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NUMERICAL SOLUTION OF A JUNCTION PROBLEM

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ABSTRACT

We consider the numerical solution of a junction problem involving bilateral restrictions and described by a variational inequality (V.I.). After a discretization phase, the resulting discrete V.I. is solved by an algorithm which combines fast methods for solving the bilateral obstacle problems and algorithms of Newton type for solving a convex optimization problem on the set $(\Re^+)^{n+1}$. The algorithm is highly efficient and finds the discrete solution in a finite number of steps.

RESUMEN

En este trabajo consideramos la solución numérica de un problema de junturas descripto por una inecuación variacional (I.V.), que involucra restricciones bilaterales. Después de una etapa de discretización, la I.V. discreta resultante es resuelta por un algoritmo que combina métodos rápidos para resolver problemas bilaterales con dos obstáculos y algoritmos de tipo Newton para resolver un problema de optimización convexa sobre $(\Re^+)^{n+1}$. El algoritmo es muy eficiente y encuentra la solución discreta en un número de pasos.

INTRODUCTION

This paper deals with the numerical computation of the state of a coupled system described by PDE's. A fruitful way to analyze those systems is the variational inequality (V.I.) approach, specially when there are state constraints or connections involving unilateral or bilateral restrictions. This approach can be seen in [1], where several cases are modelized and analyzed with the V.I. method. In this paper we study the numerical analysis of a junction problem involving bilateral restrictions. Using the V.I. formulation, we obtain through a discretization procedure a numerical method to compute the state of the coupled system.
The original problem can be solved by a decomposition–coordination method (see [2] and [3]; the method itself stems from the theory analyzed in [4]). Also the discrete problem can be solved by a method of this type. Our procedure solves the coupled problem through the solution of two simple independent problems – one of them a bilateral obstacle problem and the other one a linear problem. These problems depend on some auxiliary variables which are modified (by a fast coordination procedure) until the desired solution is obtained.

The contents of this paper can be outlined us follows: Section 2 contains the description of the problem and the characterization of its solution. Section 3 presents the relation between the V.I. system and a minimum problem. Section 4 describes the discretization and the numerical algorithm. In Section 5 we present an example of application.

CONTINUOUS PROBLEM DESCRIPTION

We consider $\Omega_1 = [-1,0] \times [0,1] \subset \Re^2$, $\Omega_2 = [0,1] \times [0,1] \subset \Re^2$, $\Gamma_i = \Omega_1 \cap \Omega_2$. Let $p : \Re^2 \mapsto \Gamma_i$ be the function defined by: $p(x_1,x_2) = (0,x_2)$. Let $m(\cdot) \in H^2(\Omega_1)$ and $M(\cdot) \in H^2(\Omega_1)$ be such that $0 \leq m(x) \leq M(x) \leq 1$, $\forall x \in \overline{\Omega}_1$, $m|_{\Gamma_i} = M|_{\Gamma_i} = 1$, $m(x) < M(x)$ in $int(\Omega_1)$. Let $K \subset H^1(\Omega_1) \times H^1(\Omega_2)$ be the convex set

$$K = \{(v_1,v_2) : m(x)\, v_2(p(x)) \leq v_1(x) \leq M(x)\, v_2(p(x)) \quad a.e.\ x \in \Omega_1\}. \qquad (1)$$

Let be $\alpha > 0$, $\beta > 0$, we define the bilinear forms a_1 and a_2:

$$a_1(u_1,v_1) = \int_{\Omega_1} (\nabla u_1 \nabla v_1 + \alpha\, u_1\, v_1)\, dx, \quad a_2(u_2,v_2) = \int_{\Omega_2} (\nabla u_2 \nabla v_2 + \beta\, u_2\, v_2)\, dx. \quad (2)$$

We will also make use of the differential operators $A_1 = -\Delta + \alpha$ and $A_2 = -\Delta + \beta$.

The variational inequality

Find $u = (u_1,u_2) \in K$ such that

$$a_1(u_1, v_1 - u_1) + a_2(u_2, v_2 - u_2) \geq (f_1, v_1 - u_1) + (f_2, v_2 - u_2), \quad \forall\, (v_1,v_2) \in K, \quad (3)$$

where $f_1 \in L^2(\Omega_1)$, $f_2 \in L^2(\Omega_2)$ and (v,w) denotes the inner product in $L^2(\Omega_1)$ or $L^2(\Omega_2)$.

Existence and uniqueness

Since K is a closed convex set and the bilinear form $a_1(u_1,v_1) + a_2(u_2,v_2)$ is coercive then there exists a unique solution $u = (u_1,u_2)$ of (3).

Characterization of the solution

As it is stated in [2] and [3] the solution can be characterized in the following way:

Conditions verified by u_1

1) Case of $u_2(p(x)) = 0 \Rightarrow u_1(x_1,x_2) = 0,\ \forall x_1 \in (-1,0)$.

2) Case of $u_2(p(x)) \neq 0$

Differential conditions

We define

$$\begin{cases} S^+ = \{x \in \Omega_1 : u_1(x) = u_2(p(x))\, M(x)\}\ , \\ S^- = \{x \in \Omega_1 : u_1(x) = u_2(p(x))\, m(x)\}\ , \\ C = \Omega_1 \backslash (S^+ \cup S^-)\ , \end{cases}$$

then the following differential relations hold

$$\begin{cases} A_1 u_1 \geq f_1 & \text{a.e. } x \in S^-, \\ A_1 u_1 \leq f_1 & \text{a.e. } x \in S^+, \\ A_1 u_1 = f_1 & \text{a.e. } x \in C. \end{cases}$$

Boundary conditions

$\forall x \in \Gamma_1 = \partial\Omega_1 \backslash \Gamma_i$ such that $u_2(p(x)) > 0$

$$\begin{cases} \dfrac{\partial u_1}{\partial n}(x) \geq 0\,, & if\ u_1(x) = m(x)\, u_2(p(x))\,, \\ \dfrac{\partial u_1}{\partial n}(x) \leq 0\,, & if\ u_1(x) = M(x)\, u_2(p(x))\,, \\ \dfrac{\partial u_1}{\partial n}(x) = 0\,, & if\ m(x)\, u_2(p(x)) < u_1(x) < M(x)\, u_2(p(x))\,. \end{cases} \tag{4}$$

Conditions verified by u_2

$$\begin{cases} A_2 u_2 = f_2 & \text{on } \Omega_2\,, \\ \dfrac{\partial u_2}{\partial \nu} = 0 & \text{on } \Gamma_2 = \partial\Omega_2 \backslash \Gamma_i\,. \end{cases}$$

The coupling equilibrium conditions at the interaction boundary Γ_i

We present here the coupling conditions that hold at each point $(0, x_2)$ of the interaction boundary Γ_i. We denote $\Omega_1(x_2) = [-1, 0] \times \{x_2\}$.
We define, if $u_2(p(x)) = 0$,

$$(E(u_1\,,\, u_2))\,(x_2) = -\frac{\partial u_2}{\partial x_1}(p(x)) - \int_{\Omega_1(x_2)} (f_1)^+\,(x)\, M(x)\; dx_1 + \int_{\Omega_1(x_2)} (f_1)^-\,(x)\, m(x)\; dx_1$$

and if $u_2(p(x)) > 0$,

$$\begin{aligned} (E(u_1\,,\, u_2))\,(x_2) = \; & -\frac{\partial u_2}{\partial x_1}(p(x)) - \int_{\Omega_1(x_2)} (A_1 u_1 - f_1)^-(x)\, M(x)\; dx_1 \\ & + \int_{\Omega_1(x_2)} (A_1 u_1 - f_1)^+(x)\, m(x)\; dx_1 \\ & - \left(\frac{\partial u_1}{\partial n}\right)^- (-1, x_2)\, M(-1, x_2) + \left(\frac{\partial u_1}{\partial n}\right)^+ (-1, x_2)\, m(-1, x_2) \\ & + \left(\frac{\partial u_1}{\partial n}\right) (0, x_2) \end{aligned} \tag{5}$$

The condition that must be satisfied at Γ_i is the following

$$\min\left(u_2(0,x_2),\left(E(u_1\,,\,u_2)\right)(x_2)\right)=0 \qquad \forall x_2\in(0,1)\,. \tag{6}$$

THE PROBLEM AS A MINIMUM PROBLEM

We will suppose that the bilinear forms a_1 and a_2 are symmetric and we define the functional $J : H^1(\Omega_1)\oplus H^1(\Omega_2)\rightarrow \Re$ in the following way:

$$J(v_1,v_2)=\tfrac{1}{2}\,a_1(v_1,v_1)-(f_1,v_1)+\tfrac{1}{2}\,a_2(v_2,v_2)-(f_2,v_2)\,. \tag{7}$$

In consequence the variational inequality (3) is equivalent to the necessary and sufficient conditions that characterize the point (u_1,u_2) which minimizes the functional J in the set K.

Solution by decomposition

A hierarchical problem

We define the set K_I and, $\forall\, u_I\in K_I$, the associated sets $K_1(u_I)$, $K_2(u_I)$

$$\begin{aligned}
&K_I=\{u_I\in H^{\frac{1}{2}}(\Gamma_i)\;:\;u_I(x)\geq 0\,,\;\forall\, x\in\Gamma_i\}\,,\\
&K_2(u_I)=\{u_2\in H^1(\Omega_2)\;:\;u_2(0,x_2)=u_I(x_2)\;,\;\forall\, x_2\in\Gamma_i\}\,,\\
&K_1(u_I)=\{u_1\in H^1(\Omega_1)\;:\;u_I(p(x))\,m(x)\leq u_1(x)\leq u_I(p(x))\,M(x)\,,\;a.e.\;x\in\Omega_1\}\,.
\end{aligned} \tag{8}$$

We introduce the notation

$$\varphi_1(u_I)=\min_{u_1\in K_1(u_I)} J(u_1,0)\,,\quad \varphi_2(u_I)=\min_{u_2\in K_2(u_I)} J(0,u_2)\,,\quad \varphi(u_I)=\varphi_1(u_I)+\varphi_2(u_I) \tag{9}$$

and to compute the functions φ_1, φ_2 we define the problems:

$$P_1(u_I)\;:\;\textit{Find}\;\bar{u}_1(u_I)\;\;\textit{such that}\;\;J(\bar{u}_1,0)=\varphi_1(u_I)\,, \tag{10}$$

$$P_2(u_I)\;:\;\textit{Find}\;\bar{u}_2(u_I)\;\;\textit{such that}\;\;J(0,\bar{u}_2)=\varphi_2(u_I)\,. \tag{11}$$

We can write $K=\bigcup_{u_I\in K_I}\left(K_1(u_I)\oplus K_2(u_I)\right)$ and in consequence

$$\min_{(u_1,u_2)\in K} J(u_1,u_2)=\min_{u_I\in K_I}\left(\min_{K_1(u_I)\oplus K_2(u_I)} J(u_1,u_2)\right). \tag{12}$$

From (7) we have

$$\begin{aligned}
\min_{K_1(u_I)\oplus K_2(u_I)} J(u_1,u_2) &= \left(\min_{u_1\in K_1(u_I)} J(u_1,0)\right)+\left(\min_{u_2\in K_2(u_I)} J(0,u_2)\right) && (13)\\
&= \varphi_1(u_I)+\varphi_2(u_I)=\varphi(u_I)\,. && (14)
\end{aligned}$$

So,

$$\min_{(u_1,u_2)\in K} J(u_1,u_2) = \min_{u_I\in K_I} \varphi(u_I), \tag{15}$$

and we conclude that problem (3) is equivalent to the following problem P_I :

$$P_I \ : \ \textit{Find } \bar{u}_I \ \textit{ such that } \ \varphi(\bar{u}_I) = \min_{u_I\in K_I} \varphi(u_I)\,. \tag{16}$$

Properties of φ

Properties of φ_1.

- φ_1 is a convex function
- φ_1 is differentiable and its derivative is Lipschitz continuous.
  We define the following operator T_1: $w_1 = T_1(v_I)$ if w_1 is the solution of the elliptic system:
  $$\begin{cases} A_1 w_1 = 0\,, & in\ C\,, \\ w_1 = v_I\, M & in\ S^+\,, \\ w_1 = v_I\, m & in\ S^-\,, \end{cases} \tag{17}$$
  with this definition it is easy to check that the (Frechet) derivative of φ_1 has the following form
  $$\langle D\varphi_1(u_I), v_I\rangle = a_1(u_1(u_I), T_1(v_I)) - (f_1, T_1(v_I)) = (A_1u_1(u_I) - f_1,\, T_1(v_I)). \tag{18}$$
  In an equivalent form we have $D\varphi_1(u_I) = T_1^*\,(A_1u_1(u_I) - f_1)\,.$
- Since $u_1(u_I)$ is a Lipschitz function of u_I, from (18) we can check that $D\varphi_1(u_I)$ is also a Lipschitz function of u_I.

Properties of φ_2.

- φ_2 is a quadratic function
- The derivative of φ_2
  We define the following operator $T_2 : w_2 = T_2(v_I)$ if w_2 is the solution of the elliptic system:
  $$\begin{cases} A_2 w_2 = 0\,, & \\ w_2 = v_I & in\ \Gamma_i\,, \\ \dfrac{\partial w_2}{\partial n} = 0 & in\ \Gamma_2\,, \end{cases} \tag{19}$$
  with this definition it is easy to check that the (Frechet) derivative of φ_2 has the following form
  $$\langle D\varphi_2(u_I), v_I\rangle = a_2(u_2(u_I), T_2(v_I)) - (f_2, T_2(v_I)) = (A_2u_2(u_I) - f_2,\, T_2(v_I)).$$
  In an equivalent form we have $D\varphi_2(u_I) = T_2^*\,(A_2u_2(u_I) - f_2)\,.$

The Hessians of φ_1 and φ_2

It can be proved that the Hessians have the following form

$$H_1 = T_1^* A_1 T_1 \quad \text{and} \qquad H_2 = T_2^* A_2 T_2 \,.$$

Necessary conditions of minimality

If Ψ is the derivative of φ, to find the minimum of φ is equivalent to find the unique value $\bar{u}_I$ such that

$$\min(\bar{u}_I, \Psi(\bar{u}_I)) = \min(\bar{u}_I, \nabla\varphi) = 0\,, \quad \forall x_2 \in (0,1)\,..$$

In fact, this condition is equivalent to condition (6).

DISCRETIZATION

The discretization procedure is similar for both domains Ω_1, Ω_2. Therefore, for the sake of briefness, we will present only the case Ω_2 and to simplify the notation we will omit the subindex 2.

- We make a partition of the domain Ω in n^2 squares with side $h = \dfrac{1}{n}$.
- Each node will be identified by the notation $x_{i,j} = (ih, jh)$ for $i, j = 0, \cdots, n$.
- We define the following characteristic functions
  - $\chi(x_1, x_2) = \chi_{[-1,1]}(x_1) \times \chi_{[-1,1]}(x_2)$
  - $\chi_{i,j}^{n,0}(x_1, x_2) = \chi(2n(x_1 - ih), 2n(x_2 - jh))$
    i.e. the characteristic function of a square with center in $x_{i,j}$ and side h
  - $\chi_{i,j}^{n,1}(x_1, x_2) = \chi(2n(x_1 - ih - \frac{h}{2}), 2n(x_2 - jh))$
  - $\chi_{i,j}^{n,2}(x_1, x_2) = \chi(2n(x_1 - ih), 2n(x_2 - jh - \frac{h}{2}))$
    i.e. the characteristic functions of squares with centers in $(ih + \frac{h}{2}, jh)$ and in $(ih, jh + \frac{h}{2})$, both of side h.
- Let $X_n = \Re^{(n+1)\times(n+1)}$ (the space of real functions defined on the discrete set

$$\{x_{i,j} : i = 0, \ldots, n, \quad j = 0, \ldots, n\}\,.$$

- We define the discrete bilinear form a_1(for $u, v \in X_n \times X_n$)

$$a_1^h(u,v) = \sum_{\substack{i=0,n \\ j=0,n-1}} (u_{i,j+1} - u_{i,j})(v_{i,j+1} - v_{i,j}) + \sum_{\substack{i=0,n-1 \\ j=0,n}} (u_{i+1,j} - u_{i,j})(v_{i+1,j} - v_{i,j}) + \alpha h^2 \sum_{\substack{i=0,n \\ j=0,n}} u_{i,j} v_{i,j}$$

  and similarly the form a_2.

- We define

$$K^h = \{(u_1, u_2) : m(x_{i,j})\, u_2(p(x_{i,j})) \le u_1(x_{i,j}) \le M(x_{i,j})\, u_2(p(x_{i,j}))\,, \forall i, j = 0, \dots, n\}$$

and $J^h : X_n \oplus X_n \to \Re$

$$J^h(u_1, u_2) = \tfrac{1}{2}\, a_1^h(u_1, u_1) - (f_1, u_1)_h + a_2^h(u_2, u_2) - (f_2, u_2)_h\,,$$

where

$$(f, u)_h = \sum_{\substack{i=0,n \\ j=0,n}} u_{i,j} \int_\Omega f \chi_{i,j}.$$

The discrete problem

In relation to (3) we define the associated discrete problem

$$P^h \; : \; Find\ (\bar{u}_1^h, \bar{u}_2^h) \;\; such\ that \;\; J^h(\bar{u}_1^h, \bar{u}_2^h) = \min_{(u_1^h, u_2^h) \in K^h} J^h(u_1^h, u_2^h)\,. \tag{20}$$

We define now the sets K_I^h and, $\forall u_I^h \in K_I^h$, the associated sets $K_1^h(u_I^h)$, $K_2^h(u_I^h)$

$$\begin{aligned} K_I^h &= \left\{ u_I^h \in \Re^{n+1} : \left(u_I^h\right)_j \ge 0\,,\ \forall j = 0, \dots, n \right\}, \\ K_1^h(u_I^h) &= \left\{ u_1^h \in X_n^0 \ : m(x_{ij})(u_I^h)_j \le (u_I^h)_{i,j} \le M(x_{ij})(u_I^h)_j\,, \forall\, i, j = 0, \dots, n \right\}, \\ K_2^h(u_I^h) &= \left\{ u_2^h \in X_n^0 \ : \left(u_2^h\right)_{0,j} = \left(u_I^h\right)_j \ ,\ \forall\, j = 0, \dots, n \right\}. \end{aligned}$$

Let us define

$$\varphi_1^h(u_I^h) = \min_{u_1^h \in K_1^h(u_I^h)} \frac{1}{2} a_1^h(u_1^h, u_1^h) - (f_1, u_1^h)_h, \qquad \varphi_2^h(u_I^h) = \min_{u_2^h \in K_2^h(u_I^h)} \frac{1}{2} a_2^h(u_2^h, u_2^h) - (f_2, u_2^h)_h.$$

With this definitions we have

$$\min_{(u_1^h, u_2^h) \in K^h} J^h(u_1^h, u_2^h) = \min_{u_I^h \in K_I^h} \varphi^h\left(u_I^h\right) = \min_{u_I^h \in K_I^h} \left(\varphi_1^h\left(u_I^h\right) + \varphi_2^h\left(u_I^h\right)\right).$$

In consequence, instead of solving problem P^h, we will solve the equivalent discrete problem P_I^h

$$P_I^h \; : \; Find\ \bar{u}_I^h \;\; such\ that \;\; \varphi^h(\bar{u}_I^h) = \min_{u_I^h \in K_I^h} \varphi^h(u_I^h)\,.$$

Remark 1 *It can be easily proved that the discrete problem P_I^h inherits the same properties of the original one, i.e.*

- *$\varphi_2^h(u_I^h)$ is a quadratic function of u_I^h (and so, the gradient is linear and the Hessian is constant).*
- *$\varphi_1^h(u_I^h)$ is a convex function (piecewise quadratic).*

- *$\varphi_1^h(u_I^h)$ is differentiable at any point.*
- *$\nabla\varphi_1^h(u_I^h)$ is Lipschitz continuous (and so, the Hessian $H_1(u_I^h)$ exists a.e. $u_I^h \in (\Re^+)^{n+1}$).*
- *The Hessian $H_1(u_I^h)$ assumes only a finite number of values (at most $3^{n\times(n+1)}$ different values).*

Also, P_I^h can be decomposed hierarchically. Our method is based in this decomposition and follows the general methodology described in [5].

Numerical methods to solve P_I^h

The problem P_I^h is solved iteratively. At each step of iteration we solve problems P_1^h and P_2^h; P_2^h is a simple linear problem (although of large dimension). P_1^h is a bilateral obstacle problem, we solve it using the fast procedure presented in [6], [7], [8]. The computation of the gradients $\nabla\varphi_1(u_I)$, $\nabla\varphi_2(u_I)$, and the Hessians $H_1(u_I)$, $H_2(u_I)$ are also computed using the obtained solutions $(u_1(u_I), u_2(u_I))$ and solving two additional simple linear problems associated to the discretization of problems (17) and (19).

Description of the Algorithm

In order to clarify the writing, from now on u_1, u_2 and u_I will be the discretized vectors. We will denote with $\dagger$ the pseudo-inverse.

Step 0 Choose initial $u_I \in (\Re^+)^{n+1}$
Step 1 Set $K = I_{n+1}$ (Identity matrix)
Step 2 Compute $u_1(u_I)$ and $u_2(u_I)$, $\varphi(u_I)$, $g = \nabla\varphi_1(u_I) + \nabla\varphi_2(u_I)$, $H = H_1(u_I) + H_2(u_I)$
$\forall\eta = 1, ..., n+1$, if $\left((u_I)_\eta = 0 \text{ and } g_\eta > 0\right)$ set $K_{\eta\eta} = 0$
Step 3 Compute Newton's direction: $d = -g\,K(K'HK)^\dagger$
Step 4 $\forall\eta = 1, ..., n+1$, if $\left((u_I)_\eta = 0 \text{ and } d_\eta > 0\right)$ set $K_{\eta\eta} = 0$
Step 5 If K has changed at **Step 4**, go to **Step 3**
Step 6 If $\|d\| = 0$ (we are at the optimal point in a submanifold), go to **Step 10**.
Step 7 If $\|d\| > 0$, set $\hat{\lambda} = \max\{\lambda : u_I + \lambda d \in (\Re^+)^{n+1}\}$, set $v_I = u_I + \hat{\lambda} d$ and compute $\varphi(v_I)$
Step 8 While $\varphi(v_I) \geq \varphi(u_I)$, set $v_I = \dfrac{u_I + v_I}{2}$ and compute $\varphi(v_I)$
Step 9 Set $u_I = v_I$. Go to **Step 2**
Step 10 Compute $g = \nabla\varphi(u_I)$ and the Hessian H
$\forall\eta = 1, ..., n+1$, if $\left((u_I)_\eta = 0 \text{ and } g_\eta > 0\right)$ set $K_{\eta\eta} = 0$
Compute $d = -g\,K(K'HK)^\dagger$
If $\|d\| = 0$, Stop (we are at the global minimum), else go to **Step 1**.

Understanding the algorithm

Problem P_I^h consists in the minimization of a C^1-piecewise quadratic function in the convex set $Q = (\Re^+)^{n+1}$. Our method applies a method of Newton type to this task. Whenever it be possible, we try to follow the Newton's directions (computed in terms of $\nabla\varphi$ and the Hessian H) to obtain a decrement of the function φ. When this is not possible (because we have arrived at the boundary of Q, i.e. some components $(u_I)_j$ are 0) we restrict the minimization to the manifold $\{v \in Q : v_j = 0\}$. In this form we obtain a decreasing collection of manifolds (each one included in the next one). As this procedure is obviously finite, the major loop of the algorithm finishes finding the minimum in a manifold (characterized by a set of indices which identifies the components of u_I with value 0). Let us denote u_I^ν, $\nu = 1, 2, \ldots$ the points which realize those minima. As the algorithm generates a strictly decreasing sequence of values $(\varphi(u_I^\nu),\ \nu = 1, 2, \ldots)$ the associated manifolds are always different. The number of possible manifolds is finite (at most 2^{n+1} manifolds) and so it is impossible to repeat the major loop an infinite number of times. We conclude that the algorithm finishes in a finite number of steps.

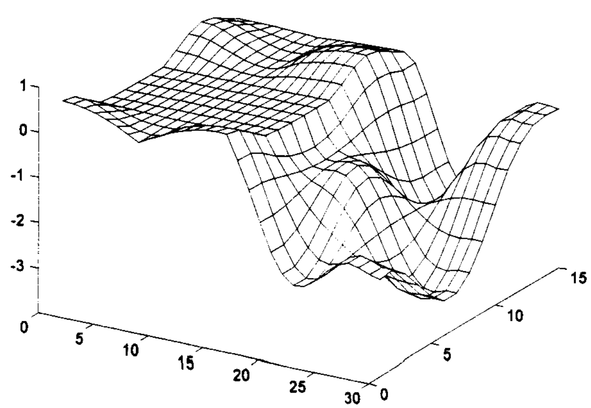
NUMERICAL EXAMPLE

We have solved an example of application where the meshes covering Ω_1 and Ω_2 have 15×15 points. The datas and the obstacles appearing in (1) are $\alpha = 2$, $\beta = 4$ and

$$f_1(x,y) = 100\,(-0.15 - sin(8x)cos(8y)) \qquad f_2(x,y) = 100\,(-0.15 - sin(8x)cos(8y))$$

$$m(x,y) = 0.5 + \frac{cos(2\pi x)}{2} \qquad M(x,y) = \sqrt{0.5 + \frac{cos(2\pi x)}{2}}$$

Figure 1 shows the solution obtained. The associated computational effort comprises 683 seconds (in a Pentium PC 133 MHz) and 4 major loops.



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