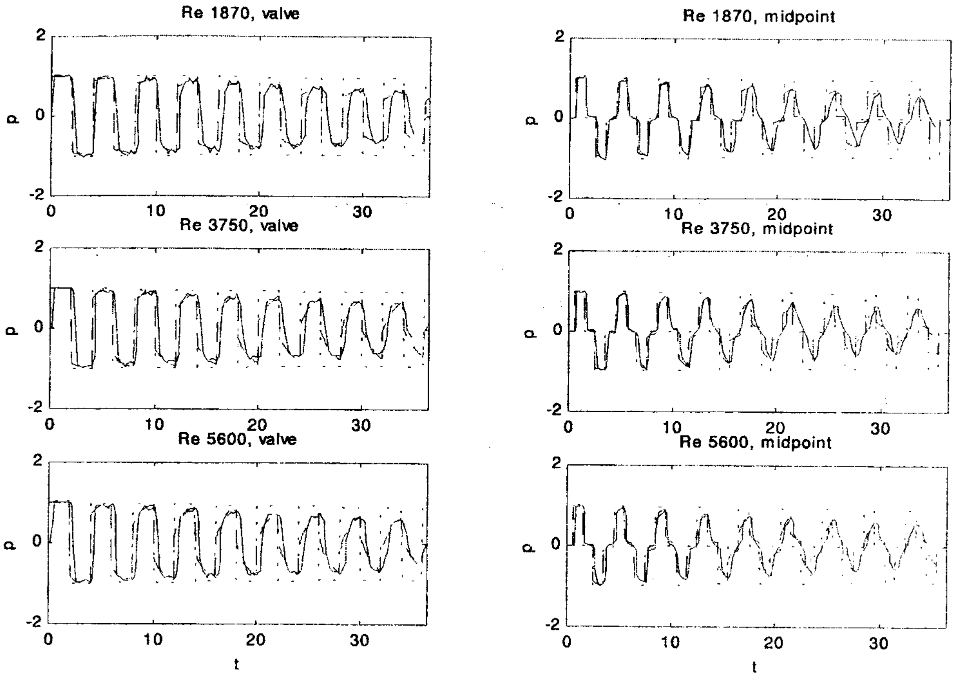

Figure 4: Case Bergant et al. Pressure histories at the valve and at the midpoint.
(—— experiment, ----- quasi-steady, — — present model $J=\{1,2,3,4\}$)

better representation of the shear stress at the wall during transients in both laminar and turbulent flow, as compared to the widely used quasi-steady method. The ability to recover the instantaneous radial profile of the mean axial velocity may also allow in the future for a smooth coupling between 1D and 2D/3D simulations of transients in pipe networks.

The numerical tests shown good agreement between numerical simulations using this method and experimental results and numerical simulations of other authors, both for laminar and low Re turbulent flow. Nevertheless, these results must be considered as preliminary, due to a couple of undesirable features regarding high sensitivity to the parameter d and the likeliness of numerical round-off errors in some cases. Solving these two problems is the main branch of future work in the subject.

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