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3D Solid Incompressible Viscoelastic Finite Element in Large Strains for the Cornea. Refractive Surgery Application.

Fabio Ariel Guarnieri * y Alberto Cardona †
INTEC - Güemes 3450 - Santa Fe 3000 - Argentina

Abstract

Several models[1,2,3,4] have been developed in order to reproduce the corneal behavior in ophthalmological procedures as tonometry, radial keratotomy and photokeratectomy with Excimer Laser. It has been found that the viscoelastic effect of a biological soft tissue, as the cornea, is negligible in tonometry[5]. Nevertheless, clinical studies on humans showed refractive changes with time in radial keratotomy. Wound healing is responsible for the long time effect (measured in years). On the other hand, the short-time effect (hours or days) has not been clarified.
In this work a 3D viscoelastic finite element model is developed taking into account incompressibility and large strains. An internal variable is introduced by means of a multiplicative decomposition of the deformation gradient. The final goal of the study is to determinate the importance of the viscoelastic effect in radial keratotomy.

Resumen

Distintos modelos [1,2,3,4] han sido desarrollados para reproducir el comportamiento corneal en los procedimientos oftalmológicos como la tonometría, la queratotomía radial y la fotoqueratectomía con Excimer Laser. Se ha encontrado que el efecto viscoelástico de los tejidos blandos biológicos, como la córnea, es despreciable en tonometría [5]. Sin embargo, estudios clínicos humanos mostraron cambios refractivos con el tiempo en la queratotomía radial. La cicatrización de la herida es responsable del efecto a largo plazo (medido en años). Por otro lado, el efecto a corto plazo (horas y días) no ha sido aclarado. En este trabajo un modelo en elementos finitos viscoelástico en 3D es desarrollado teniendo en cuenta incompresibilidad y grandes deformaciones. Una variables interna es introducida por medio de una descomposición multiplicativa del gradiente de deformación. El objetivo final del estudio es determinar la importancia del efecto viscoelástico en la queratotomía radial.

1 Introduction

In order to simulate the viscoelastic effect in the cornea after a refractive procedure, e.g. radial keratotomy, we developed a three-dimensional incompressible viscoelastic finite element with large strains.
We follow the approach of Le Tallec [6,7] where a differential form is chosen which introduces an internal variable through a multiplicative decomposition of the deformation gradient. This model is

*Becario Perfeccionamiento del *Consejo Nacional de Investigaciones Científicas y Técnicas de la República Argentina.*

†Profesor, Universidad Nacional del Litoral, Argentina, e Investigador del *Consejo Nacional de Investigaciones Científicas y Técnicas de la República Argentina.*

thermodinamically consistent, preserving incompressibility, and easy to solve the initial hyperelastic model.
The viscoelastic incompressibility constraints the admissible solution of the internal variable.
The final mixed problem in three variables displacements-pressure-internal variable is reduced to an standard mixed problem displacements-pressure by eliminating the internal variable.

2 Mathematical Formulation

2.1 Kinematics and weak form of equilibrium equations

Let φ be a deformation of a body Ω with boundary $\partial\Omega$ onto a region of E, the three-dimensional Euclidean space where the displacements field u is defined by

$$u(X) = \varphi(X) - X$$

and where X is a material point and $x = \varphi(X)$ a spatial point.
The deformation gradient is defined

$$F(X) = D\varphi(X) = \frac{\partial\varphi(X)}{\partial X} = I + \text{GRAD } u$$

where

$$\text{GRAD } u(X) = \text{grad } u_\varphi(x)F$$

The right Cauchy-Green tensor $C = F^T F$ measures the length of an elementary vector δx after deformation in the reference configuration. So if

$$\delta x = F\delta X$$

then

$$|\delta x|^2 = \delta X \cdot C \cdot \delta X$$

Similarly, the volume element dX is transformed after deformation into

$$dx = \det F(X)dX = JdX$$

with J the determinant of the transformation.
Finally, an area element NdA is transformed after deformation into

$$nda = JF^{-T}NdA = \text{cof } F \cdot NdA$$

where N and n are the normal vectors to the reference and deformed surface, respectively, and cof F is the cofactor of F.
The weak form of the equilibrium equations (without body forces) can be written as

$$\int_\Omega P : \text{GRAD } \eta \; dV = \int_{\partial_t\Omega} (\bar{t}^N \cdot \eta) \; dV, \qquad \text{in } \Omega$$

where the surface tractions are

$$t^N = PN = \bar{t}^N, \qquad \text{on } \partial_t\Omega$$

P is the non-symmetric Pioia-Kirchhoff stress tensor, and η is a material virtual displacements or variation, such that

$$\eta_\varphi \in H_\varphi = \{\eta_\varphi : \varphi(\Omega) \to E,\ \eta_\varphi(x) = 0\ \forall x \in \partial_\varphi \Omega\}$$

are the admissible variations of φ, and

$$\eta(X) = \eta_\varphi(\varphi(X))$$

$$\text{GRAD}\ \eta = \text{grad}\ \eta_\varphi\ F$$

The Kirchhoff, Cauchy, and Piola-Kirchhoff stress tensors are related by

$$\tau = J\sigma = PF^T$$

2.2 Hyperelastic incompressible constitutive law

A material is said to be *elastic* if and only if its stress tensor P at a point X is a function of X and of F at X. This means it exists a function $\hat{P}$ such that

$$P(X) = \hat{P}(X, F)$$

where

$$\hat{P} : \Omega \times M^3_+ \to M^3,$$
$$M^3_+ = \{F \in M^3 = E \otimes E, \det F > 0\}$$

This material is *homogeneous* if

$$P(X) = \hat{P}(F)$$

An elastic material is called *hyperelastic* if for any admissible deformation field $\varphi(X,t) = F(t)X + c(t)$, the work developed by the stresses P during one time period is equal to zero:

$$W_{cycle} = \int_0^T \int_\Omega \hat{P}(F) : \dot{F}\ \ dX dt = 0$$

These are the only elastic materials usually considered in practice.
For incompressible materials, any deformation is volume preserving. In this case

$$\hat{P}(F) : \Omega \times M^3_1 \to M^3,$$
$$M^3_1 = \{F \in M^3 = E \otimes E, \det F = 1\}$$

So, there exists a function W and an arbitrary scalar p (the *hydrostatic pressure*) such that [13]

$$\hat{P}(F) = \frac{\partial W}{\partial F}(F) - p\frac{\partial \det F}{\partial F} = \frac{\partial W}{\partial F}(F) - pF^{-T}$$

By replacing the stress tensor P in the weak form of the equilibrium equations, and writing the incompressibility condition in a weak form we have

$$\int_\Omega \left[\frac{\partial W}{\partial F}(I + \text{GRAD}\ u) - p\frac{\partial \det(I + \text{GRAD}\ u)}{\partial F}\right] : \text{GRAD}\ \eta\ dV = \int_{\partial_t \Omega} (\bar{t}^N \cdot \eta)\ dV$$

$$\int_\Omega q(\det(I + \text{GRAD}\ u) - 1) dV = 0$$

where p and q are pressure Lagrange multipliers.

2.3 Viscoelastic constitutive law

2.3.1 Small Strain

When a viscoelastic material is subjected to a step loading, there exist both an instantaneous and a long term equilibrium response. The standard linear solid or Kelvin model is composed of combinations of linear springs with spring constant μ and dashpots with coefficient of viscosity ν . It is able to model the two responses in a simple way.
If we add the force exerted in the elastic branch with the force in the viscoelastic branch we have

$$f = \mu_0 e + \mu e_e$$

The force in the viscoelastic branch can be written as

$$v\ \dot{e}_v = \mu e_e$$

We solve this linear rate equation obtaining the elongation response function (creep):

$$e(t) = e_e + e_v = \frac{f}{\mu_0}\left[1 + \left(\frac{\mu_0}{\mu + \mu_0} - 1\right) e^{-t/\tau}\right]$$

where $E_0 = \mu + \mu_0$ measures the instantaneous elastic stiffness, $E_\infty = \mu_0$ measures the long term elastic stiffness, and $\tau = (\nu/\mu)\left[1 + (\mu/\mu_0)\right]$ is a characteristic relaxation time which indicates how long it takes for the material to reach its long term equilibrium response.

2.3.2 Finite Strains. Generalized Linear Model.

We generalize the above simple model to three-dimensional situations involving isochoric large deformations.
In the finite strains case, the right Cauchy-Green tensors C, C_e, and C_v measure the total deformation, the elastic part and the viscous part of the viscous branch, respectively. Variable C_v is an internal viscoelastic variable.
In analogy with the small strains case we assume a free energy potential of the form:

$$\Psi(C, C_v) = \Psi_0(C) + \Psi_e(C_e)$$

where Ψ_0 measures the stored energy of the elastic branch (long term behavior) and Ψ_e measures the stored energy of the viscous branch which, disappears in relaxation.
The intrinsic dissipation in the dashpot must satisfy the Clausius-Duhem inequality (2^0 law of thermodynamics),

$$\Phi(\dot{C}_v) : \dot{C}_v \geq 0$$

We choose, $\Phi(\dot{C}_v) = -\nu\ \dot{\overline{C_v^{-1}}}$ where ν is a symmetric definite-positive tensorial viscosity and $\dot{\overline{C_v^{-1}}}$ is the partial derivative of the inverse of C_v.
If $\Psi_e(C_e) = \mathrm{tr}(C_e K C_e)$ and assuming compressibility, we have as in small strains the linear rate equation:

$$\nu\ \dot{C}_v + K(C_v - C) = 0$$
$$C_v(t) = \tfrac{K}{\nu} \int_0^{+\infty} e^{-sK/\nu} C(t-s) ds$$

2.3.3 Finite Strains. Multiplicative decomposition of the deformation

In the finite strains case the additive decomposition of the deformation is equivalent to a multiplicative decomposition of the stretch $\lambda = \lambda^e \lambda^v$ [11].
If we decompose $F = RU$ and $R = I$ then $U = U_e U_v = U_v U_e$. But if U_e and U_v are not coaxial then $U_e U_v \neq U_v U_e$ and this is not physically true in crystals. So the addtive decomposition of the deformation in finite strains is not physically true [11].
A better approach is taken into account. In crystals we were allowed to formally decompose the deformation at microscopic level into

$$\varphi = \varphi_e \circ \varphi_v$$

Therefore the deformation gradient and right Cauchy-Green strain tensor are

$$F = F_e F_v$$
$$C = F_v^T C_e F_v$$
$$J = J_e J_v$$

where $J = \det(F) > 0$. This idea is taken from plasticity where $F = F_e F_p$ [10].
So, by applying the first principle of thermodynamics in an isothermic process we have

$$work - free\ energy = dissipated\ heat$$

Then, in a reference configuration we have

$$\frac{1}{2} S : dC - d\Psi = \Phi(\dot{C}_v) : dC_v$$

where S is the symmetric Piola-Kirchhoff stress tensor. By introducing the isotropic components, the viscoelastic constitutive laws are

$$S = 2\frac{\partial \Psi(C, C_v)}{\partial C} - pC^{-1}$$
$$\det(C) = 1$$

$$\Phi(\dot{C}_v) = -\frac{\partial \Psi(C, C_v)}{\partial C_v} + qC_v^{-1}$$
$$\det(C_v) = 1$$

The first equation is a standard hyperelastic constitutive law with C_v as a constitutive parameter. The third equation is a first order differential equation in time where the variable C_v introduces the time dependence in the model. The other two equations are the incompressibility relations. If the material were elastically compressible the second relation would be dropped.

2.4 Equilibrium equations. Variational formulation

By writing the weak form of the equilibrium equations in a fixed reference configuration Ω, neglecting the body forces, considering $\bar{t}_p$ pressure external forces in $\partial\Omega$, and choosing an adequate dissipation form ($\Phi(\dot{C}_v) = -v\ \overline{\dot{C}_v^{-1}}$) we obtain the classical variational formulation together with the dissipative constitutive laws

$$\int_\Omega P : \text{GRAD}\ \eta\ dV = \int_{\partial\Omega} (\bar{t}_p \cdot \eta)\ dV$$

$$\int_\Omega q(\det F - 1) dV = 0$$

$$v\,\overline{C_v^{-1}} - \frac{\partial \Psi(C,C_v)}{\partial C_v} + qC_v^{-1} = 0$$
$$\det(C_v) = 1$$

where $C_v(\cdot, t_0)$ = *given_value* with t_0 the initial time. We call the third equation as *dissipation equation.*

3 Approximation in space

3.1 Continuous problem

When the internal variable C_v is given the continuous problem reduces to a standard well-posed mixed problem where

$$\eta \in H = \{w \in W^{1,S}(\Omega; E), w|_{\Gamma_0} = 0\},$$

and

$$q \in P = L^{S_*}(\Omega; R),$$
$$\frac{1}{3S} + \frac{1}{S_*} = 1$$

The number $S \geq 1$ is such that the integrals in the weak form of the equilibrium equations make sense for any choice of u and η. For example $W^{1,S} = H^1$ and $L^{S*} = L^2$.
We will not use the mixed form of the incompressibility constraint $\det(C_v) = 1$. Instead of this we propose a constrained space of the internal variable such that

$$C_v \in A = \{A \in L^2(\Omega), \det(A) = 1\}$$

The incompressibility condition can be rewritten by developing the expression of the determinant in the third line:

$$a_{31}\,\mathrm{cof}\,_{31}A + a_{32}\,\mathrm{cof}\,_{32}A + a_{33}\,\mathrm{cof}\,_{33}A = 1$$

Since C_v is positive definite, the diagonal cofactors are different from zero. Then we can write

$$a_{33} = \frac{1 - a_{31}\,\mathrm{cof}\,_{31}A - a_{32}\,\mathrm{cof}\,_{32}A}{\mathrm{cof}\,_{33}A}$$

So in the dissipation equation we have five unknowns (the components of C_v, taking into account that is symmetric, except a_{33}, which is given by the above equation), and we have five equations since the six components of the dissipation equation are not independent because of the incompressibility condition. That is $A = \{a_1, a_2, a_3, a_4, a_5, a_6\}$ where $a_6 = a_{33} = a_6(a_1, a_2, a_3, a_4, a_5)$.

3.2 Associated discretized problem

We now use a standard finite element procedure to approximate the continuous problems. More precisely, we suppose that Ω is a polyhedral domain in R^3 which can be decomposed into a finite number N_h of hexahedra Ω_l such that:

$$\overline{\Omega} = \bigcup_{l=1}^{N_h} \overline{\Omega_l}$$

3.2.1 Displacement/pressure mixed approximation

We introduce a space H_h for displacements built with hexahedral isoparametric finite elements of order 2 (Q2),

$$\eta_h \in H_h = \{w_h : \Omega \to E; continuous, w_h|_{\Gamma_0} = 0, w_h|_{B_l} = w_l \circ \varphi_l^{-1}, w_l \in [Q_2(\widehat{\Omega})]^N, \forall l = 1, ..., N_h\},$$

Here φ_l is the mapping from the reference element $\widehat{\Omega} = (-1, +1)^N$ into Ω_l defined by

$$\varphi_l(X) = \sum_{\alpha=1}^{2^N} N^\alpha(X) x^{\alpha l}$$

and $Q_2(\widehat{\Omega})$ is the usual space of biquadratic polynomials.
So, one has the general interpolations of the displacements over an element

$$u_h(X) = \sum_{\alpha=1}^{m} N^\alpha(X) u^\alpha$$

where $u^\alpha = \{u_1^\alpha, u_2^\alpha, u_3^\alpha\}$ are the displacements components at node α, and N^α are the standard element shape functions of the node α with values 1 at vertex α, 1/2 at mid-edges in the neighbourhood of α, and 0 at all other vertices. From this we have the discrete version of the deformation gradient,

$$F_h = I + \sum_{\alpha=1}^{m} \overline{B} u^\alpha$$

where the matrix $\overline{B}$

$$\overline{B} = \text{GRAD } N^\alpha(X) = \frac{\partial N^\alpha(X)}{\partial X}$$

The space P_h for pressures is made of first order polynomials (P1) defined independently on each hexahedron (hence pressures are discontinuous) sampled at each element center.

$$p_h \in P_h = \{q_h : \Omega \to R, q_{h|\Omega_l} \in P_1(\Omega_l), \forall l = 1, ..., N_h\}$$

The displacement-pressure mixed variational problem with a Q2/P1 element verifies the LBB (Ladyz enskaya-Babuska-Brezzi) condition, then the problem is well-posed.

3.2.2 Approximation of the viscoelastic internal variable

For the approximation of the viscoelastic internal variable we choose piecewise constant functions belonging to the space

$$C_{vh} \in A_h = \{A_h : \Omega \to (E \otimes E)_{sym}, A|_{c_i} = \text{constant}, \ \det(A) = 1\}$$

with c_i a given partition of Ω.
The analysis of the linearized case indicates that the choice of the cells c_i can be made independently of the finite element spaces H_h and P_h [7]. Nevertheless, to keep the local structure of the finite element matrix and for stability and cost-effectiveness, a natural way of defining this partition is to associate a cell c_{jl} to every integration point x_j^l of a given integration rule

$$\int_{\Omega_l} f(x)dx = \sum_{j=1}^{NG} \omega_j f(x_j^l)$$

defined on each finite element Ω_l. For second order elements we take eight cells per element.

Then the discrete internal variable is

$$A_h(X) = \sum_{\gamma=1}^{NG} a_l^\gamma 1^\gamma(X) M_l$$

where $\gamma \in \{1, ..., NG\}$ is the cell number and NG is the number of cell (number of gauss points), 1^γ is the indicator function of the cell $c^\gamma, l \in \{1, ..., 6\}$ is the component number of H_h in the space of square symmetric matrices of order 3 and M_l is the canonical basis of this space. By taking into account the incompressibility constraint we have

$$A_h = a_1 e_1 \otimes e_1 + a_2 e_2 \otimes e_2 + \frac{(1 + a_1 a_3^2 - 2a_3 a_4 a_5 + a_2 a_4^2)}{(a_1 a_2 - a_5^2)} e_3 \otimes e_3$$
$$+ a_3(e_2 \otimes e_3 + e_3 \otimes e_2) + a_4(e_1 \otimes e_3 + e_3 \otimes e_1) + a_5(e_1 \otimes e_2 + e_2 \otimes e_1)$$

4 Approximation in time

Since the main cost is associated with the solution of the first equation giving u as a function of the viscoelastic variable, either an explicit or an implicit scheme have a similar cost per time step.
We choose an implicit scheme because it is unconditionally stable. This is important in situations where the time scales are of different order of magnitude.

The Euler scheme is chosen; it is not second order accurate as the midpoint rule, but it requires less computer memory and has very nice stiff stability and long term convergence properties [6]. Let interval-time $\Delta t > 0$; for each iteration $n \geq 0$, we have to solve

$$\int_{\Omega_h} F_h^{n+1} \left(2\frac{\partial \Psi}{\partial C}(C_h^{n+1}, C_{vh}^{n+1}) - p_h^{n+1}(C^{n+1})^{-1} \right) : \eta_h \, dV = \int_{\delta\Omega} (\bar{t}_p \cdot \eta_h) \, dV$$

$$\int_{\Omega_h} q_h (\det F_h^{n+1} - 1) dV = 0$$

$$v\frac{(C_v^{n+1})^{-1} - (C_v^n)^{-1}}{\Delta t} - \frac{\partial \Psi(C^{n+1}, C_v^{n+1})}{\partial C_v} - q(C_v^{n+1})^{-1} = 0$$
$$\det(C_v^{n+1}) = 1$$

with $C_v(\cdot, t_0) = C_v^0 = given\_value$.

5 Numerical solution

5.1 Algebraic problem

From our choice of the approximated spaces (H_h, P_h, A_h) the discrete system forms a non-linear algebraic system of N equations with N unknowns, $N = n_u + n_p + n_a$:

$$\begin{cases} \mathbf{g}(\mathbf{u}, \mathbf{p}, \mathbf{a}) = 0, & n_u \text{ equilibrium equations} \\ \mathbf{k}(\mathbf{u}) = 0, & n_p \text{ incompressibility conditions} \\ \mathbf{h}(\mathbf{u}, \mathbf{a}) = 0, & n_a \text{ dissipation equations} \end{cases}$$

The linearization of the system gives

$$\begin{bmatrix} \frac{\partial \mathbf{g}}{\partial \mathbf{u}} & \frac{\partial \mathbf{g}}{\partial \mathbf{p}} & \frac{\partial \mathbf{g}}{\partial \mathbf{a}} \\ \frac{\partial \mathbf{k}}{\partial \mathbf{u}} & 0 & 0 \\ \frac{\partial \mathbf{h}}{\partial \mathbf{u}} & 0 & \frac{\partial \mathbf{h}}{\partial \mathbf{a}} \end{bmatrix} \cdot \begin{bmatrix} \Delta \mathbf{u} \\ \Delta \mathbf{p} \\ \Delta \mathbf{a} \end{bmatrix} = - \begin{bmatrix} \mathbf{g}(\mathbf{u}, \mathbf{p}, \mathbf{a}) \\ \mathbf{k}(\mathbf{u}) \\ \mathbf{h}(\mathbf{u}, \mathbf{a}) \end{bmatrix}$$

Since

$$\left[\frac{\partial \mathbf{g}}{\partial \mathbf{a}}\right] \neq \left[\frac{\partial \mathbf{h}}{\partial \mathbf{u}}\right]$$

the tangential stiffness matrix of the linearized problem is non-symmetric.
The submatrix

$$\begin{bmatrix} \frac{\partial \mathbf{g}}{\partial \mathbf{u}} & \frac{\partial \mathbf{g}}{\partial \mathbf{p}} \\ \frac{\partial \mathbf{g}}{\partial \mathbf{u}} & 0 \end{bmatrix}$$

is the tangential stiffness matrix of the incompressible hyperelastic problem with the internal variable acting as a given parameter. This matrix is symmetric and banded. So, we can eliminate the internal variable leading to the usual incompressible hyperelastic strategy and a much cheaper algorithm.
If we find a solution $\mathbf{a}(\mathbf{u})$ to the system $\mathbf{h}(\mathbf{u}, \mathbf{a}) = \mathbf{0}$, then we can eliminate $\Delta \mathbf{a}$, to reduce the system to

$$\begin{bmatrix} \left[\frac{\partial \mathbf{g}}{\partial \mathbf{u}} - \left(\frac{\partial \mathbf{g}}{\partial \mathbf{a}} \cdot \left(\frac{\partial \mathbf{h}}{\partial \mathbf{a}}\right)^{-1} \cdot \frac{\partial \mathbf{h}}{\partial \mathbf{u}}\right)\right] & \frac{\partial \mathbf{g}}{\partial \mathbf{p}} \\ \frac{\partial \mathbf{k}}{\partial \mathbf{u}} & 0 \end{bmatrix} \cdot \begin{bmatrix} \Delta \mathbf{u} \\ \Delta \mathbf{p} \end{bmatrix} = - \begin{bmatrix} \mathbf{g}(\mathbf{u}, \mathbf{p}, \mathbf{a}(\mathbf{u})) \\ \mathbf{k}(\mathbf{u}) \end{bmatrix}$$

We take the symmetric part of the coupling of the equilibrium equations, it says

$$\left(\frac{\partial \mathbf{g}}{\partial \mathbf{u}}\right)_m = \frac{\partial \mathbf{g}}{\partial \mathbf{u}} - \left(\frac{\partial \mathbf{g}}{\partial \mathbf{a}} \cdot \left(\frac{\partial \mathbf{h}}{\partial \mathbf{a}}\right)^{-1} \cdot \frac{\partial \mathbf{h}}{\partial \mathbf{u}}\right)_{sym}$$

5.2 Algorithm

1. $(\mathbf{u}, \mathbf{p})$ given, $\Delta \mathbf{u} = \Delta \mathbf{p} = \mathbf{0}$;
2. compute $\mathbf{a}$ such that $\mathbf{h}(\mathbf{u}, \mathbf{a}) = \mathbf{0}$;
3. compute $\mathbf{r} = -\mathbf{g}(\mathbf{u}, \mathbf{a}(\mathbf{u}))$;
4. test $\|r\|, \|\Delta u\|, \|\Delta p\|$;stop on convergence ;
5. compute $\left(\frac{\partial \mathbf{g}}{\partial \mathbf{u}}\right)_m = \frac{\partial \mathbf{g}(\mathbf{u}, \mathbf{p}, \mathbf{a})}{\partial \mathbf{u}} - \left(\frac{\partial \mathbf{g}(\mathbf{u}, \mathbf{a})}{\partial \mathbf{a}} \cdot \left(\frac{\partial \mathbf{h}}{\partial \mathbf{a}}\right)^{-1} \cdot \frac{\partial \mathbf{h}(\mathbf{u}, \mathbf{a})}{\partial \mathbf{u}}\right)_{sym}$
6. assemble $\mathbf{K} = \begin{bmatrix} \left(\frac{\partial \mathbf{g}}{\partial \mathbf{u}}\right)_m & \frac{\partial \mathbf{g}}{\partial \mathbf{p}} \\ \frac{\partial \mathbf{k}}{\partial \mathbf{u}} & 0 \end{bmatrix}$;
7. solve $\mathbf{K} \cdot \Delta \mathbf{u} = \mathbf{r}$
8. update $u = u + \Delta u; p = p + \Delta p$;
9. goto 1

6 Conclusions

We have formulated the finite element developed by Le Tallec. The implementation is performed in Oofelie, an object oriented finite element program [12]. This software enhances the simplicity of the formulation.
Numerical results applied to the refractive surgery will be showed at the oral presentation.

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