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SOME APPLICATIONS OF DECOMPOSITION TECHNIQUES TO SYSTEMS OF COUPLED VARIATIONAL INEQUALITIES

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Abstract

We consider a junction problem described by a variational inequality framework, which involves a coupling restriction defined in terms of the common values of the system variables at the interface. To solve the original problem, a decomposition–coordination method is proposed, where the global equilibrium condition plays an essential role in the coordination phase.

Resumen

Consideramos un problema de junturas analizado por medio de una inecuación variacional. El tipo de juntura considerado involucra una restricción definida en términos del valor común de las variables del sistema en la interfase. Para resolver el problema original, un método de descomposición–coordinación es propuesto, donde la condición de equilibrio global juega un rol fundamental en la fase de coordinación.

1 Introduction

1.1 A brief description of junction problems

This work originates in what is known as *junction problems* – we can see [1], [2], [3], [5], [6] and the bibliography therein, for a more detailed description of these problems and the analysis of some related topics. Specifically, our work analyzes some issues that appear when the variational inequality approach is used to analyze these problems (see [7]). In order to fix the ideas, we consider the geometrical situation shown in Fig. 1. In $\Re^3$ we have two open domains Ω_2 and Ω_3, of dimensions 2 and 3 respectively; Ω_2 is in the plane $x_3 = 0$. The boundary $\partial\Omega_2$ of Ω_2 consists of Γ_{2i}, where i stands for *interface or junctions* and Γ_{2e}, where e stands for exterior. The domain Ω_3 is an open domain in $\Re^3$; we denote by $\Gamma_3 = \partial\Omega_3$ its boundary.

1.2 Mathematical description of the original problem

1.2.1 System State

The state of the system (which is defined by a scalar o vector function and may represent some variables of interest: temperature, deplacement, etc) is given by a real function $(u_2, u_3) : \Omega_2 \times \Omega_3 \rightarrow$

$\Re$. We set

$$X_2 = H^1(\Omega_2), \qquad X_3 = H^1(\Omega_3), \quad X = X_2 \oplus X_3\,,$$
$$\partial\Omega_2 = \Gamma_{2i} \cup \Gamma_{2e}\,, \quad \partial\Omega_3 = \Gamma_3\,. \tag{1}$$

1.2.2 Connections: Operators M_i

Around the interface Γ_{2i}, both in Ω_2 and in Ω_3, *connections* are established between u_2 and u_3. These connections may be local or not; they are defined in terms of linear continuous operators $M_i \in \mathcal{L}(X_i, \mathcal{H})$, $i = 2,3$, where $\mathcal{H}$ is a given Hilbert space. Specifically, we consider a closed convex subset $\mathcal{K}$ of $\mathcal{H}$ and we define the *connection* in the following form:

$$M_2 u_2 - M_3 u_3 \in \mathcal{K} \subset \mathcal{H}\,. \tag{2}$$

We define the set of *admissible states* as the set of states that verify the connection (2), i.e.

$$K = \{(v_2, v_3) \in X_2 \oplus X_3 \,:\, M_2 v_2 - M_3 v_3 \in \mathcal{K}\}\,. \tag{3}$$

Obviously, as M_i are linear continuous operators, the set K is a closed convex subset of X.

1.2.3 The energy functional: The bilinear forms

As it is usual in physical problems, we find the state of the system looking for the admissible state that minimizes a functional of energy, which in our problem will be described in terms of a couple of bilinear forms a_2, a_3. We will suppose that these bilinear forms are symmetric and we define the functional $J : X_1 \oplus X_2 \to \Re$ in the following way:

$$J(u_2, u_3) = \tfrac{1}{2}\, a_2(u_2, u_2) - (f_2, u_2) + \tfrac{1}{2}\, a_3(u_3, u_3) - (f_3, u_3)\,, \tag{4}$$

where the bilinear forms have the following expressions ($\alpha > 0$, $\beta > 0$)

$$a_2(u_2, v_2) = \int_{\Omega_2} (\nabla u_2 \nabla v_2 + \alpha\, u_2 v_2)\, dx\,, \tag{5}$$

$$a_3(u_3, v_3) = \int_{\Omega_3} (\nabla u_3 \nabla v_3 + \beta\, u_3 v_3)\, dx\,. \tag{6}$$

We associate to (5), (6) the differential operators A_2 and A_3 :

$$A_2 = -\Delta + \alpha\,, \qquad A_3 = -\Delta + \beta\,.$$

In the example analyzed in this paper, we will restrict the study to bilinear forms associated to simple second order differential operators, although an extension to more general operator are straightforward and without difficulties.
We also define the functional operators $\mathcal{A}_2$ and $\mathcal{A}_3$ such that

$$a_2(u_2, v_2) = \langle \mathcal{A}_2 u_2, v_2 \rangle \text{ and } a_3(u_3, v_3) = \langle \mathcal{A}_3 u_3, v_3 \rangle\,, \tag{7}$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product in $H^1(\Omega_2)$ or $H^1(\Omega_3)$.

1.2.4 Coupled variational inequalities

We must minimize the functional (4) in the set of admissible states K or, in a equivalent way, we will consider the variational inequality: Find $u = (u_2, u_3) \in K$ such that

$$a_2(u_2, v_2 - u_2) + a_3(u_3, v_3 - u_3) \geq (f_2, v_2 - u_2) + (f_3, v_3 - u_3), \quad \forall\, (v_2, v_3) \in K\,, \tag{8}$$

where $f_2 \in L^2(\Omega_2)$, $f_3 \in L^2(\Omega_3)$ and (v, w) denotes the inner product in $L^2(\Omega_2)$ or $L^2(\Omega_3)$. By virtue of (5), (6), this variational inequality has a unique solution (see [9]).

Remark 1 *By assumption, the bilinear forms a_2 and a_3 are symmetric and in consequence, the inequality* (8) *is the necessary condition that must hold at the point that realizes the minimum of the functional J on the set K.*

Now we introduce the following example, which will be solved by decomposition techniques in Section 3.

1.2.5 The example

We use the general framework above presented. Then, the problem will be completely defined once we have specified $\mathcal{H}$, $\mathcal{K}$, M_2 and M_3.
We take $\mathcal{H} = H^{1/2}(\Gamma_3)$, $\mathcal{K} = 0$ and we define the operator M_3 to be $M_3\, v_3 = \gamma_3\, v_3$ (the trace of v_3 on Γ_3).
We consider now the trace operator $\phi \longrightarrow \phi|_{\Gamma_{2i}}$, which is linear and continuous in $H^1(\Gamma_3) \longrightarrow H^{1/2}(\Gamma_{2i})$. This operator has inverse, an operator R such that

$$R \in \mathcal{L}(H^{1/2}(\Gamma_{2i}); H^1(\Gamma_3))\,, \qquad R\,\phi|_{\Gamma_{2i}} = \phi \qquad \forall \phi \in H^{1/2}(\Gamma_{2i})\,. \tag{9}$$

We introduce now the operator $E \in \mathcal{L}(H^{1/2}(\Gamma_{2i}); H^{1/2}(\Gamma_3))$; we can take, for example, $E = R$. Let $\gamma_{2i}\, v_2$ be the trace of v_2 on Γ_{2i}, and so we have $\gamma_{2i}\, v_2 \in H^{1/2}(\Gamma_{2i})$. We define $M_2\, v_2 = E\, \gamma_{2i}\, v_2$ and we consider the convex set of admissible states

$$K = \{(v_2, v_3) \in X_2 \oplus X_3 \,:\, M_2\, v_2 - M_3\, v_3 = 0\}\,. \tag{10}$$

In this case, the connection condition

$$M_2\, v_2 = M_3\, v_3\,, \tag{11}$$

means that we consider an extension to Γ_3 of the trace $\gamma_{2i}\, v_2$ defined on Γ_{2i} and that extension must be equal to the trace of v_3.

2 A decomposition method

The idea of solving (8) by hierarchical optimization and decomposition stems from the fact that the solution $u = (u_2, u_3)$ depends on the values at some intermediate space (in the case of the example that space is $H^{1/2}(\Gamma_{2i})$) and if we choose the correct (and unique) intermediate value, then the original problem (8) is solved. The procedure chooses a value u_I of the intermediate space and – after solving some naturally defined separated problems – that value is corrected until condition (8) is verified.

Definition 1 Decomposition of the convex K
We introduce a linear space X_I and a convex set K_I. Also, $\forall\, u_I \in K_I$, we consider the associated closed convex sets $K_2(u_I)$, $K_3(u_I)$ with the property

$$K = \bigcup_{u_I \in K_I} \left(K_2(u_I) \oplus K_3(u_I)\right). \tag{12}$$

Definition 2 The separated problems
We introduce the notation

$$\varphi_2(u_I) = \min_{u_2 \in K_2(u_I)} J(u_2\,, 0)\,, \tag{13}$$

$$\varphi_3(u_I) = \min_{u_3 \in K_3(u_I)} J(0, u_3)\,, \tag{14}$$

and to compute the functions φ_2 and φ_3 we define the problems $\mathcal{P}_2(u_I)$ and $\mathcal{P}_3(u_I)$:
Problem $\mathcal{P}_2(u_I)$

$$\text{Find } \bar{u}_2(u_I) \in K_2(u_I) \ \text{ such that } \ J(\bar{u}_2\,, 0) = \varphi_2(u_I)\,, \tag{15}$$

Problem $\mathcal{P}_3(u_I)$

$$\text{Find } \bar{u}_3(u_I) \in K_3(u_I) \ \text{ such that } \ J(0, \bar{u}_3) = \varphi_3(u_I)\,. \tag{16}$$

Remark 2 *By* (5) *and* (6), *there exists unique solution of* (15) *and* (16).

Definition 3 The hierarchical optimization problem
Let us define the auxiliary function φ

$$\varphi(u_I) = \varphi_2(u_I) + \varphi_3(u_I)\,. \tag{17}$$

By virtue of (12) *we have*

$$\min_{(u_2,u_3)\in K} J(u_2\,, u_3) = \min_{u_I \in K_I} \left(\min_{K_2(u_I)\oplus K_3(u_I)} J(u_2\,, u_3) \right). \tag{18}$$

But from (4) *we have*

$$\begin{aligned} \min_{K_2(u_I)\oplus K_3(u_I)} J(u_2\,, u_3) &= \left(\min_{u_2 \in K_2(u_I)} J(u_2\,, 0) \right) + \left(\min_{u_3 \in K_3(u_I)} J(0, u_3) \right) \\ &= \varphi_2(u_I) + \varphi_3(u_I) = \varphi(u_I)\,. \end{aligned} \tag{19}$$

Then

$$\min_{(u_2,u_3)\in K} J(u_2\,, u_3) = \min_{u_I \in K_I} \varphi(u_I) \tag{20}$$

and we conclude that problem (8) *is equivalent to the following problem $\mathcal{P}_I$:*

$$\mathcal{P}_I\,:\ \text{Find } \bar{u}_I \in K_I \ \text{ such that } \ \varphi(\bar{u}_I) = \min_{u_I \in K_I} \varphi(u_I)\,. \tag{21}$$

Remark 3 *From* (18)-(21) *it follows that the new problem $\mathcal{P}_I$ is a decomposable hierarchical optimization problem* [8].

The keystone of the decomposition method is the fact that the function φ appearing in (21) *verify* (17) *and that (φ_2, φ_3) can be computed by solving two separated optimization problems.*

In the following we will suppose that the hypotheses of Theorem 1 hold and in consequence, φ_2 and φ_3 are convex functions. Finally, from (17), φ is also convex.

Theorem 1 *Let X_I, X be vector spaces, K_I a closed convex subset of X_I, $\phi(\cdot)$ a strictly convex and continuous function defined on X_I and (for each $v \in K_I$) a closed convex subset $K^v \subset X$ such that we can define on K_I the function*

$$\psi(v) = \inf_{x \in K^v} \phi(x)\,, \tag{22}$$

and such that there exists a unique $\bar{u}(v)$ with the property

$$\phi(\bar{u}(v)) = \psi(v). \tag{23}$$

Assume the multivalued mapping $v \to K^v$ is a convex mapping in the sense that

$$\lambda\, K^v + (1-\lambda)\, K^{\tilde{v}} \subset K^{\lambda v + (1-\lambda)\tilde{v}}\,, \tag{24}$$

then ψ is a convex function.

Proof: Let v and $\tilde{v}$ be elements of K_I and u, $\tilde{u}$ such that

$$u = \bar{u}(v)\,, \qquad \tilde{u} = \bar{u}(\tilde{v}), \tag{25}$$

then, ψ is convex because

$$\begin{aligned} \psi(\lambda\, v + (1-\lambda)\,\tilde{v}) &\leq \phi(\lambda\, u + (1-\lambda)\,\tilde{u}) && \text{by (22) and (24)} \\ &\leq \lambda\, \phi(u) + (1-\lambda)\,\phi(\tilde{u}) && \text{by the convexity of } \phi \\ &= \lambda\, \psi(v) + (1-\lambda)\,\psi(\tilde{v}) && \text{by (23) and (25).} \end{aligned}$$

□

Remark 4 $(\bar{u}_2(\bar{u}_I), \bar{u}_3(\bar{u}_I))$ *is the solution of the variational inequality* (8) *when $\bar{u}_I$ realizes the minimum of φ.*

Remark 5 *In general the problem $\mathcal{P}_I$ is easier than* (8), *because*

1. *The dimension of Ω_I (where X_I is defined) is smaller than the dimensions of Ω_2, Ω_3.*
2. *X_I has a simpler structure.*
3. *Problem $\mathcal{P}_I$ is the minimum of a convex function on the set K_I which involves sometimes simpler restrictions.*
4. *By solving the problems $\mathcal{P}_2$ and $\mathcal{P}_3$ it is possible to compute a descent direction for φ.*

Definition 4 The auxiliary function g
Let us suppose that φ_2, φ_3 are differentiable and let us denote g_2, g_3 the corresponding derivatives. For each $v_I \in X_I$, we define

$$g(v_I) = g_2(v_I) + g_3(v_I)\,. \tag{26}$$

Definition 5 The equilibrium condition
The optimality condition (that we will say it is the <u>*equilibrium condition*</u>*) for* $\bar{u}_I$ *to be the solution of Problem* $\mathcal{P}_I$ *is:*

$$(g(\bar{u}_I), v_I - \bar{u}_I) \geq 0, \qquad \forall v_I \in K_I. \tag{27}$$

Remark 6 *As* φ *is convex this condition is also a sufficient condition of optimality. In addition, if* φ *is strictly convex, the point* $\bar{u}_I$ *that verifies this optimality condition is unique.*

Definition 6 An iterative algorithm
The above mentioned unique value $\bar{u}_I$ *can be found iteratively using the algorithm described below, which employs the information given by the function* g. *The algorithm generates a sequence* $(\bar{u}_2(v_I^\nu), \bar{u}_3(v_I^\nu))$, *which starts at an initial pair* $\bar{u}_2(v_I^0)$, $\bar{u}_3(v_I^0)$ *and that converges to the solution* (u_2, u_3) *of* (8).

Let $\{\gamma_\nu : \nu = 0, \ldots\}$ *be a sequence of positive numbers such that* $\lim_{\nu \to +\infty} \gamma_\nu = 0$ *and* $\sum_{\nu=0}^{\infty} \gamma_\nu = +\infty$.
Also, let us denote by $P_{K_I}(q)$ *the projection of* q *on* K_I.

ALGORITHM

Step 1: $\nu = 0,\ v_I^0 \in K_I$.

Step 2: Solve problems $\mathcal{P}_2(v_I^\nu)$ and $\mathcal{P}_3(v_I^\nu)$. Set $\gamma = \gamma_\nu$.

Step 3: $\hat{v}_I = P_{K_I}(v_I^\nu - \gamma g(v_I^\nu))$.

Step 4: If $\varphi(\hat{v}_I) < \varphi(v_I^\nu) + \frac{1}{2}(g(v_I^\nu), \hat{v}_I - v_I^\nu)$, set $v_I^{\nu+1} = \hat{v}_I$, $\nu = \nu + 1$ and go to Step 2;

else, $\gamma = \dfrac{\gamma}{2}$, and go to Step 3.

3 Application of the decomposition method

3.1 Elements of the decomposition

We deal with the problem presented in Section 1.2.5. In this case we define:

$$\left|\begin{array}{l} X_I = H^{1/2}(\Gamma_{2i}), \\ K_2(v_I) = \{v_2 \in X_2 \,:\, \gamma_{2i}\, v_2(x) = v_I(x),\ a.e.\ x \in \Gamma_{2i}\}, \\ K_3(v_I) = \{v_3 \in X_3 \,:\, \gamma_3\, v_3(x) = (E(v_I))\,(x),\ a.e.\ x \in \Gamma_3\}. \end{array}\right. \tag{28}$$

With these data, we consider the problem $\mathcal{P}_2(v_I)$ defined in (15), which is equivalent to find the solution $\bar{u}_2(v_I)$ of the Dirichlet-Neumann problem:

$$\left\{\begin{array}{l} A_2\bar{u}_2 = f_2, \\ \bar{u}_2|_{\Gamma_{2i}} = v_I, \\ \dfrac{\partial \bar{u}_2}{\partial n}|_{\Gamma_{2e}} = 0. \end{array}\right. \tag{29}$$

To make explicit the dependence on v_I of the solution $\bar{u}_2(v_I)$, we introduce the following definitions: Let w_2 be the solution of

$$\begin{cases} A_2 w_2 = f_2 \,, \\ w_2|_{\Gamma_{2i}} = 0 \,, \\ \dfrac{\partial w_2}{\partial n}|_{\Gamma_{2e}} = 0 \,, \end{cases} \tag{30}$$

and let G_2 be the operator

$$\begin{aligned} H^{1/2}(\Gamma_{2i}) &\longrightarrow H^1(\Omega_2)\,, \\ \phi &\longrightarrow G_2\phi\,, \end{aligned} \tag{31}$$

such that

$$\begin{cases} A_2(G_2\phi) = 0 \,, \\ (G_2\phi)|_{\Gamma_{2i}} = \phi \,, \\ \dfrac{\partial (G_2\phi)}{\partial n}|_{\Gamma_{2e}} = 0 \,. \end{cases} \tag{32}$$

By virtue of (29), (30) and (32), we have

$$\bar{u}_2(v_I) = G_2 v_I + w_2 \,. \tag{33}$$

In consequence, for the auxiliary function $\varphi_2(v_I)$ we have

$$\varphi_2(v_I) = \tfrac{1}{2}\langle A_2(G_2 v_I + w_2), G_2 v_I + w_2\rangle - (f_2, G_2 v_I + w_2)\,, \tag{34}$$

then, for the derivative (in the Frechet sense) we obtain: $\forall \zeta \in X_I$

$$\langle D\varphi_2(v_I), \zeta\rangle = \langle G_2^* A_2 G_2 v_I + G_2^*(A_2 w_2 - \hat{f}_2)\,, \zeta\rangle. \tag{35}$$

Note 1 *We have used the notation: $\hat{f}$ for the element given by the Riesz representation theorem, i.e. $\langle \hat{f}_2\,, v_2\rangle = (f_2\,, v_2)\,,\ \forall v_2 \in X_2$. A similar notation is used below.*

Now, we consider the problem $\mathcal{P}_3(v_I)$ defined in (16). In this case, the solution $\bar{u}_3(v_I)$ is given by the solution of the Dirichlet problem:

$$\begin{cases} A_3 \bar{u}_3 = f_3 \,, \\ \bar{u}_3|_{\Gamma_3} = E v_I \,. \end{cases} \tag{36}$$

Let be w_3 the solution of

$$\begin{cases} A_3 w_3 = f_3 \,, \\ w_3|_{\Gamma_3} = 0 \,, \end{cases} \tag{37}$$

and let G_3 be the operator

$$\begin{aligned} H^{1/2}(\Gamma_3) &\longrightarrow H^1(\Omega_3)\,, \\ \phi &\longrightarrow G_3\phi\,, \end{aligned} \tag{38}$$

such that

$$\begin{cases} A_3(G_3\phi) = 0\,, \\ (G_3\phi)|_{\Gamma_2} = \phi\,. \end{cases} \tag{39}$$

By virtue of (36), (37) and (39), we have

$$\bar{u}_3(v_I) = G_3 E v_I + w_3\,. \tag{40}$$

Then, we obtain

$$\varphi_3(v_I) = \tfrac{1}{2}\langle A_3(G_3 E v_I + w_3)\,,\, G_3 E v_I + w_3\rangle - (f_3\,,\, G_3 E v_I + w_3)\,, \tag{41}$$

Therefore, for the derivative (in the Frechet sense) we have: $\forall \zeta \in X_I$

$$\langle D\varphi_3(v_I), \zeta\rangle = \langle (E^* G_3^* A_3 G_3 E) v_I + E^* G_3^* (A_3 w_3 - \hat{f}_3)\,,\, \zeta\rangle\,. \tag{42}$$

The solution $\bar{u}_I$ is given by solving the equation $g(v_I) = 0$, where $g(v_I) = D\varphi_2(v_I) + D\varphi_3(v_I)$. By virtue of (35) and (42), we have

$$G_2^* A_2 G_2 \bar{u}_I + G_2^*(A_2 w_2 - \hat{f}_2) + (E^* G_3^* A_3 G_3 E)\bar{u}_I + E^* G_3^*(A_3 w_3 - \hat{f}_3) = 0\,. \tag{43}$$

In consequence we get

$$\bar{u}_I = Q\left(G_2^*(A w_2 - \hat{f}_2) + E^* G_3^*(A_3 w_3 - \hat{f}_3)\right)\,, \tag{44}$$

where

$$Q = -(G_2^* A_2 G_2 + E^* G_3^* A_3 G_3 E)^{-1}\,. \tag{45}$$

So, the solution is given by

$$\boxed{\begin{cases} u_2(\bar{u}_I) = G_2 Q\left(-G_2^*(A_2 w_2 - \hat{f}_2) - E^* G_3^*(A_3 w_3 - \hat{f}_3)\right) + w_2\,, \\ u_3(\bar{u}_I) = G_3 E\, Q\left(-G_2^*(A_2 w_2 - \hat{f}_2) - E^* G_3^*(A_3 w_3 - \hat{f}_3)\right) + w_3\,. \end{cases}} \tag{46}$$

Conclusions

We have solved in this paper a junction problem involving coupling restrictions. Using a variational inequality formulation, we have obtained both the differential equations and the global equilibrium condition which identify the solution. We have proposed a procedure to obtain the solution via a *decomposition–coordination* method – the basic elements of the methodology used in our work can be seen in [4], [8]; for more recent developments in this field we refer the reader to [10], [11]. We have devised this procedure because it allows us to solve the coupled problem through the solution of simple independent problems – in general, they are *linear* problems or simple *obstacle* problems. These problems depend on some auxiliary variables which are modified (by a coordination procedure) until the desired solution is obtained. In the example here presented, we have given an explicit form of the solution in terms of some simple operators associated to classical problems of Dirichlet and Neumann type.

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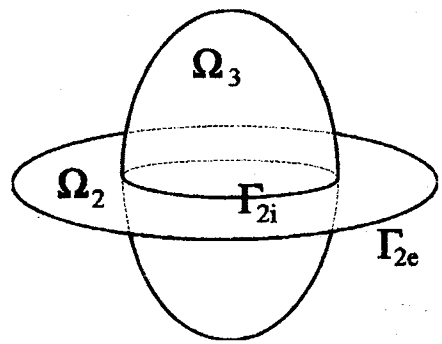

Figure 1: A 2-3 dimensional coupled domain