FLEXIBLE MULTIBODY ANALYSIS
USING IMPEDANCE AND/OR ADMITTANCE MODELS

Alberto Cardona †
INTEC (CONICET/UNL)
Güemes 3450, 3000 Santa Fe, Argentina

Michel Géradin ‡
LTAS, University of Liège
Rue E. Solvay 21, B-4000 Liège, Belgium

RESUMEN

Analizamos varias formulaciones de superelementos para modelar miembros de mecanismos. En particular, se comparan los formalismos en impedancia y en admitancia, con detalles sobre cómo formular modelos para describir miembros flexibles en un sistema multicuerpos.

ABSTRACT

Several formulations of superelements for modeling mechanism members are reviewed. The impedance and admittance formalisms are analyzed and compared, with details on how to formulate models for describing flexible members in a multibody system.

INTRODUCTION

Multibody dynamics problems are highly nonlinear, the nonlinearities being due to the large relative rotations between bodies. In fact, in many cases the deformation effects inside each body are small enough to consider that its elastic behavior remains linear in a local frame. Then, we may say that in some sense the nonlinearities are concentrated at the joints. This fact allows the development of methods for modeling complex elastic mechanism members based on the linear expansion of the elastic displacements field in a basis of deformation modes of the body.

Several authors have proposed forms of the component-mode method for analyzing multibody systems [1-13]. Most of them used vibration modes to model the dynamics of flexible multibodies with the limitation that the bodies are modeled by finite elements embedded in the mechanism analysis program. Ref. [4-6] also included bodies whose flexibility effects are represented by modal superposition, but they were oriented to the modeling of multibody systems in tree topology. Refs. [9,10] presented a multibody formalism in which also static correction modes are included to account for effects of local deformation due to kinematics constraints. In ref. [11], geometric non-linear effects are accounted for by partitioning the elastic member into several linear substructures (or superelements).

---

† Profesor, Universidad Nacional del Litoral, y Miembro de la Carrera del Investigador del Consejo Nacional de Investigaciones Científicas y Técnicas de la República Argentina.

‡ Professor, University of Liège, Belgium, and Conseiller Scientifique, ONERA, Chatillon, France.

This paper is based on an implementation we have presented for flexible multibody analysis in references [12,13]. The formulation is now extended to account for linear models describing the behavior of individual flexible bodies using either impedance or admittance forms. All terms for the transient analysis are derived from information contained in the expressions of impedance or admittance matrices. The matrices could have been derived either analytically [14,15], numerically using a finite element code or they can come from identification from experimental measures [16]. The dynamic analysis is made in the mechanism analysis package [18], linking the body to the rest of the system by the selected joints. The degrees of freedom of the superelement are the translations and the rotations at boundaries, plus a given number of internal mode amplitudes.

The inertia is computed using a corotational approximation, in which the velocities interpolation is not completely consistent with the displacements one. This way of computing the inertia has been chosen for the simplicity of formulation and easy interfacing with linear descriptions of individual bodies.

KINEMATICS HYPOTHESES

Let $\mathbf{x}$ be the position of an arbitrary point P of the flexible body; we write it in terms of variables in a local reference frame of the body:

$$\mathbf{x} = \mathbf{x}_O + \mathbf{R}_O(\mathbf{X} + \mathbf{u}) \tag{1}$$

where $\mathbf{x}_O$ is the position of the local reference frame, $\mathbf{R}_O$ is the rotation of the local frame about the global one, $\mathbf{X}$ is the position of point P in the local frame and $\mathbf{u}$ is the elastic displacement of P measured in the local frame (see figure 1).
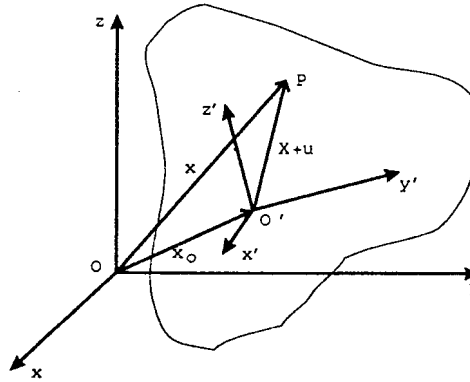

Figure 1 : Flexible body kinematics

After time-differentiating equation (1) , the virtual displacement, the velocity and the acceleration at point P result:

$$\delta\mathbf{x} = \delta\mathbf{x}_O + \mathbf{R}_O\delta\widetilde{\Theta}_O(\mathbf{X} + \mathbf{u}) + \mathbf{R}_O\delta\mathbf{u}$$

$$\dot{\mathbf{x}} = \dot{\mathbf{x}}_O + \mathbf{R}_O\widetilde{\Omega}_O(\mathbf{X} + \mathbf{u}) + \mathbf{R}_O\dot{\mathbf{u}} \tag{2}$$

$$\ddot{\mathbf{x}} = \ddot{\mathbf{x}}_O + \mathbf{R}_O(\widetilde{\Omega}_O^2 + \widetilde{\mathbf{A}}_O)(\mathbf{X} + \mathbf{u}) + 2\mathbf{R}_O\widetilde{\Omega}_O\dot{\mathbf{u}} + \mathbf{R}_O\ddot{\mathbf{u}}$$

with $\Omega_O, \mathbf{A}_O$ being the material angular velocities and accelerations of the local frame and $\delta\Theta_O$ the variation of angular displacements of the local frame [12,13,19]. Rotations can also be expressed as increments with respect to a reference value, giving

$$\Psi = \Psi_O \circ \psi \tag{3}$$

where the operation $\circ$ symbolizes the composition of rotations [12].

From equations (1, 3), we can then compute the relative displacements and slopes inside the elastic body in terms of absolute positions and rotations:

$$\left\{ \begin{array}{c} \mathbf{u} \\ \psi \end{array} \right\} = \left\{ \begin{array}{c} \mathbf{R}_O^T(\mathbf{x} - \mathbf{x}_O) - \mathbf{X}) \\ (-\Psi_O) \circ \Psi \end{array} \right\} \tag{4}$$

Let us assume that the elastic displacements and slopes in the local reference frame of each frame are small compared to the unity:

$$\frac{\|\mathbf{u}\|}{\|\mathbf{X}\|} , \|\psi\| \ll 1 \tag{5}$$

These requirements imply a geometric linearity condition in the local frame; that is to say, although the superelement as a whole undergoes finite rotations in the three-dimensional space, the displacements in a local frame remain small enough to assure the linearity of relations between local forces and local displacements. Because of this assumption, we will be able to express the dynamic equilibrium equations at the local frame using impedance and/or admittance formulations, which are inherently linear.

Local values of displacements and rotations at the boundary nodes of the body will be grouped in the vector $\mathbf{q}_{loc}$:

$$\mathbf{q}_{loc} = \left\{ \begin{array}{c} \mathbf{u}_B \\ \psi_B \end{array} \right\} \tag{6}$$

In order to connect the body to the rest of the system, we will use global positions and rotations. These quantities, evaluated at the nodes of the body, will be grouped into the vector $\mathbf{q}_{abs}$:

$$\mathbf{q}_{abs}^T = \langle \mathbf{x}_O^T \quad \Psi_O^T \quad \mathbf{x}_B^T \quad \Psi_B^T \rangle \tag{7}$$

where $\mathbf{x}_O, \Psi_O$ give the reference position and rotation, and $\mathbf{x}_B, \Psi_B$ group positions and rotations at the boundary nodes.

Global and local values are related kinematically according to the nonlinear relations above (4).

Remarks

- The choice of the local reference frame is not unique. In this paper, we develop two different models: one based on impedances and a second one based on an admittance model of the substructure. In the former case we use a local reference frame rigidly attached to one node of the body, while in the latter case we adopt a reference frame whose location and orientation is defined as coincident with the *principal axes* for the (fixed) reference configuration.

  Both frames above are attached to the reference configuration. Many authors have used floating local frames coincident with the current principal axes, i.e. *Tisserand axes*. They have the advantage of fully uncoupling the nonlinear inertia of the body; however, they require a rather cumbersome algebraic manipulation and the access to the global nonlinear stiffness and mass matrices.

- Because of the particular form of the nonlinear kinematic relations between local and global variables, values at the boundaries should be given by triplets of positions and/or rotations at each node. Therefore, the vector $\left\{ \begin{array}{c} \mathbf{x}_B \\ \Psi_B \end{array} \right\}$ is in fact an abbreviate notation for:

$$\left\{ \begin{array}{c} \mathbf{x}_B \\ \Psi_B \end{array} \right\} = \left\{ \left\{ \left\{ \begin{array}{c} \mathbf{x}_1 \\ \Psi_1 \\ \mathbf{x}_2 \\ \Psi_2 \\ \vdots \end{array} \right\} \right\}^T \quad \left\{ \begin{array}{c} \mathbf{x}_{i1} \\ \mathbf{x}_{i2} \\ \vdots \end{array} \right\}^T \quad \left\{ \begin{array}{c} \Psi_{j1} \\ \Psi_{j2} \\ \vdots \end{array} \right\}^T \right\}^T \tag{8}$$

At nodes $i1, i2, \ldots$ of the boundary, only the translation degrees of freedom have been retained to form the superelement, while at nodes $j1, j2, \ldots$ only the rotation terms are conserved.

REDUCED IMPEDANCE MODEL

Let the reduced impedance matrix $\mathbf{Z}(\omega^2)$ of a body be such that

$$\mathbf{Z}(\omega^2)\,\mathbf{q} \;=\; \mathbf{f} \tag{9}$$

where $\mathbf{q}$ is a vector of displacements and rotations at a given set of nodes of the body, measured in a local frame; and $\mathbf{f}$ is a vector of conjugated applied forces and torques. Vector $\mathbf{q}$ groups terms at a particular node O, which is taken as reference node, and terms $\mathbf{q}_{loc}$ evaluated at the boundary. The spectral expansion of the impedance can be written:

$$\mathbf{Z}(\omega^2) \;=\; \mathbf{K}^* \;-\; \omega^2\mathbf{M}^* \;-\; \omega^4\sum_{i=1}^{r}\frac{\mathbf{M}_i}{\omega_i^2-\omega^2} \tag{10}$$

$\mathbf{K}^*$ is the reduced stiffness matrix, $\mathbf{M}^*$ the reduced mass matrix and $\mathbf{M}_i$ the set of modal gain matrices (note that matrix $\mathbf{K}^*$ is singular since the body is free). Modal gains are rank-1 matrices, which can be expressed in terms of the reactions $\mathbf{h}_i$ produced at the boundary by the (clamped) eigenmodes $\boldsymbol{\phi}_i$

$$\mathbf{M}_i \;=\; \frac{\mathbf{h}_i\mathbf{h}_i^T}{\omega_i^4\mu_i} \tag{11}$$

E.g. if the modal gains have been computed from a finite element mesh, they take for expression $\mathbf{h}_i = (\mathbf{K}_{BI} - \omega_i^2\mathbf{M}_{BI})\boldsymbol{\phi}_i$, where $\mathbf{K}_{BI}, \mathbf{M}_{BI}$ are the stiffness and mass submatrices corresponding to the boundary/internal dofs partition.

By defining an appropriate set of additional variables $\mathbf{y}$, the system (9) can be transformed into a second order equation in ω^2. First replace (10) and (11) into (9) to give:

$$\mathbf{f} \;=\; \mathbf{f}_0 - \sum_{i=1}^{r}\mathbf{f}_i \tag{12}$$

where

$$\begin{aligned}
\mathbf{f}_0 &\;=\; (\mathbf{K}^* - \omega^2\mathbf{M}^*)\,\mathbf{q} \\
\mathbf{f}_i &\;=\; \frac{\omega^4\mathbf{h}_i\mathbf{h}_i^T\mathbf{q}}{\omega_i^4\mu_i(\omega_i^2-\omega^2)}
\end{aligned} \tag{13}$$

If we now define the modal amplitudes y_i in the form:

$$y_i \;=\; \frac{\omega^2\mathbf{h}_i^T\mathbf{q}}{(\omega_i^2-\omega^2)\mu_i\omega_i^2} \tag{14}$$

such that

$$\mathbf{f}_i \;=\; \omega^2\frac{\mathbf{h}_i}{\omega_i^2}y_i \tag{15}$$

Then, replace equations (13) and (15) into (12) to get the system of equations:

$$\begin{cases}
(\mathbf{K}^* - \omega^2\mathbf{M}^*)\mathbf{q} \;+\; \omega^2\sum_{i=1}^{r}\dfrac{\mathbf{h}_i}{\omega_i^2}y_i \;=\; \mathbf{f} \\[2mm]
\mu_i(\omega_i^2-\omega^2)y_i \;-\; \omega^2\dfrac{\mathbf{h}_i^T}{\omega_i^2}\mathbf{q} \;=\; 0 \qquad i=1,\ldots r
\end{cases} \tag{16}$$

Finally, by taking the inverse Laplace transform $(s = j\omega)$, we obtain the following system of second order ordinary differential equations:

$$\begin{bmatrix} \mathbf{M}^* & \mathbf{H} \\ \mathbf{H}^T & \mathrm{diag}(\mu_i) \end{bmatrix}\begin{Bmatrix} \ddot{\mathbf{q}} \\ \ddot{\mathbf{y}}_E \end{Bmatrix} \;+\; \begin{bmatrix} \mathbf{K}^* & \mathbf{0} \\ \mathbf{0} & \mathrm{diag}(\mu_i\omega_i^2) \end{bmatrix}\begin{Bmatrix} \mathbf{q} \\ \mathbf{y}_E \end{Bmatrix} \;=\; \begin{Bmatrix} \mathbf{f} \\ \mathbf{0} \end{Bmatrix} \tag{17}$$

with $\mathbf{H} = \begin{bmatrix} \frac{\mathbf{h}_1}{\omega_1^2} & \frac{\mathbf{h}_2}{\omega_2^2} & \dots & \frac{\mathbf{h}_r}{\omega_r^2} \end{bmatrix}$, and $\mathbf{y}_E^T = \langle\, y_1 \quad y_2 \quad \dots \quad y_r \,\rangle$ the (elastic) mode amplitudes vector.

Remarks

• The matrices above take the same form as in the Craig-Bampton condensation method as expected.

• The local degrees of freedom will be partitioned into reference node values $\mathbf{q}_O$ and boundary values $\mathbf{q}_{loc}$. The strain energy will be a quadratic function of the boundary displacements $\mathbf{q}_{loc}$.

<div align="center">

RESTRICTED ADMITTANCE MODEL

</div>

Let the restricted admittance matrix $\mathbf{Y}(\omega^2)$ of a body be such that

$$\mathbf{q}_{loc} = \mathbf{Y}(\omega^2)\,\mathbf{f}_{loc} \tag{18}$$

$\mathbf{q}_{loc}$ is the vector of displacements and rotations in the local frame at a given set of points of the body and $\mathbf{f}_{loc}$ the vector of conjugated applied forces and torques. The truncated spectral expansion of the admittances can be written in any of the following forms:

$$\mathbf{Y}(\omega^2) = -\frac{\mathbf{B}}{\omega^2} + \mathbf{G} + \omega^2 \sum_{i=m+1}^{r} \frac{\mathbf{B}_i}{\omega_i^2(\omega_i^2 - \omega^2)} = -\frac{\mathbf{B}}{\omega^2} + \mathbf{G}_r + \sum_{i=m+1}^{r} \frac{\mathbf{B}_i}{\omega_i^2 - \omega^2} \tag{19}$$

where $\mathbf{B}$ is the rigid mobility matrix, $\mathbf{G}$ is the (restricted) flexibility matrix, $\mathbf{B}_j$ is the modal mobility matrix, m the number of rigid body modes, r the total number of modes retained in the modal expansion (including the rigid body modes), and $\mathbf{G}_r$ the residual flexibility matrix. The rigid mobility matrix can be expressed in terms of rigid body modes in the form:

$$\mathbf{B} = \sum_{i=1}^{m} \frac{\phi_i \phi_i^T}{\mu_i} \tag{20}$$

while flexible mobility matrices are expressed in terms of flexible vibration modes as

$$\mathbf{B}_i = \frac{\phi_i \phi_i^T}{\mu_i} \qquad i = m+1, \dots r \tag{21}$$

The residual flexibility matrix can be computed by subtracting the modal contribution to the flexibility from the full flexibility as follows:

$$\mathbf{G}_r = \mathbf{G} - \sum_{i=m+1}^{r} \frac{\mathbf{B}_i}{\omega_i^2} \tag{22}$$

The rigid body modes can be considered as modes with null eigenfrequency ($\omega_i^2 = 0$), therefore the alternate form of the truncated admittance can be rewritten

$$\mathbf{Y}(\omega^2) = \mathbf{G}_r + \sum_{i=1}^{r} \frac{\mathbf{B}_i}{\omega_i^2 - \omega^2} \tag{23}$$

We now replace the latter equation into equation (18), to get:

$$\mathbf{q}_{loc} = \mathbf{G}_r \mathbf{f}_{loc} + \sum_{i=1}^{r} \frac{\phi_i \phi_i^T \mathbf{f}_{loc}}{\mu_i(\omega_i^2 - \omega^2)} \tag{24}$$

Then define mode amplitudes y_i in the form:

$$y_i = \frac{\phi_i^T \mathbf{f}_{loc}}{\mu_i(\omega_i^2 - \omega^2)} \tag{25}$$

so that equation (24) reads:

$$q_{loc} = G_r f_{loc} + \sum_{i=1}^{r} \phi_i y_i \tag{26}$$

Equations (25, 26) can be grouped into a single matrix equation:

$$\begin{bmatrix} G_r & \Phi \\ \Phi^T & \text{diag}\left(\mu_i\left(\omega_i^2 - \omega^2\right)\right) \end{bmatrix} \begin{Bmatrix} f_{loc} \\ y \end{Bmatrix} = \begin{Bmatrix} q_{loc} \\ 0 \end{Bmatrix} \tag{27}$$

where Φ is the matrix of modes:

$$\Phi = [\phi_1, \phi_2, \ldots \phi_r] \tag{28}$$

The local boundary forces at equation (27) can be explicitly given in terms of local displacements:

$$\begin{bmatrix} G_r^{-1} & -G_r^{-1}\Phi \\ -\Phi^T G_r^{-1} & \Phi^T G_r^{-1}\Phi + \text{diag}\left(\mu_i\left(\omega_i^2 - \omega^2\right)\right) \end{bmatrix} \begin{Bmatrix} q_{loc} \\ y \end{Bmatrix} = \begin{Bmatrix} f_{loc} \\ 0 \end{Bmatrix} \tag{29}$$

Then, by making an inverse Laplace transform we get the following system of second order differential equations:

$$\begin{aligned} \begin{bmatrix} G_r^{-1} & -G_r^{-1}\Phi \\ -\Phi^T G_r^{-1} & \Phi^T G_r^{-1}\Phi + \text{diag}(\mu_i\,\omega_i^2) \end{bmatrix} \begin{Bmatrix} q_{loc} \\ y \end{Bmatrix} & \\ + \begin{bmatrix} 0 & 0 \\ 0 & \text{diag}(\mu_i) \end{bmatrix} \begin{Bmatrix} \ddot{q}_{loc} \\ \ddot{y} \end{Bmatrix} & = \begin{Bmatrix} f_{loc} \\ 0 \end{Bmatrix} \end{aligned} \tag{30}$$

Remarks

- For reasons that will become apparent when computing the strain energy of the superelement, we will partition the modal matrix Φ into two parts: one concerning the rigid body modes Φ_R and a second one grouping the elastic modes Φ_E. Then, equations (30) will be written in the following form, after interchanging rows and columns:

$$\begin{aligned} \begin{bmatrix} \Phi_R^T G_r^{-1}\Phi_R & -\Phi_R^T G_r^{-1} & -\Phi_R^T G_r^{-1}\Phi_E \\ -G_r^{-1}\Phi_R & G_r^{-1} & -G_r^{-1}\Phi_E \\ -\Phi_E^T G_r^{-1}\Phi_R & -\Phi_E^T G_r^{-1} & \Phi_E^T G_r^{-1}\Phi_E + \text{diag}(\mu_{E\,i}\,\omega_i^2) \end{bmatrix} \begin{Bmatrix} y_R \\ q_{loc} \\ y_E \end{Bmatrix} & \\ + \begin{bmatrix} \text{diag}(\mu_{R\,i}) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \text{diag}(\mu_{E\,i}) \end{bmatrix} \begin{Bmatrix} \ddot{y}_R \\ \ddot{q}_{loc} \\ \ddot{y}_E \end{Bmatrix} & = \begin{Bmatrix} 0 \\ f_{loc} \\ 0 \end{Bmatrix} \end{aligned} \tag{31}$$

- In order to correctly account for the nonlinear kinematic relations between global and local variables, the rigid body modes will be recombined to the following (canonic) form:

$$\Phi_R = \begin{bmatrix} \begin{bmatrix} 1 & \tilde{X}_1 \\ 0 & 1 \end{bmatrix} \\ \begin{bmatrix} 1 & \tilde{X}_2 \\ 0 & 1 \end{bmatrix} \\ \vdots \\ \begin{bmatrix} 1 & \tilde{X}_r \\ 0 & 1 \end{bmatrix} \end{bmatrix} \tag{32}$$

where X_i gives the coordinates of node i in the reference frame and where the reference frame is selected as coincident with the principal axes of inertia of the body (clearly, in this frame and using the canonic expression of the rigid body modes, the terms $\mu_{R\,i}$ are the total mass and principal moments of inertia of the body).

- The mass and stiffness matrices described in this section take the same form as in the Mac Neal condensation method. Note that the mass matrix has not full rank. The Rubin substructuring method, in which a full rank mass matrix is obtained, and which furnish better convergence properties, can be retrieved by taking into account higher order terms in the development of the admittance (equation (19)).

COMPUTATION OF THE STRAIN ENERGY

The energy of deformation of the body can be directly obtained as a quadratic form in terms of local values of displacements and of the amplitudes of (elastic) modes:

$$\pi = \tfrac{1}{2} \left\{ \begin{matrix} \mathbf{q}_{loc} \\ \mathbf{y}_E \end{matrix} \right\}^T \mathbf{S} \left\{ \begin{matrix} \mathbf{q}_{loc} \\ \mathbf{y}_E \end{matrix} \right\} \tag{33}$$

with $\mathbf{q}_{loc}$ the nodal displacements vector at the boundaries measured in the local frame, $\mathbf{y}_E$ the (elastic) mode amplitudes vector and $\mathbf{S}$ the reduced stiffness matrix computed in either form according to the procedures outlined in the previous sections. Note that $\mathbf{S}$ is in fact the submatrix corresponding to the vector of local displacements $\mathbf{q}_{loc}$ and elastic mode amplitudes $\mathbf{y}_E$; clearly, the reference values amplitudes does not contribute to the deformation energy.

By taking into account equation (5) which expresses the condition of small displacements and rotations in the local frame, the variation of generalized displacements can be computed in terms of the variation of the superelement degrees of freedom $\mathbf{q}$ as follows:

$$\left\{ \begin{matrix} \delta\mathbf{q}_{loc} \\ \delta\mathbf{y}_E \end{matrix} \right\} = \left\{ \begin{matrix} \delta\mathbf{u}_B \\ \delta\psi_B \\ \delta\mathbf{y}_E \end{matrix} \right\} \simeq \begin{bmatrix} -\mathbf{R}_O^T & \widetilde{\mathbf{X}}_B & \mathbf{R}_O^T & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \left\{ \begin{matrix} \delta\mathbf{x}_O \\ \delta\Theta_O \\ \delta\mathbf{x}_B \\ \delta\Theta_B \\ \delta\mathbf{y}_E \end{matrix} \right\} = \Upsilon\,\delta\mathbf{q} \tag{34}$$

where $\delta\Theta_O, \delta\Theta_B$ are respectively the angular displacements variations at the reference frame and at the boundary nodes and where the vector of superelement degrees of freedom $\mathbf{q} = \left\{ \begin{matrix} \mathbf{q}_{abs} \\ \mathbf{y}_E \end{matrix} \right\}$ embodies positions and rotations at the global frame and modal displacements.

The internal forces vector of the superelement is then calculated as follows:

$$\delta\pi = \delta\mathbf{q}^T \Upsilon^T \mathbf{S} \left\{ \begin{matrix} \mathbf{q}_{loc} \\ \mathbf{y}_E \end{matrix} \right\} = \delta\mathbf{q}^T \mathbf{G}_{int} \tag{35}$$

By differentiating the internal forces and by neglecting the derivatives of Υ, we arrive at the expression of the stiffness matrix of the superelement:

$$DG_{int} \cdot \Delta\mathbf{q} \simeq \Upsilon^T \mathbf{S} \Upsilon \Delta\mathbf{q} = \mathbf{S}_{sup} \Delta\mathbf{q} \tag{36}$$

with

$$\mathbf{S}_{sup} = \Upsilon^T \mathbf{S} \Upsilon \tag{37}$$

COMPUTATION OF THE KINETIC ENERGY

The kinetic energy of the superelement can be computed from the velocities rotated to the material frame, in a corotational approach:

$$T = \tfrac{1}{2} \left\{ \begin{matrix} \left\{ \begin{matrix} \mathbf{R}_O^T \dot{\mathbf{x}}_O \\ \Omega_O \end{matrix} \right\} \\ \left\{ \begin{matrix} \mathbf{R}_O^T \dot{\mathbf{x}}_B \\ \Omega_B \end{matrix} \right\} \\ \dot{\mathbf{y}}_E \end{matrix} \right\}^T \mathbf{M} \left\{ \begin{matrix} \left\{ \begin{matrix} \mathbf{R}_O^T \dot{\mathbf{x}}_O \\ \Omega_O \end{matrix} \right\} \\ \left\{ \begin{matrix} \mathbf{R}_O^T \dot{\mathbf{x}}_B \\ \Omega_B \end{matrix} \right\} \\ \dot{\mathbf{y}}_E \end{matrix} \right\} \tag{38}$$

Here, $\mathbf{R}_O$ gives the rotation at the reference frame, attached to the elastic body at node O. The translation material velocities are computed by projection over the reference frame, while material velocities are the true material velocities at the considered node. The vector $\left\{ \begin{matrix} \mathbf{R}_O^T \dot{\mathbf{x}}_O \\ \mathbf{\Omega}_O \end{matrix} \right\}$ is the vector of (material) velocities at the reference frame, $\dot{\mathbf{y}}_E$ are the time derivatives of the internal (elastic) mode amplitudes, and $\left\{ \begin{matrix} \mathbf{R}_O^T \dot{\mathbf{x}}_B \\ \mathbf{\Omega}_B \end{matrix} \right\}$ is the vector of (material) velocities at the boundary nodes of the superelement which when expanded can be written in the form:

$$\left\{ \begin{matrix} \mathbf{R}_O^T \dot{\mathbf{x}}_B \\ \mathbf{\Omega}_B \end{matrix} \right\} = \left\{ \left\{ \left\{ \begin{matrix} \left\{ \begin{matrix} \mathbf{R}_O^T \dot{\mathbf{x}}_1 \\ \mathbf{\Omega}_1 \end{matrix} \right\} \\ \left\{ \begin{matrix} \mathbf{R}_O^T \dot{\mathbf{x}}_2 \\ \mathbf{\Omega}_2 \end{matrix} \right\} \\ \vdots \end{matrix} \right\} \right\}^T \quad \left\{ \begin{matrix} \left\{ \mathbf{R}_O^T \dot{\mathbf{x}}_{i1} \right\} \\ \left\{ \mathbf{R}_O^T \dot{\mathbf{x}}_{i2} \right\} \\ \vdots \end{matrix} \right\}^T \quad \left\{ \begin{matrix} \left\{ \mathbf{\Omega}_{j1} \right\} \\ \left\{ \mathbf{\Omega}_{j2} \right\} \\ \vdots \end{matrix} \right\}^T \right\}^T \tag{39}$$

Note that at nodes $i1, i2, \ldots$ of the boundary, only the translation degrees of freedom have been retained to form the superelement, while at nodes $j1, j2, \ldots$ only the rotation terms are conserved.

Equation (38) is not fully consistent with the approximation we followed to evaluate the strain energy. However, it leads to accurate results in accordance to the local linearity assumption. Even when the flexible body suffers large rotations, the material velocities pattern does not change; then, the kinetic energy expression continues to be valid when the system configuration changes.

Inertia forces are computed by differentiating the kinetic energy. Its first variation is :

$$\delta T = \delta \dot{\mathbf{q}}^T \, \mathbf{M} \, \dot{\mathbf{q}} \tag{40}$$

The vector of variations of generalized velocities reads :

$$\delta \dot{\mathbf{q}} = \left\{ \begin{matrix} \left\{ \begin{matrix} \mathbf{R}_O^T \delta \dot{\mathbf{x}}_O - \delta \mathbf{\Theta}_O \times (\mathbf{R}_O^T \dot{\mathbf{x}}_O) \\ \delta \dot{\mathbf{\Theta}}_O + \mathbf{\Omega}_O \times \delta \mathbf{\Theta}_O \end{matrix} \right\} \\ \left\{ \begin{matrix} \mathbf{R}_O^T \delta \dot{\mathbf{x}}_B - \delta \mathbf{\Theta}_O \times (\mathbf{R}_O^T \dot{\mathbf{x}}_B) \\ \delta \dot{\mathbf{\Theta}}_B + \mathbf{\Omega}_B \times \delta \mathbf{\Theta}_B \end{matrix} \right\} \\ \delta \dot{\mathbf{y}}_E \end{matrix} \right\} \tag{41}$$

By introducing the latter expression into (8) and by integrating by parts, we get

$$\begin{aligned} \delta T &= -\delta \mathbf{q} \cdot \mathbf{G}_{iner} \\ &= -\delta \mathbf{q} \cdot \left(\mathcal{R} \, \mathbf{M} \, \mathcal{R}^T \, \ddot{\mathbf{q}} \, - \, \mathcal{R} \left(\mathbf{M} \mathcal{W} + \mathcal{W}^T \mathbf{M} + \mathcal{U}^T \mathbf{M} \right) \mathcal{R}^T \, \dot{\mathbf{q}} \right) \end{aligned} \tag{42}$$

where the generalized velocities and variation of displacements of the superelement are

$$\begin{aligned} \dot{\mathbf{q}}^T &= \langle \dot{\mathbf{x}}_O^T \quad \mathbf{\Omega}_O^T \quad \dot{\mathbf{x}}_B^T \quad \mathbf{\Omega}_B^T \quad \dot{\mathbf{y}}_E^T \rangle \\ \delta \mathbf{q}^T &= \langle \delta \mathbf{x}_O^T \quad \delta \mathbf{\Theta}_O^T \quad \delta \mathbf{x}_B^T \quad \delta \mathbf{\Theta}_B^T \quad \delta \mathbf{y}_E^T \rangle \end{aligned} \tag{43}$$

and where

$$\mathcal{R} = \begin{bmatrix} \mathbf{R}_O & & & \\ & 1 & & \\ & & \mathbf{R}_O & \\ & & & 1 \\ & & & & 1 \end{bmatrix} \qquad \mathcal{W} = \begin{bmatrix} \tilde{\mathbf{\Omega}}_O & & & \\ & 0 & & \\ & & \tilde{\mathbf{\Omega}}_O & \\ & & & 0 \\ & & & & 0 \end{bmatrix}$$

$$\mathcal{U} = \begin{bmatrix} 0 & (\widetilde{\mathbf{R}_O^T \mathbf{x}_O}) & & & \\ & \tilde{\mathbf{\Omega}}_O & & & \\ & (\widetilde{\mathbf{R}_O^T \mathbf{x}_B}) & 0 & & \\ & & & \tilde{\mathbf{\Omega}}_B & \\ & & & & 0 \end{bmatrix} \tag{44}$$

Then, by differentiating the inertia forces with respect to the generalized accelerations in the global frame $\ddot{\mathbf{q}}$ we get the superelement tangent mass matrix $\mathbf{M}_{sup}$:

$$\mathbf{M}_{sup} = \mathcal{R} \, \mathbf{M} \, \mathcal{R}^T \tag{45}$$

The inertia forces also depend on the velocities $\dot{\mathbf{q}}$. In order to get full quadratic convergence rate, it will be necessary in some cases to compute the gyroscopic matrix of derivatives of the inertia forces with respect to velocities. This is a non symmetric matrix, which proved to be of value for improving convergence in several examples.

$$\mathbf{C}_{sup} = \mathcal{R} \left(\underbrace{[\mathbf{M}\mathcal{U} - \mathcal{U}^T\mathbf{M} - \mathcal{V}]}_{\text{skew symm.}} - \underbrace{[\mathbf{M}\mathcal{W} + \mathcal{W}^T\mathbf{M}]}_{\text{symm.}} \right) \mathcal{R}^T \tag{46}$$

where

$$\mathcal{V} = \begin{bmatrix} \mathbf{0} & \tilde{\mathbf{v}}_{u\,0} & & & \\ \tilde{\mathbf{v}}_{u\,0} & \tilde{\mathbf{v}}_{\Psi\,0} & \tilde{\mathbf{v}}_{u\,B} & & \\ & \tilde{\mathbf{v}}_{u\,B} & \mathbf{0} & & \\ & & & \tilde{\mathbf{v}}_{\Psi\,B} & \\ & & & & \mathbf{0} \end{bmatrix} \tag{47}$$

and where vectors $\mathbf{v}_{u\,i}, \mathbf{v}_{\Psi\,i}$ are computed as follows:

$$\left\{ \begin{array}{c} \left\{ \begin{array}{c} \mathbf{v}_{u\,0} \\ \mathbf{v}_{\Psi\,0} \end{array} \right\} \\ \left\{ \begin{array}{c} \mathbf{v}_{u\,B} \\ \mathbf{v}_{\Psi\,B} \end{array} \right\} \\ \mathbf{v}_E \end{array} \right\} = \mathbf{M}\mathcal{R}^T\dot{\mathbf{q}} \tag{48}$$

We note that the pseudo-damping matrix $\mathbf{C}_{sup}$ is formed by adding up two terms: a symmetric and a skew symmetric matrices which are clearly identified in equation (14). We finally remark that in this formulation, all contributions to the inertia terms (inertia forces $\mathbf{G}_{iner}$, mass matrix $\mathbf{M}_{sup}$ and pseudo-damping matrix $\mathbf{C}_{sup}$ are evaluated directly from the reduced mass matrix $\mathbf{M}$, the projection over the modal basis of the finite element mass matrix. In this way, we can very easily interface the vibration analysis code and the mechanism analysis module. It should also be noted that, contrary to what happened with the technique we proposed in ref.[12], it is not necessary to include all degrees of freedom at each boundary node but only the triplets of positions and/or rotations.

Remarks

- Note that when using the matrices derived from the admittance model (Mac Neal substructuring method), the mass matrix takes a particularly simple form. In this case, the inertia terms can be simplified to the form:

$$\delta T = -\delta\mathbf{q} \cdot \mathbf{G}_{iner} = -\delta\mathbf{q} \cdot \left(\mathbf{M}\,\ddot{\mathbf{q}} - \mathcal{R}\mathcal{U}_O^T\mathbf{M}\mathcal{R}^T\,\dot{\mathbf{q}}\right) \tag{49}$$

with

$$\mathcal{U}_O = \begin{bmatrix} \mathbf{0} & \widetilde{(\mathbf{R}_O^T\mathbf{x}_O)} & & & \\ & \Omega_O & & & \\ & & \mathbf{0} & & \\ & & & \mathbf{0} & \\ & & & & \mathbf{0} \end{bmatrix} \tag{50}$$

The tangent matrices read in this case:

$$\begin{aligned} \mathbf{M}_{sup} &= \mathbf{M} = \text{diag}(\mu_i) \\ \mathbf{C}_{sup} &= \mathcal{R} \underbrace{[\mathbf{M}\mathcal{U}_O - \mathcal{U}_O^T\mathbf{M} - \mathcal{V}]}_{\text{skew symm.}} \mathcal{R}^T \end{aligned} \tag{51}$$

EXAMPLE

The equations of motion we get by following the techniques described in the preceding paragraphs, are time-integrated using a particular implementation of the Hilbert-Hughes-Taylor algorithm to solve mechanism equations. This implementation of the HHT integrator has special provisions to treat large rotations and the equations of constraint, and is fully described in references [12,20,21].
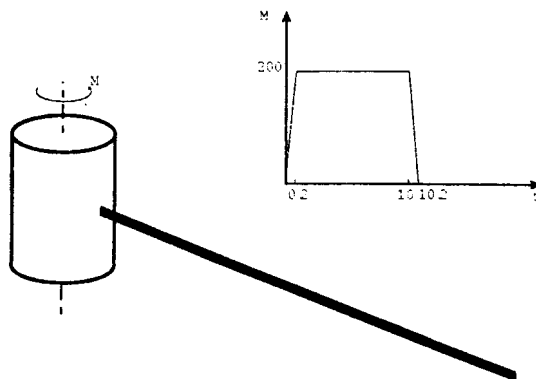
Figure 2 : Articulated beam problem

The example we considered is an articulated-free beam, initially at rest, submitted to a time-varying torque at its base. The beam is articulated to the foundation through a hinge joint (figure 2). The beam physical properties are: length 141.42, mass density 7.8×10^{-3}, cross section 9.0, moment of inertia 6.75, Young modulus $E = 2100000$ and Poisson ratio 0.3.
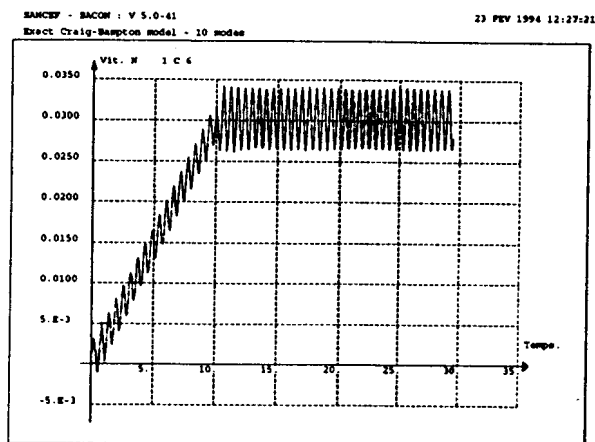
Figure 3: Exact impedance model (10 modes)
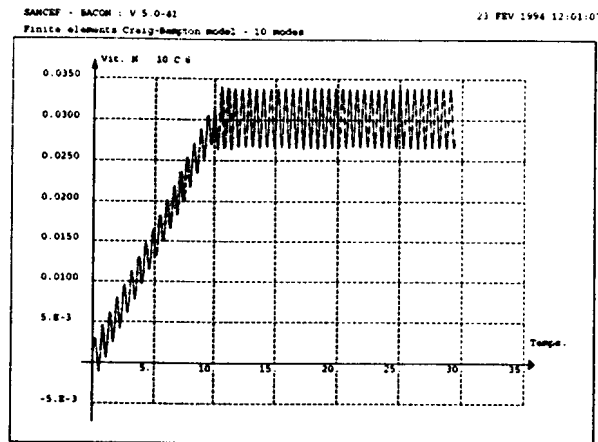*Angular velocity at the base of the beam*

Figure 4: Approximate impedance model (10 modes)
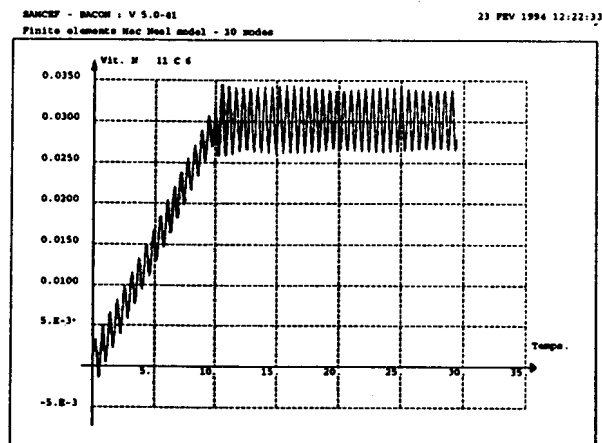_Angular velocity at the base of the beam_

Figure 5: Approximate admittance model (10 modes)
Angular velocity at the base of the beam

The dynamic response was computed using the following substructuring models:

a) An exact impedance model derived using the procedure outlined in reference [14]. The reduced model is formed by retaining translations and rotations at the two extremes of the beam, and by including 10 elastic vibration modes, resulting in a 22 degrees-of-freedom model.

b) An approximate impedance model computed using the finite element method (Craig-Bampton substructuring method). The reduced model is formed by retaining translations and rotations at the base node, and by including 10 elastic vibration modes, resulting in a 16 degrees of freedom model. The finite element mesh used to compute the elastic vibration modes was formed by ten equally spaced beam
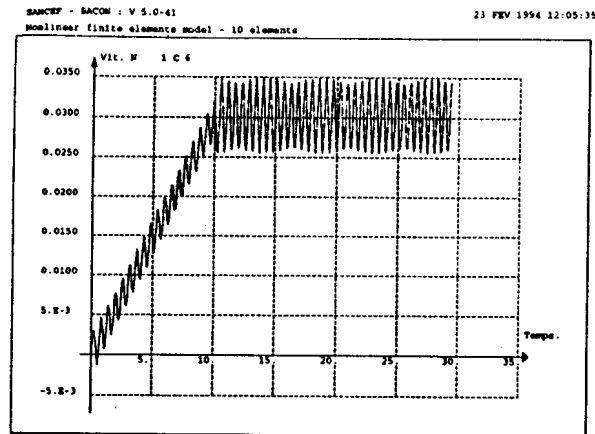
Figure 6: Nonlinear finite elements model
Angular velocity at the base of the beam

elements.

c) An approximate admittance computed from a finite element mesh (Mac-Neal substructuring method). The reduced model is formed by retaining translations and rotations at the base node, and by including 10 modes (the 6 rigid body modes plus 4 elastic free vibration modes), resulting in a 16 degrees of freedom model. The finite element mesh used to compute the elastic vibration modes was formed by ten equally spaced beam elements.

Results obtained using the three models above were compared to those obtained using a model formed by 10 equally-spaced nonlinear beam finite elements (reference [22]).

Figures 3 to 6 display the evolution in time of the angular velocity at the base node for the different models. We can appreciate that results are in almost complete accordance, both from the point of view of global and elastic motion.

CONCLUDING REMARKS

Several formulations of superelements for modeling mechanism members have been reviewed and discussed. We have placed emphasis into comparing the impedance and admittance formalisms, and into obtaining models that describe the flexible members behavior in a multibody analysis from the information contained in either of the above mentioned forms.

The impedance or the admittance expressions could have been derived analytically, numerically or from experimental measures. In particular, we have compared results from a simulation in which a single flexible beam is modeled in three ways:

a) Using an spectral expansion derived from the exact expression of the impedance.

b) Using an spectral expansion derived from an approximate expression of the impedance (built upon a finite element model of the beam).

c) Using an spectral expansion derived from an approximate expression of the admittance (built upon a finite element model of the beam).

Results were finally compared to those obtained from a nonlinear finite element analysis of the same problem.

The methods we have obtained allow in either case an easy interfacing to existing vibration analysis codes,

without requiring the computation of extra-terms within them. This simple interfacing is obtained thanks to a corotational formulation of the inertia terms.

REFERENCES

1. SHABANA A.A. and WEHAGE R.A., *A coordinate reduction technique for dynamic analysis of spatial substructures with large angular rotations* , J. Struct. Mech., Vol. 11 , pp. 401-431 (1983).

2. SHABANA A.A., *Substructure synthesis methods for dynamic analysis of multi-body systems* , Computers and Structures, Vol. 20 , pp. 737-744 (1985).

3. SHABANA A.A., *Dynamics of multibody systems*, J. Wiley & Sons (1989).

4. GERADIN M., ROBERT G. and BERNARDIN C., *Dynamic modeling of manipulators with flexible members*, in *Advanced Software in Robotics*, ed. by A. Danthine and M. Géradin, North-Holland (1984).

5. SINGH R.P., Vander VOORT R.J., and LIKINS P.W., *Dynamics of flexible bodies in tree-topology – A computer-oriented approach* , J. Guidance and Control, Vol. 8 , pp. 584-590 (1985).

6. GERADIN M., BERNARDIN C. and ROBERT G., *A computer approach to dynamic modeling of robot manipulators with flexible members*, Report RA-3, LTAS, University of Liege (1986).

7. AGRAWAL O.P. and SHABANA A.A., *Application of deformable-body mean axis to flexible multibody system dynamics* , Comp. Meth. Appl. Mech. and Engng. , Vol. 56 , pp. 217-245 (1986).

8. SHABANA A.A., *Thermal analysis of viscoelastic multibody systems* , Mechanism and Machine Theory, Vol. 21 , pp. 231-242 (1986).

9. YOO W.S. and HAUG E.J., *Dynamics of articulated structures. Part I. Theory* , J. Struct. Mech, Vol. 14 , pp. 105-126 (1986).

10. YOO W.S. and HAUG E.J., *Dynamics of articulated structures. Part II. Computer implementation and applications* , J. Struct. Mech, Vol. 14 , pp. 177-189 (1986).

11. WU S.C. and HAUG E.J., *Geometric non-linear substructuring for dynamics of flexible mechanical systems*, Int. Jour. Num. Meth. Engng. Vol. 26, pp. 2211-2226 (1988).

12. CARDONA A. and GERADIN M., *A superelement formulation for mechanism analysis*, Comp. Meth. Appl. Mech. Engng., Vol 100, pp. 1-29 (1992).

13. CARDONA A. and GERADIN M., *Modeling of superelements in mechanism analysis*, Int. J. Num. Meth. Engng., Vol. 32, pp. 1565-1593 (1991).

14. THONON C., RIXEN D., GERADIN M. and FARHAT C., *Unification of the impedance and component mode formulations in the assembling of flexible structures*, LTAS Report VA-112, Université de Liège (1993).

15. POELAERT D., *Some properties of the spectral expansion of dynamic impedances and their use in the DISTEL program*, ESA Workshop on *Modal representation of Flexible Structures by Continuum Methods*, ESTEC, Noordwijk, The Netherlands, 15-16 June 1989, Proceedings ESA WPP-09, pp. 71-89.

16. ADMIRE J.R., TINKER M.L. and IVEY E.W., *Residual flexibility test method for verification of constrained structural models*, AIAA Journal, Vol. 32, pp. 170-175 (1994).

17. *SAMCEF, Manuel d'utilisation* LTAS report, University of Liège, Belgium (1987).

18. CARDONA A. and GERADIN M., *SAMCEF: module d'analyse de mécanismes MECANO (manuel d'utilisation)* , LTAS report, University of Liège, Belgium (1987).

19. GERADIN M., PARK K.C. and CARDONA A., *On the representation of finite rotations in spatial kinematics* , LTAS report, University of Liège, Belgium (1988).

20. CARDONA A. and GERADIN M., *Time integration of the equations of motion in mechanism analysis*, Computers and Structures, Vol. 33, pp. 801-820 (1989).

21. CARDONA A. and GERADIN M., *Numerical integration of Second Order Differential-Algebraic Systems in Flexible Mechanism Dynamics*, in: *Computer Aided Analysis of Rigid and Flexible Mechanical Systems*, ed. by M. Seabra Pereira and J. Ambrósio, Kluwer Academic Publishers, pp. 501-529 (1994).

22. CARDONA A. and GERADIN M., *A beam finite element non linear theory with finite rotations*, Int. Jour. Num. Meth. Engng. Vol. 26, pp. 2403-2438 (1988).