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NATURAL FREQUENCIES OF A TIMOSHENKO BEAM:
Exact values by means of a generalized solution

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RESUMEN

El presente trabajo trata la determinación de las frecuencias naturales de vibración de una viga Timoshenko con un método alternativo al clásico. Se utiliza una solución generalizada para obtener los valores exactos de las frecuencias. La metodología está basada en el uso de secuencias minimizantes construidas con series de Fourier. El problema se reduce así a resolver una única y simple ecuación. Se incluye un apéndice teórico con la demostración de la exactitud de los autovalores. Son presentados ejemplos de vigas con varias condiciones de borde, resueltos numéricamente, y que muestran la convergencia del método.

ABSTRACT

The present paper deals with the determination of the natural frequencies of vibration of a Timoshenko beam with an alternative method to the classical approach. Here a generalized solution is used to achieve the exact values of the frequencies. The methodology is based upon the use of minimizing sequences made up of Fourier series. The problem is then reduced to solve a single simple equation. A theoretical appendix with the demonstration of the exactness of the eigenvalue is included. Cases of beams with various end conditions are numerically solved as examples, showing the rate of convergence of the method.

INTRODUCTION

The theory proposed by Timoshenko [1] takes into account the effect of the shear and rotary inertia on the transverse deflection. It applies mainly to the case of short beams. Normally two differential equations arise, involving the variables of the problem: the transverse displacement y and the flexural rotation ϕ. Many authors have dealt with the subject. Lately Laura *et al.* have published a monograph [2] reviewing the related work and containing an extensive bibliography. Cortínez in [3] shows some results as application of the methodology to the Timoshenko beam without theoretical analysis. The case of beams of constant cross-section and for certain end conditions has been solved by the classical approach yielding closed solutions. However, the resulting characteristic equations are in general rather cumbersome.

The methodology used herein was developed and applied to different situations by Filipich and Rosales [4,5,6]. It consists in proposing a minimizing sequence of Fourier series to represent the functions of the problem under study, i.e. y and ϕ. Afterwards the governing functional of the problem (the *Lagrangean*) is minimized with respect to the unknowns. The essential boundary conditions (BC) are satisfied by the entire sequence (unlike the Raleigh-Ritz method in which each base function must satisfy the essential BC). The non-satisfied conditions are taken into account, for convenience, by Lagrange multipliers. In the present case, a single characteristic equation is easily obtained. This methodology is far explained in [6] and its foundation along with the corresponding demonstrations in [4]. However, a theoretical appendix addressed to the interested reader, is included to provide the demonstration of some of the main assessments made in the approach. The procedure is applied to beams with various BC, diverse values of slenderness ratio and as well as shear coefficient k. The latter is sometimes taken as 0.6666 [1] or 0.8333 (both in beams of rectangular cross-section). Rossi *et al.* have analyzed its influence in [7].

The present approach provides a simple procedure to obtain the characteristic equation. Furthermore other complexities such as concentrated masses, intermediate supports, elastic boundaries, varying cross-section, etc. can be handled without additional difficulties.

THEORETICAL CONCEPTS

Conceptually the methodology consists in choosing minimizing sequences to represent the relevant functions of the problem under study, in this case y and ϕ. The complete sequence is required to satisfy the essential (geometric) boundary conditions; when those conditions are not satisfied identically by the sequence, use is made of an extended functional with Lagrange multipliers. At the same time the sequence converges in the mean to the classical solution. Afterwards the procedure consists in writing the Lagrangean functional of the problem; then the functions y and ϕ are replaced by their representation in Fourier series. Finally an extreme condition is placed upon the functional, condition which conduces to the exact eigenvalues of the vibration.

In the Appendix II it is shown that the obtained results are not approximations to the exact eigenvalues but the exact values of them with the desired accuracy. The interested reader may found in reference [8] the demonstrations of the necessary and sufficient conditions, in order that the method leads toward the exact solution.

PRACTICAL DETERMINATION OF THE EXACT EIGENVALUES

Let us now apply the above described procedure to the vibration of a Timoshenko beam. First, the unknowns representing the total displacement $v(x,t)$ and the flexural rotation $\vartheta(x,t)$ are written as

$$v(x,t) = y(x)\, e^{i\lambda t} \quad ; \quad \vartheta(x,t) = \phi(x)\, e^{i\lambda t} \qquad (1a,b)$$

where λ are the sought frequencies. The governing functional $\bar{\mathcal{L}}$ is

$$\bar{\mathcal{L}} = U - T \qquad (2)$$

in which the strain and kinetic energy expressions, U and T respectively, are included in Appendix I. Let us now propose the minimizing sequences (see Appendix II)

$$z_M(x) = \sum_{i=1}^{M} \frac{C_i}{\beta_i} \sin \beta_i x + C_o x + D \qquad ; \ i=1,2,3,\ldots,M \tag{3}$$

$$\sigma_M(x) = -\sum_{i=0}^{M} B_i \cos \beta_i x \qquad ; \ i=0,1,2,\ldots,M \tag{4}$$

where $\beta_i = i\pi/L$. It is verified that $\|z'_M - y'\|^2 < \varepsilon^2$ and $\|\sigma'_M - \phi'\|^2 < \delta^2$ for $\varepsilon \longrightarrow 0$ and $\delta \longrightarrow 0$ for $M \longrightarrow \infty$.

Now the geometric (essential) BC of the beam under study are imposed to (3) and (4). In the case that one or more BC are not satisfied, use is made -for convenience- of Lagrange multipliers yielding an extended functional $\mathcal{L}$, i.e.

$$\mathcal{L} = \overline{\mathcal{L}} - \sum_{j=1}^{N} \tau_j \ (NSBC) \tag{5}$$

where *NSBC* stands for non-satisfied essential BC; N is the number of NSBC and τ_j are the Lagrange multipliers. In the next step an extreme condition is imposed on the extended functional as follows

$$\frac{\partial \mathcal{L}}{\partial W_i} = 0 \ ; \qquad \frac{\partial \mathcal{L}}{\partial \tau_j} = 0 \tag{6}$$

with $W_i = (B_i, C_i, D)$, $i=0,1,2,\ldots,M$ and $j=1,2,\ldots N$. After some algebra manipulation, the frequency equation is obtained for each beam case, as will be shown in the next section.

TIMOSHENKO BEAM EXAMPLES: VARIOUS END CONDITIONS

1. Free-free beam.

In this case there are no essential BC, then $\overline{\mathcal{L}} \equiv \mathcal{L}$. On performing the derivation procedure (6) of the previous section a homogeneous system of five equations in (B_i, C_i, D) is obtained. After some algebraic steps the following characteristic equation yields

$$(K S_{12} - Z)(1 - S_{21}) + K (\tfrac{1}{2} - S_{11})^2 = 0 \tag{7}$$

in which

$$K = 1 - \Omega^2 \alpha_2/\alpha_1 ; \qquad Z = \frac{1}{3} + \alpha_2 - \frac{\alpha_2 \Omega^2}{3 \alpha_1}$$

$$S_{11} = \sum_{i=1}^{M} \frac{(-1)^{i+1} F_i}{(i\pi)^2 \varphi_i} \ ; \ S_{12} = \sum_{i=1}^{M} \frac{(-1)^{i+1} G_i}{(i\pi)^2 \varphi_i} \ ; \ S_{21} = \sum_{i=1}^{M} \frac{[1-(-1)^i] F_i}{(i\pi)^2 \varphi_i}$$

and

$$F_i = \frac{2\Omega^2}{(i\pi)^2} [(-1)^i - 1] \ ; \ G_i = \frac{2\Omega^2}{(i\pi)^2} (-1)^i \ ;$$

$$\varphi_i = \alpha_1 - \frac{\Omega^2}{(i\pi)^2} - \frac{\alpha_1^2}{\left[(i\pi)^2+\alpha_1-\Omega^2\alpha_2\right]}$$

See Appendix I for further definitions. Furthermore considering odd and even modes, the frequency equation may be written as

$$[K\,(S_I+S_P) - Z][1 - 4\,S_I] + K\,(\tfrac{1}{2} - 2\,S_I)^2 = 0 \qquad (8)$$

where
$$S_I = -\frac{2\Omega^2}{\pi^4}\sum_{(I)}^{M}\frac{1}{i^4\varphi_i}; \quad S_P = -\frac{2\Omega^2}{\pi^4}\sum_{(P)}^{M}\frac{1}{i^4\varphi_i}$$

with I = 1,3,5,... and P = 2,4,6,... Equation (8) may be factored, yielding

$$(1 - 4\,S_I) = 0; \qquad K\,(1 + 4\,S_P) - 4\,Z = 0 \qquad (9,10)$$

or in an alternative notation

$$1 + \frac{8\Omega^2}{\pi^4}\sum_{(I)}^{M}\frac{1}{i^4\varphi_i} = 0 \qquad (11)$$

$$\frac{2\Omega^2}{\pi^4}\left(\sum_{(P)}^{M}\frac{1}{i^4\varphi_i}\right)\left[1 - \Omega^2\frac{\alpha_2}{\alpha_1}\right] - \frac{1}{12}\left[\Omega^2\frac{\alpha_2}{\alpha_1} - 1 - 12\alpha_2\right] = 0 \qquad (12)$$

respectively. Equation (9) (or (11)) yields frequencies corresponding to the odd modes and equation (10) (or (12)) to the even ones.

The numerical results obtained for the first four frequencies of the free-free beam are depicted in Table I for the first four frequencies, r/L = 0.05, 0.10 and k= 0.50, 0.85 and 1.00 ($k = 1/\varkappa$). It should be mentioned that it is always possible to find a closed expression to the summations involved in the frequency equations (diverse combinations of trigonometric and hyperbolic functions). However, the fast convergence of the sums as well as the speed of the computers do not make it necessary.

Table I: Values of the first four frequency coefficients in the case of a free-free Timoshenko beam; N=10000 (5000 terms); ν=0.3.

k	Ω_1	Ω_2	Ω_3	Ω_4
r/L = 0.05				
0.50	19.8636	45.8225	75.1703	104.6442
0.85	20.3517	49.0264	83.4221	119.8195
1.00	20.4602	49.7858	85.5024	123.8422
r/L = 0.10				
0.50	15.8998	29.5351	43.9566	46.4790
0.85	16.8194	33.9617	51.8110	59.5283
1.00	17.0280	35.1035	54.1428	64.0625

2 Simply supported-free beam.
The procedure is analogous to the one described for the free-free beam except that in this case the existence of an essential BC yields D = 0. The functional is then written in terms of z_M and σ_M. After the minimization w.r.t. $\{B_i, C_i\}$ and some algebra manipulation the following frequency equation is obtained

$$K\left[\alpha_1 - \frac{\Omega^2}{3}\right] - \alpha_1 + \frac{2\Omega^4 K}{\pi^4} \sum_{i=1}^{M} \frac{(-1)^{2i+1}}{i^4 \varphi_i} = 0 \qquad (13)$$

where K and φ_i are defined in the previous sub-section 1. The numerical results are shown in Table II.

Table II: Values of the first four frequency coefficients in the case of a simply supported-free Timoshenko beam; N=5000; υ = 0.3.

k	Ω_1	Ω_2	Ω_3	Ω_4
r/L = 0.05				
0.50	14.0068	38.4443	67.0022	96.5597
0.85	14.3632	41.1490	74.5785	110.8906
1.00	14.4428	41.7972	76.5096	114.7359
r/L = 0.10				
0.50	11.4556	26.1610	39.7635	46.0809
0.85	12.2566	29.9548	47.9874	58.9092
1.00	12.4464	30.9605	50.3475	63.5424

3 Simply supported-clamped beam.
The essential BC of this case yield C_o = 0 and D = 0 as well as a non-satisfied BC (NSBC)

$$\sum_{i=0}^{M} B_i = 0 \qquad (14)$$

As was stated in (5), such condition is taken into account by extending the functional

$$\mathcal{L} = \overline{\mathcal{L}} - \tau \sum_{i=0}^{M} B_i \qquad (15)$$

The minimization is performed w.r.t. $\{B_o, B_i, C_i, \tau\}$ and the frequency equation yields

$$\frac{1}{\alpha_1 K} + 2 \sum_{i=1}^{M} \frac{Q}{\left[(i\pi)^2 + \alpha_1 K\right] Q - \alpha_1^2} = 0 \qquad (16)$$

where K is defined in section 4.1 and

$$Q = \alpha_1 - \frac{\Omega^2}{(i\pi)^2}$$

The numerical results for various k and r/L ratios are depicted in Table III.

Table III: Values of the first four frequency coefficients in the case of a simply supported-clamped Timoshenko beam; N=5000; υ = 0.3.

k	Ω_1	Ω_2	Ω_3	Ω_4
r/L = 0.05				
0.50	13.1294	35.4356	61.9113	90.1354
0.85	13.8798	39.2289	70.8463	105.6079
1.00	14.0573	40.2094	73.3124	110.0575
r/L = 0.10				
0.50	9.7589	22.9468	37.1218	46.9708
0.85	11.1303	27.2748	45.1466	59.7442
1.00	11.5057	28.5752	47.6252	64.3683

4 Clamped-clamped beam.
Due to the non-satisfied BC the functional is extended

$$\mathscr{L} = \overline{\mathscr{L}} - \tau \sum_{i}^{M} B_i \tag{17}$$

with i = 0,1,2,...,M. Two frequency equations are obtained, one for the symmetric modes and another for the anti-symmetric ones, respectively, i.e.

$$\sum_{(I)}^{M} \frac{Q}{\left[(i\pi)^2+\alpha_1 K\right]Q - \alpha_1^2} = 0 \tag{18}$$

$$\frac{1}{\alpha_1 K} + 2\sum_{(P)}^{M} \frac{Q}{\left[(i\pi)^2+\alpha_1 K\right]Q - \alpha_1^2} = 0 \tag{19}$$

where I=1,3,5,...,M; P=2,4,6,...,M and k, Q as previously defined. Table IV contains the numerical results.

Table IV: Values of the first four frequency coefficients in the case of a clamped-clamped Timoshenko beam; N=10000 (5000 terms); υ=0.3.

k	Ω_1	Ω_2	Ω_3	Ω_4
r/L = 0.05				
0.50	17.3246	39.0358	64.5615	91.7873
0.85	18.8886	44.5213	75.4728	109.0995
1.00	19.2801	46.0230	78.6436	114.3008
r/L = 0.10				
0.50	11.7067	23.5807	37.4938	50.7726
0.85	13.9160	28.7170	45.9947	62.3184
1.00	14.5691	30.3603	48.7144	66.1329

CONCLUSIONS

The solution of the vibration problem of the Timoshenko beam has been addressed by means of -what the authors call- a generalized solution making use of Fourier series. The eigenvalues found by means of this methodology are exact and the theoretical demonstration of this fact is included in Appendix II. Throughout the paper use is made of normalized functions but obviously it is possible to dispense with the normalization. Four cases of beams with diverse support conditions were considered and numerical results presented. The valued of M was chosen to be 5000 because it yields satisfactory accuracy. It should be noted that this is not a Rayleigh-Ritz method. In effect, the boundary conditions are imposed to the sequence and not to each base function. As mentioned before it is possible to find a closed expression to the summations involved in the frequency equations (diverse combinations of trigonometric and hyperbolic functions). However, the fast convergence of the sums as well as the speed of the computers do not make it necessary. Furthermore the computers find the value of trigonometric and hyperbolic functions by means of summations of finite number of terms.

REFERENCES

1. TIMOSHENKO, S.P. "On the Correction for Shear of the Differential Equation for Transverse Vibration of Prismatic Beams", Philosophical Magazine 41, 1921, p.744-746.
2. LAURA P.A.A., MAURIZI M.J. and ROSSI R. "Vibrating Timoshenko Beams", Instituto de Mecánica Aplicada (IMA-CONICET), No.92-15, 1992, Bahía Blanca, Argentina.
3. CORTINEZ V.H., "Soluciones Generalizadas para el Problema de Vibraciones Transversales de Vigas Tipo Timoshenko", Departamento de Ingeniería, Universidad Nacional del Sur, 1992 Argentina.
4. FILIPICH, C.P. and ROSALES, M.B. "A proposal for finding exact solutions in boundary value problems". Internal Report 5/90, Dep. de Ingeniería, Universidad Nac. del Sur, 1990 Argentina.
5. FILIPICH, C.P.and ROSALES, M.B. "Exact frequencies of beams and plates: a generalized solution". Int. Congress on Numerical Methods in Engineering and Applied Sciences, Concepción, Chile, 16-20 Nov. 1992.
6. FILIPICH, C.P. and ROSALES, M.B. "Beams and Arcs: exact values of frequencies via a generalized solution". Journal of Sound and Vibration 170(2), 1994, p263-269.
7. ROSSI R.E., LAURA P.A.A. and MAURIZI M.J. "Numerical Experiments on the Effect of the Value of the Shear Coefficient upon the Natural Frequencies of a Timoshenko beam".J. of Sound and Vibration 154(2), 1992, p.374-378.
8. FILIPICH, C.P. and ROSALES, M.B. "Theoretical Analysis of the Generalized Solution: The Problem of the Timoshenko Beam". Internal Report 2/93, Depto. de Ingeniería, Univ.Nac. del Sur, 1993 Argentina.
9. Rektorys K., "Variational Methods in Mathematical Sciences and Engineering". D.Reidel Publ.Co., 1977.

APPENDIX I: FUNCTIONAL

The Timoshenko beam of length L, constant cross-section A, moment of inertia J and material with Young modulus of elasticity E and transverse modulus of elasticity G is governed by the following system of differential equations

$$\alpha_1(y'' + \phi') + y\,\Omega_o^2 = 0 \qquad \text{(AI.1)}$$

$$-\phi'' + \alpha_1(y' + \phi) - \alpha_2\,\phi\,\Omega_o^2 = 0 \qquad \text{(AI.2)}$$

where $y = y(x)$ and $\phi = \phi(x)$ are the shape functions representing the displacement and the flexural rotation of the beam (the classical solution); Ω_o is the exact non-dimensional frequency of the beam.

Introducing χ which depends on the shape of the cross-section, and r the radius of gyration and the shear coefficient k the following coefficients result

$$\alpha_1 = \frac{GAL^2}{EJ\chi} \quad ; \quad \alpha_2 = \frac{J}{AL^2} \; ; \quad \frac{r}{L} = \sqrt{\alpha_2}; \quad k = \frac{1}{\chi}$$

In the present work one searches the eigenvalues and the eigenfunctions of the vibration problem through the generalized solution approach. The solution is found by making stationary a certain functional, i.e. finding the extreme of the functional under certain conditions [9]. The functional to be employed here may be motivated by observing that the inner product of equation (AI.1) times y plus the inner product of equation (AI.2) times ϕ equals zero, i.e.

$$-\left\{\left[\alpha_1(y'' + \phi') + y\,\Omega_o^2\right], y\right\} + \left\{\left[-\phi'' + \alpha_1(y' + \phi) - \alpha_2\,\phi\,\Omega_o^2\right], \phi\right\} = 0 \quad \text{(AI.3)}$$

which, after an integration by parts and the imposition of the BC, may be written in terms of norms as

$$\mathcal{L}[y, \phi, \Omega_o] = \|\phi'\|^2 + \alpha_1\|y' + \phi\|^2 - \Omega_o^2\left[\|y\|^2 + \alpha_2\|\phi\|^2\right] = 0. \quad \text{(AI.4)}$$

Alternatively, as is usually found in classical mechanics the functional may be though as the *Lagrangean* $\overline{\mathcal{L}} = U - T$ being U the strain energy and T the kinetic energy,

$$U = \frac{1}{2}\left\{EJ\,\|\phi'\|^2 + \frac{GA}{\chi}\,\|y' + \phi\|^2\right\}; \quad T = \frac{1}{2}\left\{\rho A\lambda^2\,\|y\|^2 + \rho J\lambda^2\,\|\phi\|^2\right\} \quad \text{(AI.5)}$$

in which ρ is the mass density of the beam and the exact dimensional frequency λ is related to the non-dimensional one by

$$\lambda = \frac{1}{L^2}\sqrt{\frac{EJ}{\rho A}}\;\Omega_o. \quad \text{(AI.6)}$$

APPENDIX II: THE FOUND EIGENVALUES ARE EXACT

Demonstration:

From Appendix I the governing functional in terms of the exact solution y and ϕ is

$$\overline{\mathcal{L}}[y, \phi, \Omega_o] = \|\phi'\|^2 + \alpha_1\|y' + \phi\|^2 - \Omega_o^2\left[\|y\|^2 + \alpha_2\|\phi\|^2\right] = 0 \quad \text{(AII.1)}$$

and consequently

$$\Omega_o^2 = \frac{\|\phi'\|^2 + \alpha_1\|y' + \phi\|^2}{\|y\|^2 + \alpha_2\|\phi\|^2}; \quad \text{(AII.2)}$$

let us now replace the functions y and ϕ by arbitrary functions z and $\sigma \in C$, $\|z'\|^2 < \infty$ and $\|\sigma'\|^2 < \infty$, respectively. An eigenvalue expression can then be written in terms of such functions as

$$\Omega^2 = \frac{\|\sigma'\|^2 + \alpha_1 \|z'+\sigma\|^2}{\|z\|^2 + \alpha_2 \|\sigma\|^2} \tag{AII.3}$$

It follows from Rayleigh's principle that

$$\Omega_o^2 \le \Omega^2. \tag{AII.4}$$

Additionally it is required that $\|y\|^2 = \|\phi\|^2 = \|z\|^2 = \|\sigma\|^2 = 1$ which is always possible by imposing, in each case, a new function which is the ratio between the old function and its norm. Then both eigenvalue expressions write respectively

$$\Omega_o^2 = \left(\frac{1}{1+\alpha_2}\right) \left[\|\phi'\|^2 + \alpha_1 \|y'+\phi\|^2\right] \tag{AII.5a}$$

$$\Omega^2 = \left(\frac{1}{1+\alpha_2}\right) \left[\|\sigma'\|^2 + \alpha_1 \|z'+\sigma\|^2\right]. \tag{AII.5b}$$

Now let us work out expression (AII.5b) as follows

$$\Omega^2 = \left(\frac{1}{1+\alpha_2}\right) \left\{A + (\phi',\sigma') + \alpha_1 [(z'+\sigma),(y'+\phi)]\right\} \tag{AII.6}$$

where $A = \|\sigma'-\phi'\|^2 + \alpha_1 \|(z'+\sigma)-(y'+\phi)\|^2 + [\phi',(\sigma'-\phi')] +$

$$+ \alpha_1 \left\{(y'+\phi),[(z'+\sigma)-(y'+\phi)]\right\}. \tag{AII.7}$$

The second term of the expression between the curly brackets in equation (AII.6) is found to be

$$(\phi',\sigma') = |\phi'\sigma|_0^1 - (\sigma,\phi''). \tag{AII.8}$$

The boundary conditions in this equation vanish since ϕ is the classical solution and σ satisfies the essential BC. After replacing it in (AII.6) and regrouping one obtains

$$\Omega^2 = \left(\frac{1}{1+\alpha}\right) \left\{A + \left[\sigma,[\alpha_1 (y'+\phi)-\phi'']\right] - [z,\alpha_1 (y''+\phi')]\right\}. \tag{AII.9}$$

Now making use of the set of differential equations of the vibrating Timoshenko beam (Appendix I), the eigenvalue Ω results

$$\Omega^2 = \left(\frac{1}{1+\alpha_2}\right) \left\{A + \Omega_o^2 \,[(y,z) + \alpha_2 (\sigma,\phi)]\right\} \tag{AII.10}$$

and applying twice the triangular inequality

$$\Omega^2 \le \left(\frac{1}{1+\alpha_2}\right) \left\{|A| + \Omega_o^2 \,[\,|(y,z)| + \alpha_2 |(\sigma,\phi)|\,]\right\}. \tag{AII.11}$$

Furthermore, due to the Cauchy-Schwarz inequality and the value unity of the norms of the functions, the eigenvalue is bounded as follows

$$\Omega^2 \le \frac{|A|}{(1+\alpha_2)} + \Omega_o^2 \tag{AII.12}$$

Consider now minimizing sequences z_M and σ_M which satisfy

$$\|z'_M - y'\|^2 < \varepsilon^2 \qquad \varepsilon \longrightarrow 0 \text{ for } M \longrightarrow \infty \text{ and } \|z_M\|^2 = 1$$

$$\|\sigma'_M - \phi'\|^2 < \delta^2 \qquad \delta \longrightarrow 0 \text{ for } M \longrightarrow \infty \text{ and } \|\sigma_M\|^2 = 1.$$

Also $\|\phi'\|^2 = Q^2$, $\|y'\|^2 = S^2$, $\|\sigma'_M\|^2 = N^2$, $\|z'_M\|^2 = R^2$, $\|z_M - y\|^2 < \vartheta^2$, $\|\sigma_M - \phi\|^2 < \gamma^2$, with $\vartheta \to 0$ and $\gamma \to 0$ when $M \to \infty$. Additionally each sequence (z_M and σ_M) satisfies the essential BC. With these sequences the term A in the expression between curly brackets in equation (AII.6) is found to be

$$|A_M| \le \|\sigma'_M - \phi'\|^2 + \alpha_1 \|(z'_M + \sigma_M) - (y' + \phi)\|^2 + |[\phi', (\sigma'_M - \phi')]| +$$

$$+ \alpha_1 \left| \left\{ (y' + \phi), [(z'_M + \sigma_M) - (y' + \phi)] \right\} \right| \qquad \text{(AII.13)}$$

Each term of the second member converges as follows

$$\|\sigma'_M - \phi'\|^2 < \delta^2 \qquad \text{(AII.14)}$$

$$\|(z'_M + \sigma_M) - (y' + \phi)\|^2 = \|(z'_M - y')\|^2 + \|\sigma_M - \phi\|^2 + 2\, [(z'_M - y')(\sigma_M - \phi)] \le$$

$$\le \varepsilon^2 + \gamma^2 + 2\, \varepsilon\gamma \qquad \text{(AII.15)}$$

$$[\phi', (\sigma'_M - \phi')]^2 \le \|\phi'\|^2 \|\sigma'_M - \phi'\|^2 \le Q^2\, \delta^2 \qquad \text{(AII.16)}$$

$$\left| \left\{ (y' + \phi), [(z'_M + \sigma_M) - (y' + \phi)] \right\} \right| \le |[y', (z'_M - y')]| + |y', (\sigma_M - \phi)| +$$

$$|\phi, (z'_M - y')| + |\phi, (\sigma_M - \phi)| \le (\varepsilon + \gamma)(1 + S) \qquad \text{(AII.17)}$$

Consequently,

$$|A_M| \le \delta^2 + \alpha_1 (\varepsilon + \gamma)^2 + R\, \delta + \alpha_1 (\varepsilon + \gamma)(1 + S) = \eta \qquad \text{(AII.18)}$$

and η tends to zero as M tends to infinity. Therefore $|A_M| \longrightarrow 0$ as $M \longrightarrow \infty$. With these results and in view of equations (AII-4) and (AII-12), the following relevant result is obtained

$$\Omega_M^2 \longrightarrow \Omega_o^2 \text{ as } M \longrightarrow \infty \qquad \text{(AII.20)}$$

Then it is demonstrated that the eigenvalues found with the minimizing sequences are exact, under the requirements that the selected functions are minimizing sequences and satisfy the essential boundary conditions.