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A GLOBAL HYBRIDIZED MIXED FINITE ELEMENT METHOD FOR INFILTRATION AND GROUNDWATER FLOW MODELLING

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RESUMEN

Se presenta un modelo numérico para la simulación de infiltración y flujo subterráneo unidimensional en medios porosos de saturación variable. El algoritmo consiste en una discretización de la ecuación de Richards que combina una linealización temporal usando un esquema de Picard con una aproximación espacial utilizando un método mixto híbrido de elementos finitos. El algoritmo es computacionalmente eficiente y conservativo. Se incluyen además algunas características relevantes del problema algebraico asociado y un ejemplo numérico de infiltración en una zona de llanura.

ABSTRACT

A numerical model for simulation of one dimensional infiltration and groundwater flow in variably-saturated porous media is presented. The algorithm consists in a discretization of Richards' equation that combines a temporal linearization using a Picard iteration with a spatial approximation employing a hybridized mixed finite element procedure. The algorithm is computationally efficient and mass conservative. Some relevant features of the associated algebraic problem and a numerical example of infiltration in a flatland region are also included.

INTRODUCTION

Prediction of water movement in variably-saturated porous media is an important problem in many branches of science and engineering. The water motion is assumed to obey Richards' equation. This equation may be written in terms of pressure head (p-based form) or water content (θ-based form) as the dependent variable. Only the p-based form of the equation can be used for simulating water flow in soils with saturated regions, but unfortunately this models are inherently non-mass-conserving([1],[2]). Celia et al. [3] greatly improved the performance of p-based models by using an appropiate temporal discretization of a mixed form of Richards' equation. The approximations that are usually applied to the spatial domain are finite difference and finite element standard methods.

The object of this work is to present a numerical model to solve the mixed form of Richards' equation based on a global hybridized mixed finite element procedure. The algorithm produces perfectly mass conservative numerical solutions and it is computationally efficient.

THE DIFFERENTIAL MODEL

We will consider the numerical simulation of underground water flow in a porous domain $\Omega = (0,1)$ with boundary $\partial\Omega = \Gamma^B \cup \Gamma^T$, where $\Gamma^B = \{z=0\}$ and $\Gamma^T = \{z=1\}$. It will be assumed that water flow obeys Richard's equation stated in the form

$$\begin{aligned} &i) \quad \frac{\partial\theta(p)}{\partial t} + \nabla\cdot\vec{q} = 0, \qquad z\in\Omega, \\ &ii) \quad \vec{q} = -K(p)\nabla(p+z), \qquad z\in\Omega, \end{aligned} \tag{1}$$

where θ and p are water content and pressure head, respectively; K is the hydraulic conductivity, which is assumed independent of p for saturated soils but varies strongly with p in unsaturated soils; z denotes the vertical dimension; and t is time.

Equation (1.i) states conservation of mass for the water phase and (1.ii) defines the water flux $\vec{q}$ in terms of Darcy's law. Equations (1) are valid under the following assumptions: the porous media is undeformable; the water density remains constant; and the air mobility is much greater than the water mobility so that the air remains at essentially atmospheric pressure.

We will consider solving (1) with the following boundary conditions:

$$\begin{aligned} \vec{q}\cdot\vec{\nu} &= q_{in}(t), \ \text{ on } \Gamma^T, \\ \vec{q}\cdot\vec{\nu} &= q_{out}(t), \ \text{ on } \Gamma^B. \end{aligned} \tag{2}$$

The function $q_{in}(t)$ represents the rainfall data, while the term $q_{out}(t)$ is used to represent the effect of the regional flow.

To solve the differential problem (1)-(2) we also need additional relations between the dependent variables θ and p. We will use the following water retention and hydraulic conductivity models proposed by van Genutchen [4]:

$$\begin{aligned} \theta(p) &= \frac{\theta_s - \theta_r}{[1+(\alpha|p|)^n]^m} + \theta_r, \\ K(p) &= K_s \frac{\{1-(\alpha|p|)^{n-1}[1+(\alpha|p|)^n]^{-m}\}^2}{[1+(\alpha|p|)^n]^{m/2}} \end{aligned} \tag{3}$$

where $m = 1 - \frac{1}{n}$; θ_r and θ_s are the residual and saturated water contens, respectively; $K_s = K_s(z)$ is the saturated hydraulic conductivity; and α and n are model parameters determined by laboratory experiments ([5],[6]).

SOLVING THE DIFFERENTIAL PROBLEM

Time Discretization

Temporal discretization of (1) using a backward Euler method coupled with a Picard iteration scheme may be written as follow:

$$\begin{aligned} &i) \quad \frac{\theta^{n+1,i+1} - \theta^n}{\Delta t} + \nabla\cdot\vec{q}^{\,n+1,i+1} = 0, \qquad z\in\Omega, \\ &ii) \quad \vec{q}^{\,n+1,i+1} = -K^{n+1,m}\nabla(p^{n+1,i+1}+z), \qquad z\in\Omega, \end{aligned} \tag{4}$$

where superscript n and i denote time and iteration level, respectively; $\Delta t = t^{n+1} - t^n$ is the time step; $\theta^{n+1,i+1} = \theta(p^{n+1,i+1})$ and $K^{n+1,i} = K(p^{n+1,i})$.

Following [3] we expand $\theta^{n+1,i+1}$ in a truncated Taylor series with respect to p,

$$\theta^{n+1,i+1} \sim \theta^{n+1,i} + C^{n+1,i}(p^{n+1,i+1} - p^{n+1,i}) \tag{5}$$

being $C^{n+1,i} = \frac{\partial\theta}{\partial p}\Big|^{n+1,i}$.

Using (5) in (4) and rewriting the equations in terms of the increment $\delta p^{i+1} = p^{n+1,i+1} - p^{n+1,i}$ we obtain:

$$\begin{aligned} &i) \quad \frac{\theta^{n+1,i} - \theta^n}{\Delta t} + \frac{C^{n+1,i}}{\Delta t}\delta p^{i+1} + \nabla \cdot \vec{q}^{\,n+1,i+1} = 0, \qquad z \in \Omega, \\ &ii) \quad \vec{q}^{\,n+1,i+1} = -K^{n+1,i}\nabla(p^{n+1,i} + \delta p^{i+1} + z), \qquad z \in \Omega. \end{aligned} \tag{6}$$

The next step will be define a spatial approximation of (6) using a global hybridized mixed finite element procedure.

A Mixed Weak Formulation

Let us introduce some notation. For all nonnegative integers s, let $(H^s(\Omega), \|\cdot\|_s)$ denote the usual Sobolev space. In particular, $H^0(\Omega) = L^2(\Omega)$ and $\| \|_0$ is the usual L^2–norm, with inner product

$$(v, w) = \int_\Omega v\; w\, dz.$$

Also, for notational convenience, let

$$\langle v, w\rangle_\Gamma = v(0)w(0) + v(1)w(1),$$

denote the inner product on $L^2(\Gamma)$, with the associated norm denoted by $|\cdot|_{0,\Gamma} = (\langle\cdot,\cdot\rangle_\Gamma)^{1/2}$. Let

$$\begin{aligned} V &= \{\vec{v} \in H(div, \Omega) : \vec{v}\cdot\vec{\nu} = 0 \text{ on } \partial\Omega\}, \\ W &= \{\psi \in L^2(\Omega)\}, \end{aligned}$$

provided with the natural norm.

Thus we can state a mixed weak formulation for problem (1)–(2) as follows: Assume that $(\vec{q}^{\,n}, p^n) \in V \times W$ are known and $\vec{q}^{\,n}\cdot\vec{\nu}$ satisfy (2). Then, given $(\vec{q}^{\,n+1,0}, p^{n+1,0}) \in V \times W$ find $(\vec{q}^{\,n+1,i+1}, \delta p^{i+1}) \in V \times W$ such that $\vec{q}^{\,n+1,i+1}\cdot\vec{\nu}$ satisfy (2) and

$$\begin{aligned} &i) \quad \left(\frac{\theta^{n+1,i} - \theta^n}{\Delta t}, \psi\right) + \left(\frac{C^{n+1,i}}{\Delta t}\delta p^{i+1}, \psi\right) + (\nabla\cdot\vec{q}^{\,n+1,i+1}, \psi) = 0, \qquad \psi \in W, \\ &ii) \quad \left(\frac{\vec{q}^{\,n+1,i+1}}{K^{n+1,i}}, \vec{v}\right) - (\delta p^{i+1}, \nabla\cdot\vec{v}) - (p^{n+1,i}, \nabla\cdot\vec{v}) + (\nabla z, \vec{v}) = 0, \qquad \vec{v} \in V \end{aligned} \tag{7}$$

In the next section we will solve approximately (7) using global and hybridized global mixed procedures.

A Global Mixed and Hybridized Mixed Procedure

Let us consider a nonoverlapping partition τ_h^{Nz} of Ω into subintervals $\Omega_k = (z_k, z_{k+1})$:

$$\Omega = \bigcup_{k=1}^{N_z} \partial\Omega_k\ ; \quad \Omega_k \cap \Omega_l = \emptyset \quad k \neq l.$$

Set $h_k = z_{k+1} - z_k$ and $h = \max_k h_k$. Let

$$\begin{aligned} V^h &= \{\vec{v} \in H(\mathrm{div}, \Omega) : \vec{v}|_{\Omega_k} \in V_k^h \quad \text{and } \vec{v} \cdot \vec{\nu} = 0 \text{ on } \Gamma\}, \\ W^h &= \{\psi \in L^2(\Omega) : \psi|_{\Omega_k} \in W_k^h\}, \end{aligned}$$

where $V_k^h = P_1(\Omega_k)$ and $W_k^h = P_0(\Omega_k)$. Here $P_m(\Omega_k)$ denotes the polynomials of degree not greater than m in Ω_k.

Then the global mixed finite element procedure for (7) can be stated as follows: Let $(\vec{q}^{\,h,n}, p^{h,n}) \in V^h \times W^h$ be given and such that $\vec{q}^{\,h,n} \cdot \vec{\nu}$ satisfy (2). Then, given $(\vec{q}^{\,h,n+1,0}, p^{h,n+1,0}) \in V^h \times W^h$, find $(\vec{q}^{\,h,n+1,i+1}, \delta p^{h,i+1}) \in V^h \times W^h$ such that $\vec{q}^{\,h,n+1,i+1} \cdot \vec{\nu}$ satisfy (2) and

$$\begin{aligned} &i) \quad \left(\frac{\theta^{h,n+1,i} - \theta^{h,n}}{\Delta t}, \psi\right) + \left(\frac{C^{h,n+1,i}}{\Delta t}\delta p^{h,i+1}, \psi\right) + (\nabla \cdot \vec{q}^{\;n+1,i+1}, \psi) = 0, \qquad \psi \in W^h, \\ &ii) \quad \left(\frac{\vec{q}^{\,h,n+1,i+1}}{K^{h,n+1,i}}, \vec{v}\right) - (\delta p^{h,i+1}, \nabla \cdot \vec{v}) - (p^{h,n+1,i}, \nabla \cdot \vec{v}) + (\nabla z, \vec{v}) = 0, \qquad \vec{v} \in V^h. \end{aligned} \tag{8}$$

In order to define a global hybridized procedure, following ([7],[8]) we will remove the constrain imposing the continuity of the normal components of the flux across the interior boundaries $\Gamma_k = \partial\Omega_{k-1} \cap \partial\Omega_k, k = 2, \cdots, N_z$. We also introduce a space of Lagrange multipliers Λ^h which elements λ^h will be associated with the pressure head values at the interior boundaries Γ_k. Thus, let

$$\begin{aligned} \Lambda^h &= \{\lambda^h : \lambda^h|_{\Gamma_k} = \lambda_k^h \in P_0(\Gamma_k), \quad k = 2, \cdots, N_z\}, \\ V_{-1}^h &= \{\vec{v} \in L^2(\Omega) : \vec{v}|_{\Omega_k} \in V_k^h \quad \text{and } \vec{v} \cdot \vec{\nu} = 0 \text{ on } \partial\Omega\}. \end{aligned}$$

The global hybridized mixed finite element procedure is defined in the following fashion : Let $(\vec{Q}^{h,n}, P^{h,n}, \lambda^h) \in V_{-1}^h \times W^h \times \Lambda^h$ be given and such that $\vec{Q}^{h,n} \cdot \vec{\nu}$ satisfy (2). Then, given $(\vec{Q}^{h,n+1,0}, P^{h,n+1,0}, \lambda^{h,n+1,0}) \in V_{-1}^h \times W^h \times \Lambda^h$, find $(\vec{Q}^{h,n+1,i+1}, P^{h,n+1,i+1}, \lambda^{h,n+1,i+1}) \in V_{-1}^h \times W^h \times \Lambda^h$ such that $\vec{Q}^{h,n+1,i+1} \cdot \vec{\nu}$ satisfy (2) and

$$\begin{aligned} &i) \quad \left(\frac{\theta^{h,n+1,i} - \theta^{h,n}}{\Delta t}, \psi\right) + \left(\frac{C^{h,n+1,i}}{\Delta t}\delta P^{h,i+1}, \psi\right) + (\nabla \cdot \vec{Q}^{\;n+1,i+1}, \psi) = 0, \quad \psi \in W^h, \\ &ii) \quad \left(\frac{\vec{Q}^{h,n+1,i+1}}{K^{h,n+1,i}}, \vec{v}\right) - (\delta P^{h,i+1}, \nabla \cdot \vec{v}) - (P^{h,n+1,i}, \nabla \cdot \vec{v}) + \sum_{k=2}^{N_z} \langle \lambda_k^{h,n+1,i+1}, \vec{v} \cdot \vec{\nu} \rangle_{\Gamma_k} \\ &\qquad + (\nabla z, \vec{v}) = 0, \qquad \vec{v} \in V^h, \\ &iii) \quad \sum_{k=2}^{N_z} \langle \mu^h, \vec{Q}^{h,n+1,i+1} \cdot \vec{\nu} \rangle_{\Gamma_k} = 0, \quad \mu^h \in \Lambda^h. \end{aligned} \tag{9}$$

It can be shown that problem (9) has a unique solution. Moreover, the solution $(\vec{Q}^{h,n+1,i+1}, P^{h,n+1,i+1}) \in V_{-1}^h \times W^h$ coincides with the solution $(\vec{q}^{\,h,n+1,i+1}, p^{h,n+1,i+1}) \in V^h \times W^h$ of problem (8) [7].

Algebraic Problem associated with the Global Hybridized Procedure

Let us describe the algebraic problem associated with (9). First note that $V_k^h = span\{\varphi_k^L, \varphi_k^R\}$, and $W_k^h = span\{\psi_k\}$, where

$$\varphi_k^L(z) = \begin{cases} -1 + \frac{z-z_k}{h_k}, & z \in \Omega_k \\ 0, & z \notin \Omega_k \end{cases} \qquad \varphi_k^R(z) = \begin{cases} \frac{z-z_k}{h_k}, & z \in \Omega_k \\ 0 & z \notin \Omega_k \end{cases} \qquad \psi_k(z) = \begin{cases} 1, & z \in \Omega_k \\ 0, & z \notin \Omega_k \end{cases}$$

Set

$$P_k^{h,n+1,i+1}(z) = P_k^{n+1,i+1}\psi_k(z), \quad k = 1, \cdots, N_z,$$
$$\vec{Q}_k^{h,n+1,i+1}(z) = Q_k^{L,n+1,i+1}\varphi_k^L(z) + Q_k^{R,n+1,i+1}\varphi_k^R(z), \quad , k = 1, \cdots, N_z.$$

Choose $\psi = \psi_k$ in $(9.i)$ to get

$$\frac{\theta^{n+1,i} - \theta^n}{\Delta t} h_k + \frac{C^{n+1,i}}{\Delta t} h_k \delta P_k^{i+1} + Q_k^{L,n+1,i+1} + Q_k^{R,n+1,i+1} = 0, \quad k = 1, \cdots, Nz \tag{10}$$

Then, take $\vec{v} = \varphi_k^L$ in $(9,i)$ and $\vec{v} = \varphi_k^R$ in $(9,ii)$, and apply a trapezoidal rule to compute the first term in $(9.ii)$ to obtain

$$i) \quad Q_k^{L,n+1,i+1} = \frac{2K_k^{L,n+1,i}}{h_k}[\delta P_k^{i+1} + P_k^{n+1,i} - \lambda_k^{n+1,i+1} + \frac{h_k}{2}], \quad k = 2, \cdots, N_z, \tag{11}$$
$$ii) \quad Q_k^{R,n+1,i+1} = \frac{2K_k^{R,n+1,i}}{h_k}[\delta P_k^{i+1} + P_k^{n+1,i} - \lambda_{k+1}^{n+1,i+1} - \frac{h_k}{2}], \quad k = 1, \cdots, N_z - 1.$$

where $K_k^{L,n+1,i}$ and $K_k^{R,n+1,i}$ are de hydraulic conductivity on the left and right borders of Ω_k evaluated using $\lambda_k^{n+1,i}$ and $\lambda_{k+1}^{n+1,i}$, respectively.
Next note that $(9.iii)$ is equivalent to

$$Q_k^{L,n+1,i+1} + Q_{k-1}^{R,n+1,i+1} = 0, \quad k = 2, \cdots, N_z. \tag{12}$$

Using (11) in (12) we get the following expresion for the the Lagrange multipliers in terms of δP_k^{i+1}

$$\lambda_k^{n+1,i+1} = A_k \left[\frac{K_k^{L,n+1,i}}{h_k}(\delta P_k^{i+1} + P_k^{n+1,i} + \frac{h_k}{2}) + \frac{K_{k-1}^{R,n+1,i}}{h_{k-1}}(\delta P_{k-1}^{i+1} + P_{k-1}^{n+1,i} - \frac{h_{k-1}}{2}) \right] \tag{13}$$

where

$$A_k = \frac{h_k h_{k-1}}{h_{k-1}K_k^{L,n+1,i} + h_k K_{k-1}^{R,n+1,i}}$$

Finally using (11) and (13) in (10) we obtain a tridiagonal system of equations for the unknowns $\delta P_k^{i+1}, k = 1, \cdots, N_z$.
The steps in a full time iteration can be indicated as follow:

i) Give as initial guess for $(P_k^{n+1,i}, \lambda_k^{n+1,i})$ the previous time solution (P_k^n, λ_k^n).
ii) Solve the tridiagonal system to obtain δP_k^{i+1}, $k = 1, \cdots, N_z$.
iii) Update Lagrange multiplier $\lambda_k^{n+1,i+1}$ using (13).
iv) Check the convergence for $P_k^{n+1,i+1}$. If it has not been achieved, shift $P_k^{n+1,i}$ using $P_k^{n+1,i} = P_k^{n+1,i} + \delta P_k^{i+1}$ and start a new iteration (go to ii)).
v) When the converge has been achieved the fluxes can be computed using (11).

We implemented a dynamic time step control which significantly improved the CPU efficiency. The time step is increased whenever the Picard scheme converge in less than 3 iterations; Δt is decreased whenever the number of iterations is greater than 10. The automatic time adjustments is stopped when the time step becames either smaller or greater than preselected minimun and maximun step sizes.

NUMERICAL EXAMPLE

The algorithm have been used to simulate the infiltration and variation of water table lavel in a flatland region in the Province of Buenos Aires.

The hydraulic parameters of the soils were taken from the example in the work by Celia et al. [3]. The domain lenght is 4 meters and the size mesh is 1 cm. The monthly average net rainfall was used at the upper boundary (q_{in}) and an extimated value of regional flux was applied at the lower boundary (q_{out}). In order to start the numerical simulation a hydrostatic initial condition with the water table at 2.6 meters from the surface was chosen. We simulate 8 years, from 1972 to 1979.

Figure 1 shows saturation profiles for severals times of the simulation. The evolution of water table levels in the period and mesured field data are shown in Figure 2. Satisfactory agreement is achieved.

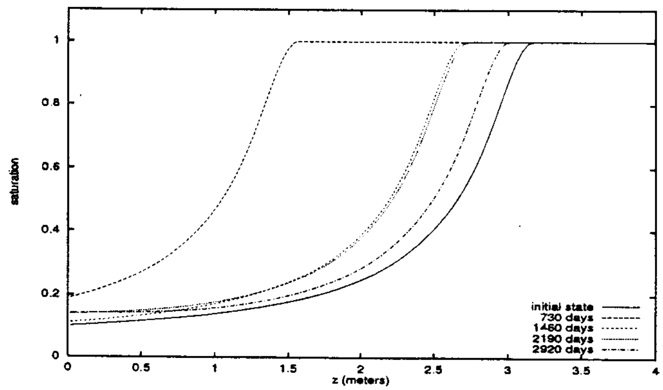

Figure 1: Saturation profiles

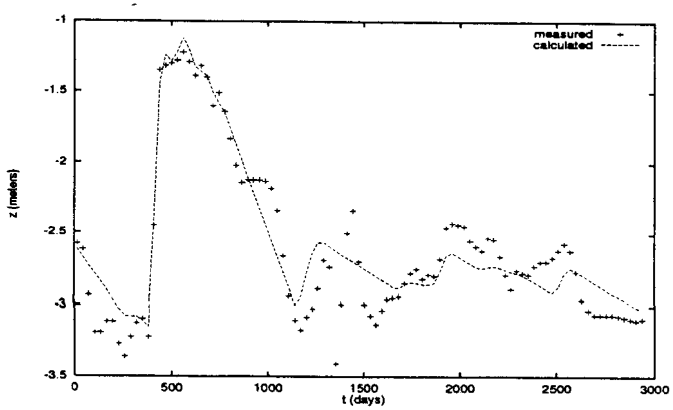

Figure 2: Time evolution of water table comparing the numerical results with measured data

No plot of mass balance as a function of time is provided because the mass balance ratio (total additional mass in the domain / total net flux into the domain) is always unity.

CONCLUSIONS

We have present a numerical algorithm for simulation of 1-D infiltration and groundwater flow in variably-saturated porous media.

The method solves the mixed form of Richards' equation using a Picard linearization in time and a global hybridized mixed finite element procedure. Numerical results show that the algorithm produces solutions that are essentially mass conservative. Implementation of dynamic time step control greatly improved the CPU efficiency.

From the numerical example we can see that the algorithm can be a powerful tool to predict water movement in flatlands regions.

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