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INTERACTION BETWEEN A VISCOUS FLUID AND A CENTRAL SOLID CYLINDRICAL CORE CONFINED IN CYLINDRICAL SHELL

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RESUMO

O movimento transiente de um vaso cilíndrico, interagindo com um fluido viscoso e um eixo cilíndrico maciço e concêntrico, é simulado usando-se o Método dos Elementos Finitos. A dinâmica do problema transiente acoplado é resolvida através do método da aceleração média de Newmark no caso da casca e pelo método da Diferença Retrógrada de Euler no caso do fluido. Cálculos são realizados para se determinar deslocamentos, velocidades e acelerações de pontos ao longo da parede do cilindro, que é livre na extremidade superior e engastado na base. Uma resposta monotonicamente decrescente, típica de sistemas amortecidos é observada, e a partir desses resultados, estimativas das frequências naturais do sistema acoplado são feitas.

ABSTRACT

The transient motion of a cylindrical vessel, interacting with a viscous fluid and central solid cylindrical core, is simulated using the Finite Element Method. The transient coupled dynamics problem is solved with the aid of the Newmark average acceleration procedure for the shell and in the case of fluid element, Euler's Backward Difference scheme was utilised. For the theoretical investigation, calculations are performed to determine the displacement, velocity and acceleration of points along the outer cylinder wall, which is closed at the bottom and open at the top, following the initial force, which is a linear function of the distance from the base. A monotonically decaying harmonic response typical of a damped system is observed and, from this, estimates of the natural frequencies of the coupled system are made.

INTRODUCTION

In the present investigation, the transient motion of a cylindrical vessel, interacting with a viscous fluid and a central solid cylindrical core, is simulated using axisymmetric two-noded shell elements for the cylindrical wall [1,2,3,4,5] and 8-noded isoparametric fluid elements for the liquid. Galerkin's weighted residual technique had been employed for Navier-Stokes equations expressed in polar coordinates in order to derive the fluid finite element equations. For the wall of the cylinder, classical linear shell theory of Novozhilov [6] had been employed. The transient coupled dynamics problem had been solved with the aid of the Newmark average acceleration procedure [7] for the shell and in case of the fluid elements, Euler's Backward Difference scheme [8] was utilised

For the theoretical investigation, calculations were performed to determine displacement, velocity and acceleration of the points along the outer cylinder wall, which is closed at the bottom and open at the top, following an initial force, which is a linear function of the distance from the base. A monotonically decaying harmonic response typical of a damped system was observed and, from this, estimates of the natural frequencies of the coupled system were made. It was found that more accurate results are obtained if the mesh refinement is confined to the areas just below the free surface. It has also been observed that the fluid equations to solve for the axisymmetric modes were essentially linear and they were not linear for the circumferential wave number n=0.

SHELL ELEMENT

Curved elements are the most appropriate type of elements to depict axisymmetric shells. Although the geometry of the vessel studied in the present work is cylindrical, and therefore the radius of curvature is taken as infinite at all points, for the sake of generality, a two node annular axisymmetric element of constant curvature as shown in Fig. 1 was adopted.

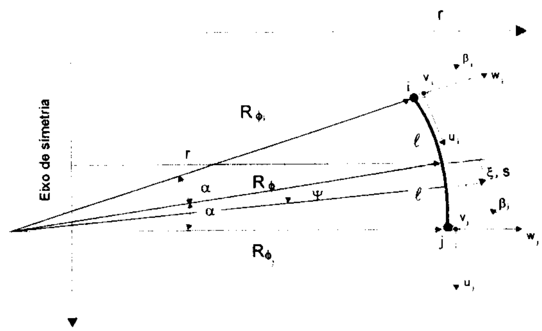

Figure 1 – Two node annular axixymmetric element of constant meridional curvature

The assumed displacement functions [1,2,3,4,5] are as follows

$$u=\frac{(1-\xi)}{2}u_i\cos(n\theta)+\frac{(1+\xi)}{2}u_j \quad \cos(n\theta)$$
$$v=\frac{(1-\xi)}{2}v_i\sin(n\theta)+\frac{(1+\xi)}{2}v_j \quad \sin(n\theta)$$
$$w=\frac{(2-3\xi+\xi^3)}{4}w_i\cos(n\theta)+\frac{(1+\xi)(1-\xi)^2}{4}l\beta_i\cos(n\theta)+\frac{(2+3\xi-\xi^3)}{4}w_j\cos(n\theta)$$

$$+\frac{(1-\xi)(1+\xi)^2}{4}l\beta_j\cos(n\theta) \tag{1}$$

where u, v, w and β are meridional, circumferential, normal and rotational displacements respectively. In the above $(u_i, v_i, w_i, \beta_i, u_j, v_j, w_j, \beta_j)$ are the nodal displacements at nodes i and j respectively and they are employed as the degrees of freedom of the shell element. N is the number of circumferential waves. θ is the angular coordinate in the circumferential direction, l is one half of the meridional length and ξ is the linear local coordinate along a meridian. It varies from –1 (at node I) to +1 (at node j).

The displacements u, v, w, and β may be expressed as a matrix product of the shape function [N] and the nodal displacement vector $\{U_k\}$ as

$$\{U\}=[N]\{U_K\} \tag{2}$$

where

$$\{U\}^T=[uvw]$$
$$\{U_K\}^T=[u_i v_i w_i \beta_i u_j v_j w_j \beta_j] \tag{3}$$

FLUID EQUATIONS

When considering the fluid contained in the vessel, certain physical and geometric assumptions are made:

(a) The fluid domain is axisymmetric,
(b) There is no heat transfer between the fluid and the outside environment,
(c) The fluid is viscous, incompressible and Newtonian,
(d) The flow is unsteady, and
(e) The mass of the system is conserved

The differential equation relating fluid velocity components, acceleration components and pressure gradients at a certain point of the domain, which ensures assumptions (a), (b), (c), and (d) are given by the Navier-Stokes momentum equations [9] expressed in cylindrical coordinates. To ensure the assumption (e), the continuity equation has to be included in the system of equations.
Considering the displacement functions given in (1), the displacements along the shell element may be expressed as:

$$u=\bar{u}(\xi,t)\ \cos(n\theta)$$
$$v=\bar{v}(\xi,t)\ \cos(n\theta)$$
$$w=\bar{w}(\xi,t)\ \cos(n\theta) \tag{4}$$

where $\bar{u}(\xi,t)$, $\bar{v}(\xi,t)$ and $\bar{w}(\xi,t)$ are the circular amplitudes of the displacements u, v and w at a given position ξ and time t. Considering that the velocities may be calculated as the time derivative of the shell displacements at the shell-fluid interface, the fluid velocity components and the shell velocity components are expected to be compatible. It follows that the fluid velocity components will also vary circumferentially in the similar manner.

$$v_r=\bar{v}_r\cos(n\theta)$$
$$v_\theta=\bar{v}_\theta\sin(n\theta)$$
$$v_z=\bar{v}_z\cos(n\theta) \tag{5}$$

where $\bar{v}_r$, $\bar{v}_\theta$ and $\bar{v}_z$ are the fluid velocity components in cylindrical coordinates.
Assuming the pressure varies as,

$$p(r,z,\theta,t)=p_o(r,z,\theta)+\bar{p}(r,z,\theta,t)\ \cos(n\theta) \tag{6}$$

where the first term in the right hand side of the equation is the static pressure and the remaining term corresponds to the pressure in dynamic situation. From the above equation, the pressure gradients components may be derived as,

$$\frac{\partial p}{\partial r}=\frac{\partial p_o}{\partial r}+\frac{\partial \bar{p}}{\partial r}\cos(n\theta)$$
$$\frac{\partial p}{\partial \theta}=\frac{\partial p_o}{\partial \theta}-n\bar{p}\sin(n\theta)$$
$$\frac{\partial p}{\partial z}=\frac{\partial p_o}{\partial z}+\frac{\partial \bar{p}}{\partial z}\cos(n\theta) \tag{7}$$

Taking into account the set of equations (5) and (7), assuming that all variables are considered in terms of the circular amplitudes (i.e. discarding the bar notation above the amplitudes in the following), the Navier-Stokes equations and the continuity equation may be written as follows,

$$\left[\rho k_1+\frac{\partial P_o}{\partial r}+\frac{\partial P}{\partial r}\cos(n\theta)-(\mu k_2+\rho g_r)\right]=0 \tag{8}$$

$$\left[\rho k_3+\frac{\partial P_o}{\partial \theta}-\frac{nP}{r}\sin(n\theta)-(\mu k_4+\rho g_\theta)\right]=0 \tag{9}$$

$$\left[\rho k_5+\frac{\partial P_o}{\partial z}+\frac{\partial P}{\partial z}\cos(n\theta)-(\mu k_6+\rho g_z)\right]=0 \tag{10}$$

$$\left(\frac{\partial v_r}{\partial r}+\frac{v_r}{r}+\frac{n}{r}v_\theta+\frac{\partial v_z}{\partial z}\right)\cos(n\theta)=0 \tag{11}$$

where μ is the viscosity, ρ is the fluid density, (g_r, g_θ, g_z) are cylindrical components of the gravitational acceleration and

$$k_1=v_r\frac{\partial v_r}{\partial r}\cos^2(n\theta)-\left(\frac{n}{r}v_\theta v_r+\frac{v_\theta^2}{r}\right)\sin^2(n\theta)+v_z\frac{\partial v_r}{\partial z}\cos^2(n\theta)+\dot{v}_r\cos(n\theta) \tag{12}$$

$$k_2=\left(\frac{1}{r}\frac{\partial v_r}{\partial r}+\frac{\partial^2 v_r}{\partial r^2}-\frac{v_r}{r^2}+\frac{n^2}{r^2}v_r+\frac{\partial^2 v_r}{\partial z^2}-\frac{2n}{r^2}v_\theta\right)\cos(n\theta) \tag{13}$$

$$k_3=\left(v_r\frac{\partial v_\theta}{\partial r}+\frac{n}{r}v_\theta^2+\frac{v_r v_\theta}{r}+v_z\frac{\partial v_\theta}{\partial z}\right)\cos(n\theta)\sin(n\theta)+\dot{v}_\theta\sin(n\theta) \tag{14}$$

$$k_4=\left(\frac{1}{r}\frac{\partial v_\theta}{\partial r}-\frac{v_\theta}{r^2}+\frac{\partial^2 v_\theta}{\partial r^2}-\frac{n^2}{r^2}v_\theta+\frac{\partial^2 v_\theta}{\partial z^2}-\frac{2n}{r^2}v_r\right)\sin(n\theta) \tag{15}$$

$$k_5=v_r\frac{\partial v_z}{\partial r}\cos^2(n\theta)-\frac{n}{r}v_\theta v_z\sin^2(n\theta)+v_z\frac{\partial v_z}{\partial z}\cos^2(n\theta)+\dot{v}_z\cos(n\theta) \tag{16}$$

$$k_6=\left(\frac{1}{r}\frac{\partial v_z}{\partial r}+\frac{\partial^2 v_z}{\partial r^2}-\frac{n^2}{r^2}v_z+\frac{\partial^2 v_z}{\partial z^2}\right)\cos(n\theta) \tag{17}$$

Assuming that the variation of the velocity, acceleration and pressure is given in terms of the assumed shape functions

$$v_r=\sum_{j=1}^{8}N_j v_{rj} \qquad v_\theta=\sum_{j=1}^{8}N_j v_{\theta j} \qquad v_z=\sum_{j=1}^{8}N_j v_{zj} \qquad P=\sum_{k=1}^{4}M_k p_k \tag{18}$$

where $v_{rj}, v_{\theta j}, v_{zj}$ and p_k are nodal variables, N_j are shape functions of the quadrilateral eight noded element and M_K are shape functions considering four corner nodes of the element only.
Using Galerkin's weighted residual procedure, the Finite Element equations are

$$\int_V N_i X dV=0 \tag{19}$$

where X is one of the Navier-Stokes momentum equations or the continuity equation. It varies according to the range of the shape functions used in the approximations, namely 8 for the three momentum equations, but 4 for the continuity equation. Here dV is equal to $rd\theta drdz$. The resulting equations, after the Galerkin's weighted residual principle is applied, can be analytically integrated in the circumferential direction. Before this integration, equations 8, 10 and 11 are multiplied by $\cos(n\theta)$ and equation 9 is multiplied by $\sin(n\theta)$. Performing the integrals and reducing the second order derivatives by partial integration, one group of equations are for $n \rhd 0$ is obtained

$$\sum_{i=1}^{8}\int_A N_i\left(\rho\sum_{j=1}^{8}N_j\dot{\bar{v}}_{rj}\right)rdrdz+\sum_{i=1}^{8}\int_A N_i\mu\left(\frac{(1+n^2)}{r^2}\sum_{j=1}^{8}N_j\bar{v}_{rj}+\frac{2n}{r^2}\sum_{j=1}^{8}N_j\bar{v}_{\theta j}\right)rdrdz$$
$$\sum_{i=1}^{8}\int_A N_i\left(\sum_{k=1}^{4}\frac{\partial M_k}{\partial r}\bar{p}_k\right)rdrdz+\sum_{i=1}^{8}\int_A \mu\left(\frac{\partial N_i}{\partial r}\sum_{j=1}^{8}\frac{\partial N_j}{\partial r}\bar{v}_{rj}+\frac{\partial N_i}{\partial z}\sum_{j=1}^{8}\frac{\partial N_j}{\partial z}\bar{v}_{rj}\right)rdrdz$$
$$-\sum_{i=1}^{8}\int_s \mu N_i\frac{\partial v_r}{\partial n}ds=0 \tag{20}$$

$$\sum_{i=1}^{8}\int_A N_i\left(\rho\sum_{j=1}^{8}N_j\dot{\bar{v}}_{\theta j}\right)rdrdz+\sum_{i=1}^{8}\int_A N_i\left(\frac{n}{r}\sum_{k=1}^{4}M_k\bar{p}_k+\right)rdrdz$$
$$+\sum_{i=1}^{8}\int_A N_i\mu\left(\frac{(1+n^2)}{r^2}\sum_{j=1}^{8}N_j\bar{v}_{\theta j}+\frac{2n}{r^2}\sum_{j=1}^{8}N_j\bar{v}_{rj}\right)rdrdz$$
$$+\sum_{i=1}^{8}\int_A \mu\left(\frac{\partial N_i}{\partial r}\sum_{j=1}^{8}\frac{\partial N_j}{\partial r}\bar{v}_{\theta j}+\frac{\partial N_i}{\partial z}\sum_{j=1}^{8}\frac{\partial N_j}{\partial z}\bar{v}_{\theta j}\right)rdrdz$$
$$-\sum_{i=1}^{8}\int_s \mu N_i\frac{\partial \bar{v}_\theta}{\partial n}ds=0 \tag{21}$$

$$\sum_{i=1}^{8}\int_A N_i\left(\rho\sum_{j=1}^{8}N_j\dot{\bar{v}}_{zj}\right)rdrdz+\sum_{i=1}^{8}\int_A N_i\left(\sum_{k=1}^{4}\frac{\partial M_l}{\partial z}\bar{p}_k\right)rdrdz$$
$$+\sum_{i=1}^{8}\int_A \mu\left(\frac{\partial N_i}{\partial r}\sum_{j=1}^{8}\frac{\partial N_j}{\partial r}\bar{v}_{zj}+\frac{\partial N_i}{\partial z}\sum_{j=1}^{8}\frac{\partial N_j}{\partial z}\bar{v}_{zj}\right)rdrdz-\sum_{i=1}^{8}\int_s \mu N_i\frac{\partial \bar{v}_z}{\partial n}ds=0 \tag{22}$$

$$\sum_{l=1}^{4}\int_A M_l\left(\sum_{j=1}^{8}\frac{\partial N_j}{\partial r}\bar{v}_{rj}\right)rdrdz+\sum_{l=1}^{4}\int_A M_l\left(+\frac{1}{r}\sum_{j=1}^{8}N_j\bar{v}_{rj}\right)rdrdz$$
$$\sum_{i=1}^{8}\int_A M_l\left(\sum_{j=1}^{8}\frac{\partial N_j}{\partial z}\bar{v}_{zj}\right)rdrdz=0 \tag{23}$$

In the above n is a normal coordinate at the external edge s.

SHELL EQUATION OF MOTION AND SHELL-FLUID COUPLING

Consider equation (4) and the relationships between stresses $\{\sigma\}$ and strains $\{\varepsilon\}$ and between strains and displacements $\{U_k\}$,

$$\{\sigma\}=[D]\{\varepsilon\} \tag{24}$$

$$\{\varepsilon\}=[B]\{U_k\} \tag{25}$$

where [D] is the elasticity matrix and [B] is a matrix relating strains to displacements. Applying the virtual work procedure [10], taking into account the work done by the stresses, by the inertia forces and by the external forces, the shell equation of motion may be obtained as follows,

$$[K]\{U_k\}+[M]\{\ddot{U}_k\}=\{R\} \tag{26}$$

where

$$[K]=\int_v [B]^T[D][B]dv$$
$$[M]=\int_v [N]^T\rho_s[N]dv$$
$$\{R\}=\int_v [N]^T\{F\}dv$$

where [M] and [K] are the mass and stiffness matrices respectively, ρ_s is the shell density and $\{F\}$ is the vector, which considers the external forces per unit volume applied over the shell wall by the fluid. The solution of the problem of a vibrating vessel filled with fluid may be stablished as an iterative solution. The fluid differential equations may be represented as

$$[A]\{\ddot{c}\}+[V]\{\dot{c}\}+[P]\{p\}=\{S\} \tag{27}$$

where $\{c\}$ and $\{\ddot{c}\}$ are the fluid displacement and acceleration vectors respectively.
The necessary boundary conditions of one equation, say (26), are the results of the other one (27) and vice versa. First the values in {R} may be set to zero and equation (26) may be solved for an initial disturbance on the sell wall. With the known velocities at the shell nodes, the boundary conditions for the fluid equation may be set, so that, equation (27) may be solved for velocities, accelerations and pressures in the fluid medium. The pressures at the boundary (shell-fluid interface) will allow (26) to be solved for the nest time step. This procedure was repeated for a defined duration of time and the response of the system may be thus analysed.
To solve shell matrix differential equation (26) the Newmark scheme may be employed [7], so that

$$\{\dot{c}\}^{n+1}=\{c\}^n+[(1-\alpha)\{\ddot{c}\}^n+\alpha\{\ddot{c}\}^{n+1}]\Delta t \tag{28}$$

$$\{c\}^{n+1}=\{c\}^n+\{\dot{c}\}^n\Delta t+[(\frac{1}{2}-\beta)\{\ddot{c}\}^n+\beta\{\ddot{c}\}^{n+1}](\Delta t)^2 \tag{29}$$

where $\alpha=0.5$ and $\beta=0.25$ give a constant average acceleration method. The superscripts n and n+1 refer to the time steps t_n and t_{n+1} with $\Delta t=t_{n+1}-t_n$.
Rearranging equations (26), (28) and (29), one arrives at

$$[\overline{K}]\{c\}^{n+1}=[\overline{R}] \tag{30}$$

where

$$[\overline{K}]=[K]+\frac{4}{\Delta t^2}[M]$$

$$\{\overline{R}\}=\{R\}^{n+1}+[M](\frac{4}{\Delta t^2}\{c\}^n+\frac{4}{\Delta t}\{\dot{c}\}^n+2\{\ddot{c}\}^n)$$

For the fluid equations, the Backward Euler's scheme [8] may be adopted, so that

$$\{\dot{c}\}^{n-1}=\frac{(\{c\}^{n+1}-\{c\}^n)}{\Delta t} \tag{31}$$

$$\{\ddot{c}\}^{n+1}=\frac{(\{\dot{c}\}^{n+1}-\{\dot{c}\}^{n})}{\Delta t} \tag{32}$$

Replacing equations (31) and (32) in the matrix fluid equation (27) it follows that

$$[\overline{V}]\{\dot{c}\}^{n+1}+[P]\{p\}^{n-1}=\{\overline{S}\} \tag{33}$$

where

$$[\overline{V}]=\frac{1}{\Delta t}[A]+[V]$$

$$\{\overline{S}\}=\{S\}^{n-1}+\frac{1}{\Delta t}[A]\{\dot{c}\}^{n}$$

The first and second terms of equation (33) may be adequately assembled in order to give a set of equations dependent on {C} and {p} simultaneously

RESULTS

Figure 2 shows a schematic drawing of the cylinder with an inner rigid core and without the inner core. Two Finite Element meshes were considered for each case. For both cases a 36 elements mesh with a

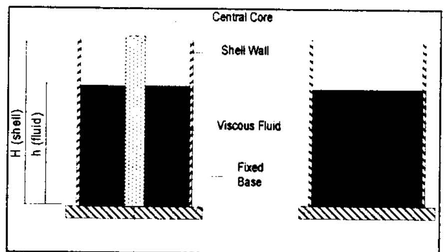

Figure 2 – Partially filled cylinder with and without inner core.

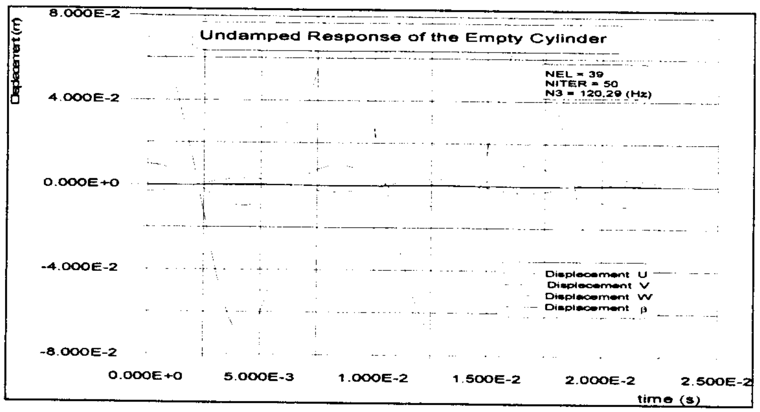

Figure 3 – Undumped response of the empty cylinder. n=3.

distribution 3 elements in the horizontal direction and a 12 element for the vertical direction were considered. The material of the cylinder was a rigid PVC with the following physical and mechanical properties: $E_{(pvc)}=2.70\mathrm{x}10^{9}(N/m^2)$, $\nu_{(pvc)}=0.38$ and $\rho_{(pvc)}=2.746\mathrm{x}10^{3}(Kg/m^3)$. The geometric characteristics of the shell-central core system was: $h_{(shell)}=2.5\mathrm{x}10^{-3}(m)$ (shell thickness), $R_{(shell)}=9.0\mathrm{x}10^{-2}(m)$, $H_{(shell)}=3.9\mathrm{x}10^{-1}(m)$ and $R_{(core)}=5.2\mathrm{x}10^{-2}(m)$. For the viscous fluid, the following property values were taken into account: $\mu_{(H_2O)}=1.005\mathrm{x}10^{-3}(N/m^2)$, $\rho_{(H_2O)}=9.982\mathrm{x}10^{2}(Kg/m^3)$ and $h_{(fluid)}=3.042\mathrm{x}10^{-1}(m)$ (fluid height).

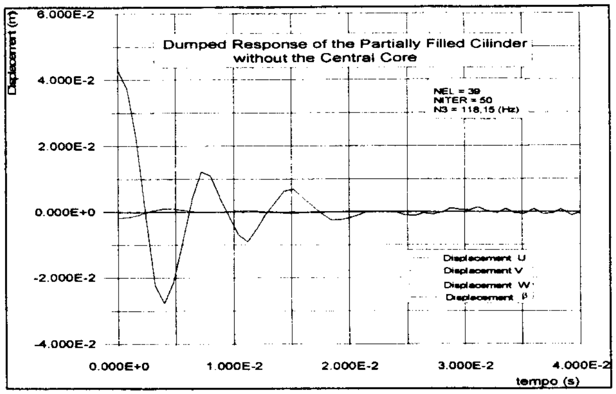

Figure 4 – Dumped response of the partially filled cylinder without the inner core. n=3.

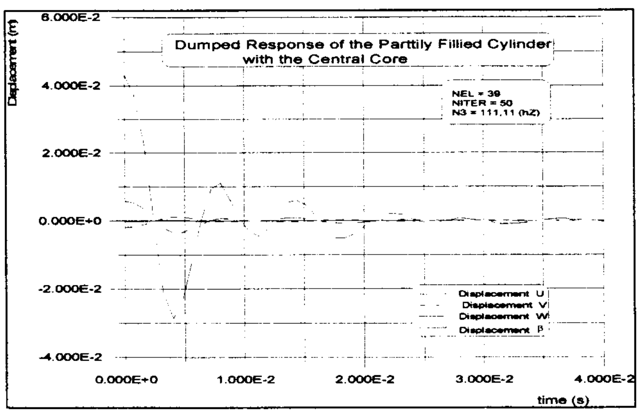

Figure 5 – Dumped response of the partially filled cylinder with the inner core. n=3.

Results were obtained for several wave numbers. As an example, Figures 3, 4 and 5 show some results for the n=3 case. These figures compare the response effects of the presence of the fluid and the inner core compared to the empty shell response. All of these three figures show the time displacements response of a node located at the top free end of the cylinder. To obtain the free vibrational response of the shell, a linear distributed force (zero at the bottom and maximum at the top) was initially applied to the shell for 14 time steps. At the initial time (time=0), the force distribution was withdrawn, the accelerations and velocities were set equal to zero and the shell was released to vibrate. The linear distributed force was applied to excite the meridional "beam mode" , which couples with the circumferential mode with n=3, for the example discussed. Figures 3,4 and 5 show the u, v and w displacements as a function of the time, and with these results the period and the natural frequency of the mode may be numerically and automatically calculated.

CONCLUSIONS

The theoretical analysis of the Navier-Stokes equation have shown that the Finite Element equations are basically linear when wave numbers greater than zero are taken into consideration. A damped type response of the shell-fluid system was observe in all theoretical simulations, which adds reliability to the assumptions made. A very important observation is that the presence of the fluid in the system makes the natural frequency to decrease, which is a very predictable behaviour. Previous results of a similar model have also shown a good degree of correlation between the theoretical predictions when compared with the experimental results [11].

In the case analysed (n=3), results show a certain influence of the presence of the inner core. For this case, without the central core, the presence of the fluid makes the frequency to be reduced from 120,29 Hz to 118,15 Hz. Considering the presence of the rigid core, the frequency is further reduced to 111,11 Hz. Other results, not presented in this article, show a different kind of behaviour, where the presence of the central core alters the frequency to higher levels. The effect of the inner core is therefore not definitive in terms of the frequency alterations.

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