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NONCONFORMING ITERATIVE DOMAIN DECOMPOSITION PROCEDURES FOR THE SIMULATION OF WAVES IN FLUID–SATURATED POROUS SOLIDS

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Abstract

The numerical simulation of waves in a dispersive porous solid saturated by either single–pahse or a two–phase fluids is accomplished by solving a collection of non-coercive elliptic problems in the space-frequency domain for a finite number of frequencies. For each frequency, the space–frequency domain solution is computed approximately using non–conforming finite element techniques. These numerical procedure is an iterative, hybridized non–conforming domain decomposition algorithm. To obtain the space-time solution, an approximation to the inverse Fourier transform is employed. Numerical examples showing the implementation of the procedure are also presented.

1. Introduction.

The study of attenuation of seismic waves yields important information about rock properties, such as saturation levels, lithology, and porosity distribution. These effects are more often described better in the space-frequency domain than in the space-time domain. The object of this paper is to present a non–conforming iterative finite element domain decomposition procedure to solve a model problem describing the propagation of pressure waves in a two-dimensional, bounded, dispersive fluid–saturated porous solid.

This formulation is designed for implementation in parallel architectures. The idea is to solve in parallel a collection of non-coercive elliptic problems subject to absorbing boundary conditions at artificial boundaries and then to obtain the space-time solution using an approximation to the inverse Fourier transform.

This approach has already been presented by the authors and some of their colleagues to find solutions for acoustic and elastic waves with another type of dissipative behaviour and another technique to solve the algebraic equations arising in the elliptic problems generated by the space-frequency domain approach; see [9],[8],[18],[5],[10] . Other domain decomposition procedures related to these include [3],[4],[6],[12],[13],[14] .

The organization of the paper is as follows. In §2 we present the differential model problem. Then in §3 we derive a formally equivalent domain decomposition at the differential

level and present an iterative hybridized non–conforming domain decomposition algorithm. Finally, in §4 we present the results of experimental calculations.

2. The Differential Model.

Let $\Omega = (0,1)^2$ and $\Gamma = \partial\Omega$.
Consider the following problem. Find $u(x,\omega)$ such that

$$\begin{aligned} &\text{i)} \quad \frac{-\omega^2 u(x,\omega)}{K(x,\omega)} - \nabla\cdot\left(\frac{1}{\rho}(x)\nabla u(x,\omega)\right) = f(x,\omega), \qquad x\in\Omega, \quad \omega\in, \\ &\text{ii)} \quad \frac{\partial u(x,\omega)}{\partial \nu} + i\omega\alpha(x,\omega)u(x,\omega) = 0, \qquad x\in\Gamma, \quad \omega\in. \end{aligned} \tag{2.1}$$

In (2.1), $u(x,\omega)$ represents the Fourier transform of the pressure $u(x,t)$, $\rho(x)$ is the density, ν denotes the unit outer normal on Γ and $K(x,\omega)$ is the complex bulk modulus of the saturated material, which is given by the formulas

$$K = K_s\cdot\frac{K_m+T}{K_s+T}, \qquad T = \frac{K_f(K_s-K_m)}{\phi(K_s-K_f)}, \tag{2.2}$$

where

$$K_s(x,\omega) = \frac{K_s(x,0)}{A_s(\omega)-iB_s(\omega)} \qquad K_m(x,\omega) = \frac{K_m(x,0)}{A_m(\omega)-iB_m(\omega)}$$

In the above relations, $\phi = \phi(x)$ denotes the effective porosity and K_f the bulk modulus of the saturant fluid. Also, $K_s(x,0)$ and $K_m(x,0)$ denote the relaxed bulk modulus of the solid grains and the the dry matrix, repectively.
The coefficients $A_j(\omega)$ and $B_j(\omega)$, $j = s,m$, associated with a continuous distribution of relaxation times, characterize the viscoelastic behaviour of the material and are given by (see [15],[16])

$$A_j(\omega) = 1 - \frac{1}{\pi S_{av,j}}\ln\frac{1+\omega^2\tau_{1,j}^2}{1+\omega^2\tau_{2,j}^2}, \tag{2.3a}$$

$$B_j(\omega) = \frac{2}{\pi S_{av,j}}\tan^{-1}\frac{\omega(\tau_{1,j}-\tau_{2,j})}{1+\omega^2\tau_{1,j}\tau_{2,j}}; \tag{2.3b}$$

In (2.3) $\tau_{1,j}$ and $\tau_{2,j}$ are given angular frequencies such that the quality factor $S_j(\omega) = \frac{B_j(\omega)}{A_j(\omega)}$ is approximately equal to $S_{av,j}$ in the range $\tau_{1,j}^{-1} \le \omega \le \tau_{2,j}^{-1}$. Realistic values for $S_{av,j}$ in rocks are in the range 40 to 1000.
For the non–disipative case formulas (2.2) reduce to those derived by Gassmann in [11], which are valid under the assumptions that in the low frequency seismic range the relative motion between the fluid and the solid frame can be ignored, the shear modulus of the saturated rock coincides with that of the solid skeleton and that all the poral space is filled with a single–phase fluid (capillary forces are ignored). Formulas (2.2) are an extension of those due to Gassmann obtained by using the Correspondence Principle [1] .

In the case in which the porous solid is saturated by a two–phase fluid, The bulk modulus of the saturated porous solid can be computed using the following expressions [17] :

$$K = K_s \cdot \frac{K_m + \widetilde{T}}{K_s + \widetilde{T}}, \qquad \widetilde{T} = \frac{\widetilde{K}_f(K_s - K_m)}{\phi(K_s - \widetilde{K}_f)},$$

where

$$\frac{1}{\widetilde{K}_f} = \frac{1}{\epsilon}\left(\frac{\delta S_n}{K_n} + \frac{S_w}{K_w}\right), \qquad \delta = \frac{1 + p_c'(S_n)S_nS_w/K_w)}{1 + p_c'(S_n)S_nS_w/K_n)},$$

and

$$\epsilon = 1 + (\delta - 1)(S_n + p_c(S_n)/p_c'(S_n)).$$

Here $p_c(S_n)$ denotes the capillary pressure function, and S_w, K_w, S_n and K_n are the saturation and bulk modulus of the wetting and non–wetting phases, respectively.
Equation (2.1.ii) is a first order absorbing boundary condition obtained by imposing the condition that the boundary Γ be transparent for normally arriving waves. Its derivation can be found in [16] . The complex coefficient $\alpha(x,\omega)$ in (2.1.ii) can be written as

$$\alpha(x,\omega) = (M(x,\omega) - iN(x,\omega)), \tag{2.4}$$

with $M(x,\omega)$ and $N(x,\omega)$ being given by

$$M(x,\omega) = C_r\big(2(C_r^4 + C_i^4)\big)^{-1/2}\left[1 + \big(1 + (C_i/C_r)^4\big)^{1/2}\right]^{1/2},$$

$$N(x,\omega) = \frac{C_i^2}{C_r}\big(2(C_r^4 + C_i^4)\big)^{-1/2}\left[1 + \big(1 + (C_i/C_r)^4\big)^{1/2}\right]^{-1/2},$$

and

$$C_r^2(x,\omega) = K_r(x,\omega)/\rho(x), \qquad C_i^2(x,\omega) = K_i(x,\omega)/\rho(x).$$

It can be shown that problem is well posed [8] .

3. An Iterative Hybridized Non–Conforming Finite Element Domain Decomposition Procedure.

We will consider a nonconforming finite element approximate solution of (2.1) using a nonconforming finite element space constructed using the following reference rectangular element

$$\widehat{R} = [-1,1]^2, \qquad Q = \mathrm{Span}\{1, x, y, (x^2 - \frac{5}{3}x^4) - (y^2 - \frac{5}{3}y^4)\},$$

with the degrees of freedom being the values at the nodal points

$$a_1 = (0,-1), \qquad a_2 = (1,0), \qquad a_3 = (0,1), \qquad a_4 = (-1,0).$$

Any $q \in Q$ is uniquely defined by its values at the nodal points $a_i, 1 \leq i \leq 4$.
Let τ^h be a partition of Ω into squares $(\Omega_j)_{1 \leq j \leq J}$ of side length h. Let $\Gamma_j = \partial\Omega_j \cap \Gamma$ denote the boundary of Ω_j and let $\Gamma_{jk} = \partial\Omega_j \cap \partial\Omega_k = \Gamma_{kj}$ denote the common face of adjacent elements Ω_j and Ω_k. (see Figure 1)

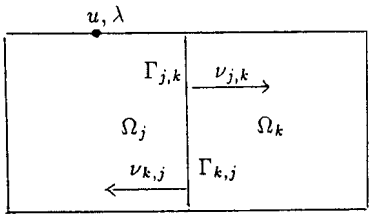

Figure 1. Mesh description

Consider the decomposition of problem (2.1) over Ω_j as follows: for $j = 1, \ldots, J$, find $u_j(x,\omega)$ satisfying

$$\begin{aligned} &\text{i)} \quad \frac{-\omega^2 u_j(x,\omega)}{K(x,\omega)} - \nabla \cdot \left(\frac{1}{\rho} \nabla u_j(x,\omega) \right) = f(x,\omega), \qquad x \in \Omega_j, \\ &\text{ii)} \quad \frac{\partial u_j(x,\omega)}{\partial \nu_{jk}} + i\omega\alpha(x,\omega) u_j(x,\omega) = 0, \qquad x \in \Gamma_j, \end{aligned} \tag{3.1}$$

with the consistency conditions

$$u_j(x,\omega) = u_k(x,\omega), \qquad x \in \Gamma_{jk}, \tag{3.2}$$

and

$$\frac{1}{\rho} \frac{\partial u_j(x,\omega)}{\partial \nu_{jk}} = -\frac{1}{\rho} \frac{\partial u_k(x,\omega)}{\partial \nu_{kj}}, \qquad x \in \Gamma_{jk}. \tag{3.3}$$

Instead of (3.2)–(3.3) we will impose the nonconforming Robin transmission boundary conditions

$$\frac{1}{\rho} \frac{\partial u_j(m_{jk})}{\partial \nu_{jk}} + i\beta_{jk} u_j(m_{jk}) = -\frac{1}{\rho} \frac{\partial u_k(m_{jk})}{\partial \nu_{kj}} + i\beta_{jk} u_k(m_{kj}),$$
$$x \in \Gamma_{jk} \subset \partial\Omega_j,$$

and

$$\frac{1}{\rho} \frac{\partial u_k(m_{kj})}{\partial \nu_{kj}} + i\beta_{jk} u_k(m_{kj}) = -\frac{1}{\rho} \frac{\partial u_j(m_{jk})}{\partial \nu_{jk}} + i\beta_{jk} u_j(m_{jk}),$$
$$x \in \Gamma_{kj} \subset \partial\Omega_k,$$

with β_{jk} being a positive function defined on the interior boundaries Γ_{jk}.

Since the object of the domain decomposition procedure is to localize the calculations, we define the iterative procedure at the differential level in the following fashion: Given $u_j^0 \in H^1(\Omega_{jk})$, find $u_j^n \in H^1(\Omega_j)$ such that

$$(-\frac{\omega^2}{K(x,\omega)}u_j^n,\varphi)_j + \left(\frac{1}{\rho}\nabla u_j^n,\nabla\varphi\right)_j + \left\langle\left\langle i\omega\frac{\alpha}{\rho}u_j^n,\varphi\right\rangle\right\rangle_{\Gamma_j}$$
$$+\sum_k \langle\langle[\frac{1}{\rho}\frac{\partial u_k^{n-1}(m_{kj})}{\partial\nu_{kj}} + i\beta_{jk}(u_j^n - u_k^{n-1})],\varphi\rangle\rangle_{\Gamma_{jk}} = (f,\varphi)_j, \quad \varphi \in H^1(\Omega_j). \tag{3.4}$$

We will denote by $(\cdot,\cdot)_j$ and $\langle\cdot,\cdot\rangle_{\Gamma_{jk}}$ the inner products in $L^2(\Omega_j)$ and $L^2(\Gamma_{jk})$, respectively. Also, let $\langle\langle\cdot,\cdot\rangle\rangle_{\Gamma_{jk}}$ denote the approximation to the inner product $\langle\cdot,\cdot\rangle_{\Gamma_{jk}}$ computed using the mid-point rule; i.e., if $m_{jk} = m_{kj}$ is the mid-point of the interface Γ_{jk}, then

$$\langle\langle u,v\rangle\rangle_{\Gamma_{jk}} = |\Gamma_{jk}|(u\overline{v})(m_{jk}),$$

where $|\Gamma_{jk}|$ denotes the measure of Γ_{jk}.

Next we will define a nonconforming hybrid finite element domain decomposition procedure motivated by (3.4). For that purpose, we introduce a new set Λ^h of Lagrange multipliers λ_{jk}^h associated with the flux values $-\frac{1}{\rho}\frac{\partial u_{jk}}{\partial\nu_{jk}}(m_{jk})$ at the mid-points m_{jk} of the interior faces Γ_{jk} as follows:

$$\Lambda^h = \{\lambda : \lambda|_{\Gamma_{jk}} = \lambda_{jk} \in P_0(\Gamma_{jk}) \equiv \Lambda_{jk}\}.$$

Also, set a nonconforming finite element space NC^h as follows.

$$NC^h = \left\{v \in L^2(\Omega) : v|_{\Omega_j} \in Q(\Omega_j)\right\}.$$

The iterative hybridized nonconforming finite element domain decomposition procedure is defined as follows: choose $\left(u_j^{h,0},\lambda_{jk}^{h,0},\lambda_{kj}^{h,0}\right) \in NC_j^h \times \Lambda_{jk} \times \Lambda_{kj}$ arbitrarily. Then, compute $\left(u_j^{h,n},\lambda_{jk}^{h,n}\right) \in NC_j^h \times \Lambda_{jk}$ as the solution of the equations

$$(-\frac{\omega^2}{K(x,\omega)}u_j^{h,n},\varphi)_j + \left(\frac{1}{\rho}\nabla u_j^{h,n},\nabla\varphi\right)_j + \left\langle\left\langle i\omega\frac{\alpha}{\rho}u_j^{h,n},\varphi\right\rangle\right\rangle_{\Gamma_j} \tag{3.5}$$
$$+\sum_k \langle\langle\lambda_{jk}^{h,n},\varphi\rangle\rangle_{\Gamma_{jk}} = (f,\varphi)_j, \quad \varphi \in NC_j^h,$$

$$\lambda_{jk}^{h,n} = -\lambda_{kj}^{h,n-1} + i\beta_{jk}[u_j^{h,n}(m_{jk}) - u_k^{h,n-1}(m_{kj})], \quad \text{on } \Gamma_{jk}. \tag{3.6}$$

It can be shown that iterative procedure is convergent [8].

The following alternatives are implemented for improving effectiveness of the algorithm:

i) Relaxation of the solution and the Lagrange multipliers then each iteration.

ii) To use an iterative red-black strips scheme for the calculations of the solution and the Lagrange multipliers.

4. Experimental Calculations.

For the numerical experiments we were chosen a reservoir model and several location parameters are shown in Figure 2.

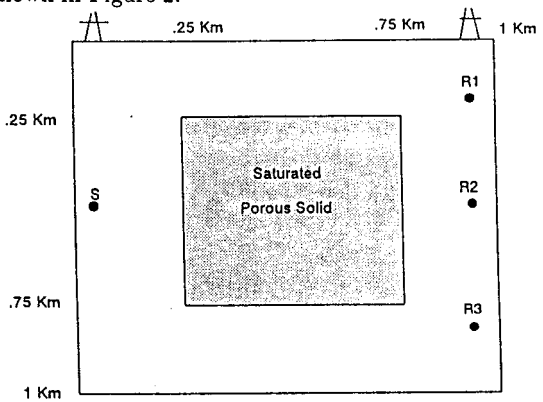

Figure 2. Reservoir model

The source function $\widehat{f}(x,\omega)$ was the Fourier transform of the function [19]

$$f(x,t) = -2\xi(t-t_0)e^{-\xi(t-t_0)^2}\delta(x_1 - x_{1s})\delta(x_2 - x_{2s}), \quad t \geq 0, \tag{4.1}$$

with $\xi = 8f_0^2$, $t_0 = .8/f_0$, $f_0 = 15$ Hz being the main source frequency; $\widehat{f}(x,\omega)$ was filtered linearly between $\omega_* = 30$ Hz and $\omega^* = 35$ Hz. The iterative domain decomposition procedure (3.5)–(3.6) was employed to compute the ap roximate solution u_{jk}^h at a finite number of frequencies between zero and ω^*, and the time domain solution was obtained using an approximation to the inverse Fourier transform. The constants Q_m, τ_1, and τ_2 in (2.3)–(2.4) were chosen to be 70, .1591 10^6 msec, and 10^3 msec, respectively, so that $Q(\omega) \approx Q_m$ in the range $[f_1, f_2] = [10^{-6}$ kHz, 10^3 kHz$]$.

Outside the saturated region the compressional velocity at zero frequency was chosen to be 4.0 Km/sec. In the saturated region the compressional velocity of the dry matrix was taken to be 3.0 Km/sec. The saturant fluids were chosen to be either gas or brine or a gas–brine inmiscible fluid with water saturation $S_w = 0.7$, and the following properties: $\rho_{gas} = 0.1$ g/cm^3, $K_{gas} = 0.022\ 10^{10}$ dyn/cm^2, $\rho_{water} = 1.0$ g/cm^3, $K_{water} = 2.4\ 10^{10}$ dyn/cm^2. For both formations, the bulk modulus K_s and density ρ_s of the solid grains were taken to be $K_s = 37.9\ 10^{10}$ dyn/cm^2 and $\rho_s = 2.65$ g/cm^3. The porosity was 0.3.

Figure 3 shows a snapshot of the real part of the solution $u_{jk}^h(x,\omega)$ at 20 Hz.

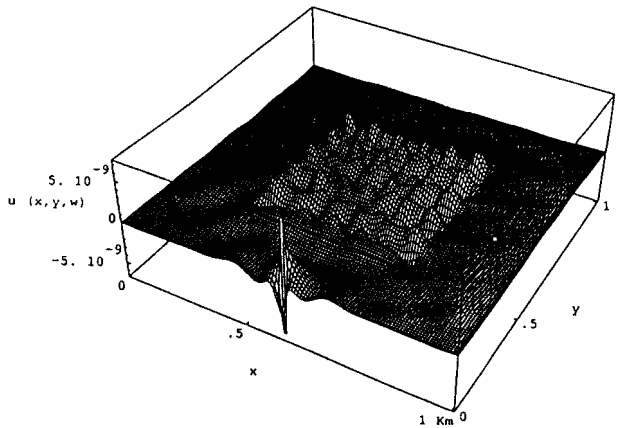

Figure 3. The real part of $u^h_{jk}(x,\omega)$

Figure 4 shows a "slice" of the same solution along the straight path connecting S and R2 for different saturant fluids. The figure show the expected change in character of the solution across the the change in the material properties.

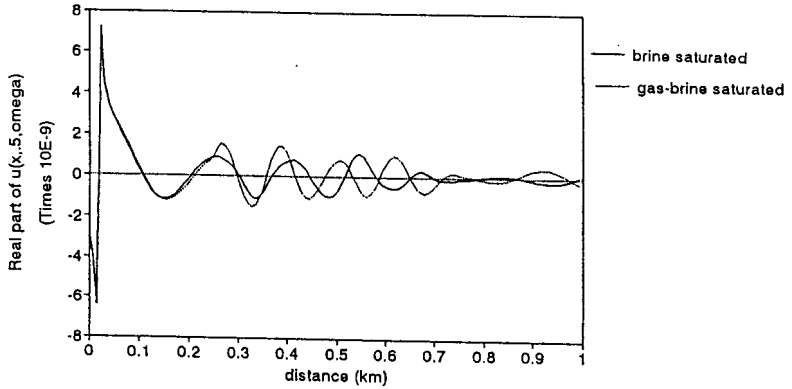

Figure 4. Slice of snapshot for 20 Hz at z=.5 Km

All the numerical tests were run in an IBM SP2 supercomputer with 16 nodes at Purdue University.

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