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ENERGY CONSERVING TIME INTEGRATION FOR MULTIBODY DYNAMICS APPLICATION TO TOP MOTION

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ABSTRACT

The paper describes a second-order accurate energy conserving method to numerically time-integrate the constrained equations of motion of multibody dynamics. The aspects of finite rotation incrementation, energy conservation in presence of algebraic constraints and numerical implementation are examined in detail. The application is made to the classical problem of a spinning symmetrical top in a gravity field to assess the numerical properties of the method.

INTRODUCTION

It is a well known fact that the time integration of the second-order, index 2 differential - algebraic equations (DAE) which arise in multibody dynamics may lead to numerical instability when using a second-order accurate method of integration of Newmark type [1]. This situation can be attributed to the defective character of the associated eigenvalue problem : the algebraic constraints are indeed responsible for the appearance of a set of infinite eigenvalues with multiplicity 2 which can be shown to induce a weak instability in the acceleration response.

One way to control this instability is to modify the algorithm so as to obtain a small amount of damping at high frequencies. A number of modifications of the classical Newmark time integrator have been proposed to introduce high frequency dissipation while retaining second-order accuracy. Worthwile mentioning methods of this type are the implicit algorithm of Hilber, Hughes and Taylor (HHT) [2-5] and the α - generalized method of Hulbert [8].

An alternate way to achieve stability is based on energy conservation. Indeed, it has been shown that the average acceleration scheme exactly preserves the total energy of the system [9].

Simo and his co-workers have introduced energy preserving algorithms for unconstrained rigid body dynamics [6] and nonlinear elastodynamics [7]. The unconditional stability results then from the energy conservation in the system. A similar energy preserving scheme has been proposed

for nonlinear elastic multibody systems by Bauchau [10]. In this scheme, the equations of motion are discretized in such a way that energy is preserved in the structural parts of the system, while the constraints are discretized in such a way that their work exactly vanishes. The combination of these two features of the discretization guarantee the stability of the numerical integration for constrained multibody systems.

The present contribution is much inspired Bauchau's work, the objective being to assess the numerical efficiency of the energy conserving method for its later application to industrial problems and its implementation in a general multibody dynamics code. The classical problem of the spinning symmetric top in a gravity field is taken as a benchmark problem to formulate in a general manner the dynamic equations of a rigid body under constraints, develop an efficient implementation of the energy preserving scheme and verify the numerical properties of the resulting algorithm.

Particular attention is brought to the parametrization of the finite rotations and their incrementation in time using the mid-point rule. An updated lagrangian point of view is adopted in which the relative rotation between two successive instants is decomposed into two equal rotations which are then expressed either in terms of the vector part of the conformal rotation vector, as proposed in [], or in terms of the vector part of Euler parameters. The representation in terms of Euler parameters is finally adopted because of its even greater simplicity.

Energy conservation is imposed by expressing the balance of énergy on one time step. It is shown that : (i) the mid-point rule automatically guarantees the conservation of the total mechanical energy, (ii) the nullity of the work produced by the constraint forces is achieved by imposing the verification of the time derivative of the contraints at mid-point, which is somewhat equivalent to reduce by one the index of the initial DAE system.

The resulting solution algorithm is expressed in terms of a non-symmetric iteration matrix which is straightforward to obtain. It is then applied to the problem of the spinning top in a gravity field. The energy conservation property is well verified in this case, giving a remarkably neat numerical solution of the problem. Most importantly, it is also experimentally verified, as it could be expected from the energy conservation property, that the drift of the constraints remains negligible throughout the period of integration.

KINEMATICS OF SPHERICAL MOTION

Spherical motion corresponds to the rotation of a rigid body about a fixed point in space. It is characterized by the facts that the length of the position vector of a given point P attached to the rigid body remains unaffected by the pure rotation, and that the relative angle between any two directions attached to the body remains constant under the transformation. To describe it into matrix form, let us define $\mathbf{X}$ the position vector of point P in the reference configuration, of cartesian components $[X_1\ X_2\ X_3]^T$, and $\mathbf{x}$ the position vector of point P after transformation, of cartesian components $[x_1\ x_2\ x_3]^T$.

The pure rotation can be expressed as a linear transformation

$$\mathbf{x} = \mathbf{R}\mathbf{X} \tag{1}$$

the rotation operator $\mathbf{R}$ being proper orthogonal

$$\mathbf{R}^T = \mathbf{R}^{-1} \qquad \text{and} \qquad \det(\mathbf{R}) = 1 \tag{2}$$

The absolute velocity vector of point P is computed in the form

$$\mathbf{v} = \dot{\mathbf{R}}\mathbf{X} = \mathbf{R}\tilde{\mathbf{\Omega}}\mathbf{X} \tag{3}$$

where $\tilde{\mathbf{\Omega}}$ is the skew-symmetric matrix of angular velocities, defined by

$$\tilde{\mathbf{\Omega}} = \mathbf{R}^T\dot{\mathbf{R}} = \begin{bmatrix} 0 & -\Omega_3 & \Omega_2 \\ \Omega_3 & 0 & -\Omega_1 \\ -\Omega_2 & \Omega_1 & 0 \end{bmatrix} \tag{4}$$

The material expression of the angular velocity vector is obtained by extracting the vector part of (4)

$$\mathbf{\Omega} = \text{vect}(\tilde{\mathbf{\Omega}}) \tag{5}$$

The absolute acceleration of point P is obtained through further differentiation

$$\mathbf{a} = \ddot{\mathbf{R}} = \mathbf{R}(\dot{\tilde{\mathbf{\Omega}}} - \tilde{\mathbf{\Omega}}^T\tilde{\mathbf{\Omega}})\mathbf{X} \tag{6}$$

The material expression of the angular acceleration vector is obtained by extracting the vector part from the angular acceleration matrix

$$\dot{\mathbf{\Omega}} = \text{vect}(\dot{\tilde{\mathbf{\Omega}}} - \tilde{\mathbf{\Omega}}^T\tilde{\mathbf{\Omega}}) = \text{vect}(\tilde{\mathbf{\Omega}}) \tag{7}$$

the symmetric term $-\tilde{\mathbf{\Omega}}^T\tilde{\mathbf{\Omega}}$ representing the centrifugal and Coriolis contributions to the acceleration.

The virtual displacement of point P is computed in the same manner as the absolute velocity

$$\delta\mathbf{x} = \delta\mathbf{R}\mathbf{X} = \mathbf{R}\delta\tilde{\mathbf{\Theta}}\mathbf{X} \tag{8}$$

where $\delta\tilde{\mathbf{\Theta}}$ is the skew-symmetric matrix of material rotation increments, defined as

$$\delta\tilde{\mathbf{\Theta}} = \mathbf{R}^T\delta\mathbf{R} = \begin{bmatrix} 0 & -\delta\Theta_3 & \delta\Theta_2 \\ \delta\Theta_3 & 0 & -\delta\Theta_1 \\ -\delta\Theta_2 & \delta\Theta_1 & 0 \end{bmatrix} \tag{9}$$

The vector of material rotation increments (or virtual angular displacements) is obtained by taking the vector part of (9)

$$\delta\mathbf{\Theta} = \text{vect}(\delta\tilde{\mathbf{\Theta}}) \tag{10}$$

The virtual velocities may be computed either through variation of (3) or through time differentiation of (8)

$$\begin{aligned} \delta\mathbf{v} &= \delta(\mathbf{R}\tilde{\mathbf{\Omega}}\mathbf{X}) = \delta\mathbf{R}\tilde{\mathbf{\Omega}}\mathbf{X} + \mathbf{R}\delta\tilde{\mathbf{\Omega}}\mathbf{X} = \mathbf{R}(\delta\tilde{\mathbf{\Theta}}\tilde{\mathbf{\Omega}} + \delta\tilde{\mathbf{\Omega}})\mathbf{X} \\ &= \frac{d}{dt}(\mathbf{R}\delta\tilde{\mathbf{\Theta}}\mathbf{X}) = \dot{\delta}\tilde{\mathbf{\Theta}}\mathbf{X} + \mathbf{R}\delta\tilde{\mathbf{\Theta}}\mathbf{X} = \mathbf{R}(\tilde{\mathbf{\Omega}}\delta\tilde{\mathbf{\Theta}} + \delta\dot{\tilde{\mathbf{\Theta}}})\mathbf{X} \end{aligned} \tag{11}$$

and therefore the virtual angular velocities are related to the time derivatives of the virtual angular displacments by

$$\delta\tilde{\mathbf{\Omega}} = \delta\dot{\tilde{\mathbf{\Theta}}} - \delta\tilde{\mathbf{\Theta}}\tilde{\mathbf{\Omega}} + \tilde{\mathbf{\Omega}}\delta\tilde{\mathbf{\Theta}} \tag{12}$$

The corresponding vector parts are simply related by

$$\delta\mathbf{\Omega} = \delta\dot{\mathbf{\Theta}} + \tilde{\mathbf{\Omega}}\delta\mathbf{\Theta} \tag{13}$$

MOTION EQUATIONS OF A TOP IN DESCRIPTOR FORM

Let us adopt the center of mass of the top as the origin of the material frame, and its attachment point as the origin of the spatial frame. The kinetic energy of the top may then be split into translation and rotation contributions

$$\mathcal{K} = \frac{1}{2}\mathbf{\Omega}^T\mathbf{J}\mathbf{\Omega} + \frac{1}{2}m\dot{\mathbf{x}}^T\dot{\mathbf{x}} \tag{14}$$

where m is the mass of the top, $\dot{\mathbf{x}}$ is the velocity vector of the center of mass expressed in the spatial frame, $\mathbf{J}$ is the inertia tensor of the top measured in material axes and $\mathbf{\Omega}$ is the material expression of the angular velocity vector.

The angular velocity vector may be extracted from the skew symmetric matrix of material angular velocities $\tilde{\mathbf{\Omega}}$, the latter being computed in terms of the rotation operator $\mathbf{R}$

$$\mathbf{\Omega} = \text{vect}(\tilde{\mathbf{\Omega}}) \qquad \text{with} \qquad \tilde{\mathbf{\Omega}} = \mathbf{R}^T\dot{\mathbf{R}} \tag{15}$$

Assuming that the reference for the potential energy is the origin of the spatial frame, the potential energy may be expressed in the form

$$\mathcal{V} = -m\mathbf{g}^T\mathbf{x} \tag{16}$$

where $\mathbf{g}$ is the acceleration vector. For example, if the gravity is acting along the negative x_3 direction, $g = [0 \quad 0 \quad g]^T$ and $\mathcal{V} = +mgx_3$.

Finally, let us denote by vector $-\mathbf{X}_g$ the location of the top attachment point in material coordinates. The center of mass is then constrained to verify at any time instant t the geometric relationship

$$\mathbf{\Phi} = \mathbf{x} - \mathbf{R}\mathbf{X}_g = 0 \tag{17}$$

The Lagrangian of the constrained system can be constructed in the form

$$\mathcal{L} = \mathcal{K} - \mathcal{V} + \boldsymbol{\lambda}^T\mathbf{\Phi} \tag{18}$$

where $\boldsymbol{\lambda}$ is a vector of lagrangian multipliers associated to the constraint. Its components may be interpreted as the reaction forces at the attachment point.

The motion equations result from the application of Hamilton's principle

$$\delta\int_{t_1}^{t_2}\mathcal{L}dt = 0 \tag{19}$$

Substtituting the explicit expression of the Lagrangian and performing the variation yields

$$\int_{t_1}^{t_2}\left[\delta\mathbf{\Omega}^T\mathbf{J}\mathbf{\Omega} + \delta\dot{\mathbf{x}}^Tm\dot{\mathbf{x}} + \delta\mathbf{x}^T(\boldsymbol{\lambda} + m\mathbf{g}) + \delta\boldsymbol{\lambda}^T(\mathbf{x} - \mathbf{R}\mathbf{X}_g) - \boldsymbol{\lambda}^T\delta\mathbf{R}\mathbf{X}_g\right]dt = 0 \tag{20}$$

where the variations of rotation operator and material angular velocities may be related to the material expression of angular virtual displacements by (9) and (13). Substituting them into (20) and performing an integration by parts yields

$$\begin{aligned}&\left[\delta\mathbf{x}^Tm\dot{\mathbf{x}} + \delta\mathbf{\Theta}^T\mathbf{J}\mathbf{\Omega}\right]_{t_1}^{t_2} + \int_{t_1}^{t_2}\left[\delta\mathbf{\Theta}^T(-\mathbf{J}\dot{\mathbf{\Omega}} - \tilde{\mathbf{\Omega}}\mathbf{J}\mathbf{\Omega} - \tilde{\mathbf{X}}_g\mathbf{R}^T\boldsymbol{\lambda})\right.\\ &\left.+\delta\mathbf{x}^T(-m\ddot{\mathbf{x}} + \boldsymbol{\lambda} + m\mathbf{g}) + \delta\boldsymbol{\lambda}^T(\mathbf{x} - \mathbf{R}\mathbf{X}_g)\right]dt = 0\end{aligned} \tag{21}$$

We get thus the motion equations in the differential-algebraic form

$$\begin{aligned} m\ddot{\mathbf{x}} - \boldsymbol{\lambda} &= m\mathbf{g} \\ \mathbf{J}\dot{\boldsymbol{\Omega}} + \tilde{\boldsymbol{\Omega}}\mathbf{J}\boldsymbol{\Omega} + \tilde{\mathbf{X}}_g\mathbf{R}^T\boldsymbol{\lambda} &= 0 \\ -\mathbf{x} + \mathbf{R}\mathbf{X}_g &= 0 \end{aligned} \tag{22}$$

It is of interest to note that by defining the spatial and linear expressions of linear and angular momenta

$$\mathbf{p} = m\dot{\mathbf{x}} \qquad\qquad \mathbf{h} = \mathbf{R}\mathbf{J}\boldsymbol{\Omega} \tag{23}$$

the first two equations (22) may still be rewritten in the simpler form

$$\dot{\mathbf{p}} - \boldsymbol{\lambda} = m\mathbf{g} \qquad\qquad \dot{\mathbf{h}} + \tilde{\mathbf{x}}\boldsymbol{\lambda} = 0 \tag{24}$$

TIME DISCRETIZATION

Both equilibrium equations (24) are expressed at mid-point, and time derivatives are computed using the trapezoidal rule

$$\begin{aligned} \frac{1}{h}(\mathbf{p}_{n+1} - \mathbf{p}_n) - \boldsymbol{\lambda}_{n+\frac{1}{2}} &= m\mathbf{g} \\ \frac{1}{h}(\mathbf{h}_{n+1} - \mathbf{h}_n) + \tilde{\mathbf{x}}_{n+\frac{1}{2}}\boldsymbol{\lambda}_{n+\frac{1}{2}} &= 0 \end{aligned} \tag{25}$$

The special treatment to be applied to the constraint equation will appear naturally later on from energy conservation considerations.

Discretization of linear momentum

The linear momentum equation is next expressed at mid-point

$$\mathbf{p}_{n+\frac{1}{2}} = \frac{1}{2}(\mathbf{p}_n + \mathbf{p}_{n+1}) = \frac{m}{h}(\mathbf{x}_{n+1} - \mathbf{x}_n) \tag{26}$$

giving

$$\mathbf{p}_{n+1} = \frac{2m}{h}(\mathbf{x}_{n+1} - \mathbf{x}_n) - \mathbf{p}_n \tag{27}$$

so that the discretized translation equilibrium equation may be put in the final form

$$\frac{2m}{h^2}(\mathbf{x}_{n+1} - \mathbf{x}_n) - \frac{2}{h}\mathbf{p}_n - \boldsymbol{\lambda}_{n+\frac{1}{2}} = m\mathbf{g} \tag{28}$$

Discretization of angular momentum

The angular momentum equation is expressed likewise

$$\mathbf{h}_{n+\frac{1}{2}} = \frac{1}{2}(\mathbf{h}_n + \mathbf{h}_{n+1}) = \mathbf{R}_{n+\frac{1}{2}}\mathbf{J}\boldsymbol{\Omega}_{n+\frac{1}{2}} \tag{29}$$

with

$$\tilde{\boldsymbol{\Omega}}_{n+\frac{1}{2}} = \mathbf{R}^T_{n+\frac{1}{2}}\dot{\mathbf{R}}_{n+\frac{1}{2}} = \frac{1}{h}\mathbf{R}^T_{n+\frac{1}{2}}(\mathbf{R}_{n+1} - \mathbf{R}_n) \qquad \text{and} \qquad \boldsymbol{\Omega}_{n+\frac{1}{2}} = \text{vect}(\tilde{\boldsymbol{\Omega}}_{n+\frac{1}{2}}) \tag{30}$$

and the discretized rotation equilibrium equation takes thus the form

$$\frac{2}{h^2}\mathbf{R}_{n+\frac{1}{2}}\mathbf{J}\text{vect}(\mathbf{R}^T_{n+\frac{1}{2}}(\mathbf{R}_{n+1} - \mathbf{R}_n)) - \frac{2}{h}\mathbf{h}_n + \tilde{\mathbf{x}}_{n+\frac{1}{2}}\boldsymbol{\lambda}_{n+\frac{1}{2}} = 0 \tag{31}$$

ROTATION PARAMETRIZATION

In order to define the configuration which is half-way between $\mathbf{R}_n$ and $\mathbf{R}_{n+1}$, let us decompose the rotation increment from $\mathbf{R}_n$ to $\mathbf{R}_{n+1}$ in the form of two successive equal rotations

$$\mathbf{R}_n^T \mathbf{R}_{n+1} = \mathbf{G}^2 \tag{32}$$

The resulting operator $\mathbf{G}$ is such that

$$\mathbf{R}_{n+\frac{1}{2}} = \mathbf{R}_n \mathbf{G} = \mathbf{R}_{n+1} \mathbf{G}^T \tag{33}$$

and verifies the orthonormality properties

$$\mathbf{G}\mathbf{G}^T = \mathbf{G}^T \mathbf{G} = \mathbf{I} \tag{34}$$

The matrix of angular velocities (30) may be put in the form

$$\tilde{\mathbf{\Omega}}_{n+\frac{1}{2}} = \frac{1}{h} \mathbf{G}^T \mathbf{R}_n^T (\mathbf{R}_n \mathbf{G}^2 - \mathbf{R}_n) = \frac{1}{h} (\mathbf{G} - \mathbf{G}^T) \tag{35}$$

so that the discretized angular velocities at mid-point are approximated by

$$\mathbf{\Omega}_{n+\frac{1}{2}} = \frac{1}{h} \text{vect}(\mathbf{G} - \mathbf{G}^T) = \frac{2}{h} \text{vect}(\mathbf{G}) \tag{36}$$

Since $\mathbf{G}$ is a rotation operator, $\text{vect}(\mathbf{G})$ has the property

$$\mathbf{G}\,\text{vect}(\mathbf{G}) = \mathbf{G}^T \text{vect}(\mathbf{G}) = \text{vect}(\mathbf{G})$$

which expresses the fact that the rotation direction

$$\mathbf{n} = \frac{\text{vect}(\mathbf{G})}{\|\text{vect}(\mathbf{G})\|} \tag{37}$$

remains invariant under the rotation.

By making use of the above property, the discretized equation of equilibrium can be rewritten in the form

$$\frac{4}{h^2} \mathbf{R}_n \mathbf{G} \mathbf{J} \mathbf{G}^T \text{vect}(\mathbf{G}) - \frac{2}{h} \mathbf{h}_n + \tilde{\mathbf{x}}_{n+\frac{1}{2}} \lambda_{n+\frac{1}{2}} = 0 \tag{38}$$

The matrix $\mathbf{G}$ describing the half rotation may be constructed in two alternative ways:

Euler Parameters

Let us describe the relative rotation from $\mathbf{R}_n$ to $\mathbf{R}_{n+1}$ in terms of its invariants $\mathbf{n}$ and ϕ ($\mathbf{n}$ being the direction of the rotation axis in frame $\mathbf{R}_n$, and ϕ being the amplitude of the rotation) :

$$\mathbf{R}_n^T \mathbf{R}_{n+1} = \mathbf{R}(\mathbf{n}, \phi) \tag{39}$$

with the general expression of the rotation operator

$$\mathbf{R} = \mathbf{R}(\mathbf{n}, \phi) = [\cos\phi \mathbf{I} + (1 - \cos\phi)\mathbf{n}\mathbf{n}^T + \sin\phi \tilde{\mathbf{n}}] \tag{40}$$

Supposing that the direction of the rotation is kept constant, (39) may then be split in two equal rotations of the form

$$\mathbf{G} = \mathbf{R}(\mathbf{n}, \frac{1}{2}\phi) \tag{41}$$

From eqn (40) expressed for $(\mathbf{n}, \frac{1}{2}\phi)$ we note that the vector part of $\mathbf{G}$ is nothing else than the vector part of Euler parameters

$$\text{vect}(\mathbf{G}) = \mathbf{n}\sin\frac{\phi}{2} = \mathbf{e} \tag{42}$$

Euler parameters may thus be used to parametrize matrix $\mathbf{G}$. Substituting eqn (42) in the explicit expression of $\mathbf{R}(\mathbf{n}, \frac{1}{2}\phi)$ yields

$$\mathbf{G} = \mathbf{R}(\mathbf{n}, \frac{1}{2}\phi) = e_0\mathbf{I} + \frac{1}{1+e_0}\mathbf{e}\mathbf{e}^T + \tilde{\mathbf{e}}$$

with

$$e_0 = \cos\frac{\phi}{2} \qquad \text{such that} \qquad e_0^2 = 1 - |\mathbf{e}|^2 \tag{43}$$

The representation involves only the three components e_1, e_2 and e_3, e_0 being computed by the above relation.

From the expression of matrix $\mathbf{G}$ it is easy to reconstruct the expression of $\mathbf{R}$ in terms of Euler parameters

$$\mathbf{R}(\mathbf{e}) = \mathbf{G}^2 = (e_0^2 - \mathbf{e}^T\mathbf{e})\mathbf{I} + 2(\mathbf{e}\mathbf{e}^T + e_0\tilde{\mathbf{e}}) \tag{44}$$

and the following property may also be verified

$$\mathbf{G}\mathbf{e} = \mathbf{G}^T\mathbf{e} = \mathbf{e} \tag{45}$$

which expresses the fact that the direction of the rotation remains unaffected by the half rotation..

The angular velocities at mid point are very simply computed by

$$\mathbf{\Omega}_{n+\frac{1}{2}} = \frac{2}{h}\text{vect}(\mathbf{G}) = \frac{2}{h}\mathbf{e} = \frac{2}{h}\mathbf{G}\mathbf{e} = \frac{2}{h}\mathbf{G}^T\mathbf{e} \tag{46}$$

Conformal Rotation Vector

An alternative and also quite elegant way to perform the decomposition of the relative rotation in two equal parts is to describe the rotation in terms of the conformal rotation vector, defined in terms of Euler parameters as

$$a_i = \frac{4e_i}{1+e_0} \qquad i = 0,1,2,3 \tag{47}$$

Its vector part is then expressed in terms of the rotation invariants $\mathbf{n}$ and ϕ by

$$\mathbf{a} = 4\mathbf{n}\tan\frac{\phi}{4} \tag{48}$$

and the scalar part is expressed in terms of the modulus of the vector part as

$$a_0 = 2 - \frac{a^2}{8} \qquad \text{with} \qquad a^2 = |\mathbf{a}|^2 \tag{49}$$

The representation involves only the three parameters a_1, a_2 and a_3, a_0 being used only for sake of simplification in the notation.

The expression of the rotation tensor is deduced from the expression (44) in terms of Euler parameters by the inverse transformation

$$e_i = \frac{a_i}{4 - a_0} \qquad i = 0,1,2,3 \tag{50}$$

and takes the form

$$\mathbf{R}(\mathbf{a}) = \frac{1}{(4 - a_0)^2}[(a_0^2 - \mathbf{a}^T\mathbf{a})\mathbf{I} + 2(\mathbf{a}\mathbf{a}^T + a_0\tilde{\mathbf{a}})] \tag{51}$$

It is easy to verify that its decomposition into two successive rotations

$$\mathbf{R}(\mathbf{a}) = \mathbf{G}^2 \tag{52}$$

yields

$$\mathbf{G}\frac{1}{(4 - a_0)}[a_0 + \tilde{\mathbf{a}} + \frac{1}{4}\mathbf{a}\mathbf{a}^T] \tag{53}$$

It also maintains the invariance of the rotation direction

$$\mathbf{G}\mathbf{a} = \mathbf{G}^T\mathbf{a} = \mathbf{a} \tag{54}$$

and the computation of its vector part yields

$$2\text{vect}(\mathbf{G}) = \text{vect}(\mathbf{G} - \mathbf{G}^T) = \frac{2}{(4 - a_0)}\mathbf{a} \tag{55}$$

Finally, it may be shown that the material components of the angular velocity vector are obtained in the form

$$\boldsymbol{\Omega} = \frac{2}{(4 - a_0)}\mathbf{G}\dot{\mathbf{a}} \tag{56}$$

In discretized form, we have

$$\boldsymbol{\Omega}_{n+\frac{1}{2}} = \frac{1}{2}(\boldsymbol{\Omega}_n + \boldsymbol{\Omega}_{n+1}) = \frac{2}{h}\text{vect}(\mathbf{G}) = \frac{2\mathbf{a}}{h(4 - a_0)} \tag{57}$$

The use of both Euler and conformal rotation parameters yields very similar expressions of the discretized equations of rotational equilibrium. Substituting (46) and (57) into (31) yields

$$\frac{4}{h^2}\mathbf{R}_n\mathbf{G}\mathbf{J}\mathbf{G}^T\mathbf{e} - \frac{2}{h}\mathbf{h}_n + \tilde{\mathbf{x}}_{n+\frac{1}{2}}\lambda_{n+\frac{1}{2}} = 0 \tag{58}$$

and

$$\frac{4}{(4 - a_0)h^2}\mathbf{R}_n\mathbf{G}\mathbf{J}\mathbf{G}^T\mathbf{a} - \frac{2}{h}\mathbf{h}_n + \tilde{\mathbf{x}}_{n+\frac{1}{2}}\lambda_{n+\frac{1}{2}} = 0 \tag{59}$$

Because of their even greater simplicity to represent the mid-point rotation, the Euler parameters of the half-rotation have been prefered to the conformal rotation vector for the numerical implementation of the method.

ENERGY CONSERVATION

In order to express the balance of energy on one time step, let us multiply the translation equilibrium equation by $h\mathbf{v}_{n+\frac{1}{2}}^T$, the rotational equilibrium equation by $h\mathbf{\Omega}_{n+\frac{1}{2}}^T\mathbf{R}_n^T = 2\text{vect}(\mathbf{G})^T\mathbf{R}_n^T$ and add both terms to compute the scalar quantity

$$\begin{aligned}\mathcal{A} =& h\mathbf{v}_{n+\frac{1}{2}}^T\left[\frac{2}{h^2}m(\mathbf{x}_{n+1}-\mathbf{x}_n)-\frac{2}{h}\mathbf{p}_n-\lambda_{n+\frac{1}{2}}-m\mathbf{g}\right]+\\ & h\mathbf{\Omega}_{n+\frac{1}{2}}^T\mathbf{R}_n^T\left[\frac{4}{h^2}\mathbf{R}_n\mathbf{G}\mathbf{J}\mathbf{G}^T\text{vect}(\mathbf{G})-\frac{2}{h}\mathbf{h}_n+\tilde{\mathbf{x}}_{n+\frac{1}{2}}\lambda_{n+\frac{1}{2}}\right]=0\end{aligned} \tag{60}$$

Let us compute successively

$$\begin{aligned}\mathcal{A}_1 =& h\mathbf{v}_{n+\frac{1}{2}}^T\left(\frac{2}{h^2}m(\mathbf{x}_{n+1}-\mathbf{x}_n)-\frac{2}{h}\mathbf{p}_n-m\mathbf{g}\right)\\ =&\frac{h}{2}(\mathbf{v}_n+\mathbf{v}_{n+1})^T\left(\frac{1}{h}m(\mathbf{v}_{n+1}+\mathbf{v}_n)-\frac{2}{h}m\mathbf{v}_n\right)-(\mathbf{x}_{n+1}-\mathbf{x}_n)m\mathbf{g}\\ =&(\mathcal{K}_{n+1}-\mathcal{K}_n)_{tr}+(\mathcal{V}_{n+1}-\mathcal{V}_n)\end{aligned}$$

$$\begin{aligned}\mathcal{A}_2 =& h\mathbf{\Omega}_{n+\frac{1}{2}}^T\left(\frac{4}{h^2}\mathbf{R}_n\mathbf{G}\mathbf{J}\mathbf{G}^T\text{vect}(\mathbf{G})-\frac{2}{h}\mathbf{h}_n\right)\\ =&2\mathbf{\Omega}_{n+\frac{1}{2}}^T\left(\mathbf{G}\mathbf{J}\mathbf{G}^T\mathbf{\Omega}_{n+\frac{1}{2}}-\mathbf{R}_n^T\mathbf{h}_n\right)\\ =&\frac{1}{2}(\mathbf{\Omega}_n+\mathbf{\Omega}_{n+1})^T\mathbf{J}(\mathbf{\Omega}_{n+1}+\mathbf{\Omega}_n)-(\mathbf{\Omega}_{n+1}+\mathbf{\Omega}_n)^T\mathbf{J}\mathbf{\Omega}_n\\ =&\frac{1}{2}(\mathbf{\Omega}_{n+1}+\mathbf{\Omega}_n)^T\mathbf{J}(\mathbf{\Omega}_{n+1}-\mathbf{\Omega}_n)\\ =&(\mathcal{K}_{n+1}-\mathcal{K}_n)_{rot}\end{aligned} \tag{61}$$

and therefore

$$\mathcal{A}_1+\mathcal{A}_2=\mathcal{K}_{n+1}-\mathcal{K}_n+\mathcal{V}_{n+1}-\mathcal{V}_n \tag{62}$$

Equation (62) expresses the conservation of energy over one time step provided that we have

$$\mathcal{A}_3=-h\mathbf{v}_{n+\frac{1}{2}}^T\lambda_{n+\frac{1}{2}}+h\mathbf{\Omega}_{n+\frac{1}{2}}^T\mathbf{G}^T\mathbf{R}_n^T\tilde{\mathbf{x}}_{n+\frac{1}{2}}\lambda_{n+\frac{1}{2}}=0 \tag{63}$$

Since

$$\tilde{\mathbf{x}}_{n+\frac{1}{2}}=\mathbf{R}_{n+\frac{1}{2}}\tilde{\mathbf{X}}_g\mathbf{R}_{n+\frac{1}{2}}^T \tag{64}$$

we get the condition

$$(-\mathbf{v}_{n+\frac{1}{2}}^T+\mathbf{\Omega}_{n+\frac{1}{2}}^T\tilde{\mathbf{X}}_g\mathbf{R}_{n+\frac{1}{2}}^T)\lambda_{n+\frac{1}{2}}=0 \tag{65}$$

which is fulfilled if

$$\mathbf{v}_{n+\frac{1}{2}}=-\mathbf{R}_{n+\frac{1}{2}}\tilde{\mathbf{X}}_g\mathbf{\Omega}_{n+\frac{1}{2}}=+\mathbf{R}_{n+\frac{1}{2}}\tilde{\mathbf{\Omega}}_{n+\frac{1}{2}}\mathbf{X}_g \tag{66}$$

The condition (66) corresponds to the time derivative of the initial constraint expressed at midpoint. Owing to (27), it can still be rewritten

$$-\mathbf{x}_{n+1}+\mathbf{x}_n-2\mathbf{R}_n\mathbf{G}\tilde{\mathbf{X}}_g\mathbf{G}^T\mathbf{e}=0 \tag{67}$$

NONLINEAR SOLUTION

In view of their numerical solution, the motion equations at $t_{n+\frac{1}{2}}$ are written in residual form

$$\mathbf{r} = \begin{bmatrix} \mathbf{r}_x \\ \mathbf{r}_e \\ \mathbf{r}_\lambda \end{bmatrix} \begin{bmatrix} \mathbf{f}_{ext} + \boldsymbol{\lambda} + \frac{2}{h}\mathbf{p}_n - \frac{2m}{h^2}(\mathbf{x}_{n+1} - \mathbf{x}_n) \\ -\tilde{\mathbf{x}}_{n+\frac{1}{2}}\boldsymbol{\lambda} + \frac{2}{h}\mathbf{h}_n - \frac{4}{h^2}\mathbf{R}_n\mathbf{G}\mathbf{J}\mathbf{e} \\ \mathbf{x}_{n+1} - \mathbf{x}_n + 2\mathbf{R}_n\mathbf{G}\tilde{\mathbf{X}}_g\mathbf{e} \end{bmatrix} = 0 \tag{68}$$

with

$$\tilde{\mathbf{x}}_{n+\frac{1}{2}} = \mathbf{R}_{n+\frac{1}{2}}\tilde{\mathbf{X}}_g\mathbf{R}^T_{n+\frac{1}{2}} \tag{69}$$

The tangent iteration matrix is the jacobian matrix of (68)

$$\mathbf{S} = -\frac{\partial(\mathbf{r}_x, \mathbf{r}_e, \mathbf{r}_\lambda)}{\partial(\mathbf{x}, \mathbf{e}, \boldsymbol{\lambda})} = \begin{bmatrix} \mathbf{S}_{xx} & 0 & \mathbf{S}_{x\lambda} \\ 0 & \mathbf{S}_{ee} & \mathbf{S}_{e\lambda} \\ \mathbf{S}_{\lambda x} & \mathbf{S}_{\lambda e} & 0 \end{bmatrix} \tag{70}$$

It is not symmetric, its different terms being given by

$$\begin{aligned}
\mathbf{S}_{xx} &= -\frac{\partial \mathbf{r}_x}{\partial \mathbf{x}} = \frac{2m}{h^2}\mathbf{I} \\
\mathbf{S}_{x\lambda} &= -\frac{\partial \mathbf{r}_x}{\partial \boldsymbol{\lambda}} = -\mathbf{I} \\
\mathbf{S}_{ee} &= -\frac{\partial \mathbf{r}_e}{\partial \mathbf{e}} = \mathbf{R}_n\left\{\frac{4}{h^2}\mathbf{G}\mathbf{J} + \sum_i\left[\frac{4}{h^2}\frac{\partial \mathbf{G}}{\partial \mathbf{e}_i}\mathbf{J} + \left(\frac{\partial \mathbf{G}}{\partial \mathbf{e}_i}\tilde{\mathbf{X}}_g\mathbf{G}^T + \mathbf{G}\tilde{\mathbf{X}}_g\frac{\partial \mathbf{G}^T}{\partial \mathbf{e}_i}\right)\mathbf{R}^T_n\boldsymbol{\lambda}\right]\mathbf{k}^T_i\right\} \\
\mathbf{S}_{e\lambda} &= -\frac{\partial \mathbf{r}_e}{\partial \boldsymbol{\lambda}} = \tilde{\mathbf{x}}_{n+\frac{1}{2}} \\
\mathbf{S}_{\lambda x} &= -\frac{\partial \mathbf{r}_\lambda}{\partial \mathbf{x}} = -\mathbf{I} \\
\mathbf{S}_{\lambda e} &= -\frac{\partial \mathbf{r}_\lambda}{\partial \mathbf{e}} = -2\mathbf{R}_n\left[\mathbf{G}\tilde{\mathbf{X}}_g + \sum_i \frac{\partial \mathbf{G}}{\partial \mathbf{e}_i}\tilde{\mathbf{X}}_g\mathbf{e}\mathbf{k}^T_i\right]
\end{aligned} \tag{71}$$

and $\mathbf{k}_i$ being the unit vector along direction i.

The time integration procedure is then as follows :

1. *Initialization*

Given : $t_0 = 0,\ \mathbf{R}_0,\ \boldsymbol{\Omega}_0$

Compute : $\mathbf{x}_0 = \mathbf{R}_0\mathbf{X}, \quad \mathbf{v}_0 = \mathbf{R}_0\boldsymbol{\Omega}_0\mathbf{X}$

$\mathbf{p}_0 = m\mathbf{v}_0, \quad \mathbf{h}_0 = \mathbf{R}_0\mathbf{J}\boldsymbol{\Omega}_0$

2. *Time integration : while* $t_n < h$ *do*

(i) *increment time :*

$$t_{n+1} = t_n + h$$

(i) *predict new solution*

$$\mathbf{x}_{n+1} = \mathbf{x}_n + \frac{1}{2}h\mathbf{v}_n\ , \qquad \mathbf{e}_{n+1} = \mathbf{e}_n\ , \qquad \boldsymbol{\lambda} = 0$$

(iii) *iterate :*

while $(\|\mathbf{r}_x\| > \epsilon_x$ .or. $\|\mathbf{r}_e\| > \epsilon_e$ .or. $\|\mathbf{r}_\lambda\| > \epsilon_\lambda)$ *do*

- *Displacement and rotation correction at* $t_{n+\frac{1}{2}}$ *:*

$$\mathbf{R}_{n+\frac{1}{2}} = \mathbf{R}_n\mathbf{G}\ , \qquad \mathbf{x}_{n+\frac{1}{2}} = \mathbf{R}_{n+\frac{1}{2}}\tilde{\mathbf{X}}_g\mathbf{R}^T_{n+\frac{1}{2}}$$

- *residual evaluation :* $\mathbf{r}_x, \mathbf{r}_e, \mathbf{r}_\lambda$
- *linear solution :* $\mathbf{S}d\mathbf{q} = \mathbf{r} \quad \Rightarrow \quad d\mathbf{x}, d\mathbf{e}, d\boldsymbol{\lambda}.$
- *incrementation :*

$$\mathbf{x}_{n+1} = \mathbf{x}_{n+1} + d\mathbf{x}\ , \qquad \mathbf{e}_{n+1} = \mathbf{e}_{n+1} + d\mathbf{e}\ , \qquad \boldsymbol{\lambda} = \boldsymbol{\lambda} + d\boldsymbol{\lambda}$$

end

(iv) *solution updating at* $t_{n+\frac{1}{2}}$ *:*

$$\mathbf{R}_{n+1} = \mathbf{R}_{n+\frac{1}{2}}\mathbf{G}$$

$$\mathbf{v}_{n+1} = \frac{2}{h}(\mathbf{x}_{n+1} - \mathbf{x}_n) - \mathbf{v}_n\ ,\ \mathbf{p}_{n+1} = m\mathbf{v}_{n+1}$$

$$\mathbf{h}_{n+1} = \frac{4}{h}\mathbf{R}_{n+\frac{1}{2}}\mathbf{J}\mathbf{e}_{n+1} - \mathbf{h}_n\ ,\ \boldsymbol{\Omega}_{n+1} = (\mathbf{R}_{n+1}\mathbf{J})^{-1}\mathbf{h}_{n+1}$$

end.

NUMERICAL APPLICATION

In order to demonstrate the numerical properties of the energy conserving methodology and algorithm, let us consider the problem of determining the trajectory of a symmetrical top in a gravity field.

It is classical to describe the instantaneous motion of the top in terms of Euler angles. To this

purpose, let us recall the expression of the rotation operator in terms of the Euler angles (ϕ, θ, ψ)

$$\begin{aligned}\mathbf{R} &= \mathbf{R}(x_3,\phi)\,\mathbf{R}(x_1',\theta)\,\mathbf{R}(x_3'',\psi)\\ &= \begin{bmatrix} \cos\phi\cos\psi - \sin\phi\cos\theta\sin\psi & -\cos\phi\sin\psi - \sin\phi\cos\theta\cos\psi & \sin\phi\sin\theta \\ \sin\phi\cos\psi + \cos\phi\cos\theta\sin\psi & -\sin\phi\sin\psi + \cos\phi\cos\theta\cos\psi & -\cos\phi\sin\theta \\ \sin\theta\sin\psi & \sin\theta\cos\psi & \cos\theta \end{bmatrix}\end{aligned}$$

where the ($'$) indicates that the current frame is modified by the previous transformation. Angle ψ corresponds to the spin of the top about its rotation axis; θ gives the inclination of the top axis with respect to the vertical, and ϕ gives the azimutal position of the top axis in the horizontal plane Ox_1x_2, describing thus the precession motion.

The inverse transformation from rotation operator to Euler angles (becoming singular when $\theta = 0$ or π) is

$$\begin{aligned}\psi &= \tan^{-1}(r_{31}, r_{32})\\ \theta &= \tan^{-1}(r_{31}\sin\psi + r_{32}\cos\psi, r_{33})\\ \psi &= \tan^{-1}(r_{21}\cos\psi - r_{22}\sin\psi, r_{11}\cos\psi - r_{12}\sin\psi)\end{aligned}$$

The time derivatives of Euler angles are related to the material angular velocities by

$$\begin{bmatrix}\dot\phi\\ \dot\theta\\ \dot\psi\end{bmatrix} = \begin{bmatrix} \frac{\sin\psi}{\sin\theta} & \frac{\cos\psi}{\sin\theta} & 0 \\ \cos\psi & -\sin\psi & 0 \\ \frac{-\sin\psi\cos\theta}{\sin\theta} & \frac{-\cos\psi\cos\theta}{\sin\theta} & 1 \end{bmatrix}\begin{bmatrix}\Omega_1\\ \Omega_2\\ \Omega_3\end{bmatrix}$$

Let us consider a symmetrical top with the following properties. Mass m : 5 kg, moments of inertia : $J_{11} = J_{22} = 0.8\ Kgm^2$, $J_{33} = 1.8\ kgm^2$, distance from CG to origin (attachment point) : $L = 1.3m$, gravity : $g = 9.81m/s^2$ (along negative x_3 axis). The initial position of the top is described in terms of Euler angles : $\phi_0 = 0$, $\theta_0 = \frac{\pi}{9}rad$, $\phi_0 = 0$.

Two response cases have been considered.

In case 1, the top is simply dropped from its initial position with a spin velocity $\dot\psi_0 = 50rad/s$. Both other angular velocities are zero ($\dot\phi_0 = \dot\theta_0 = 0$). In case 2, the top is thrown from its initial position with a spin velocity $\dot\psi_0 = 50rad/s$, a precession angular velocity of $\dot\phi_0 = -10rad/s$ and a zero nutation angular velocity ($\dot\theta_0 = 0$).

Figures 1 and 2 display both computed responses in various forms and in terms of different kinematic and kinetic quantities, namely : (a) vertical displacement versus time, (b) three-dimensional trajectory of the CG, (c) Euler angle θ versus time, (d) Euler angle ϕ versus time, (e) time evolution of the kinematic constraints, (f) phase diagram of the non cyclic variable θ, (g) relative energy variation $E/E_0 - 1$ versus time, (h) transverse angular velocity Ω_1 versus time. The total energy of the system is obviously conserved during the period of observation of the motion as it could be expected from the very design of the integration algorithm. All the time evolutions demonstrate also that the periodic character of the motion is perfectly preserved, which can be regarded as a direct consequence of the energy conservation property. Of interest also is the time evolution of the kinematic constraints : despite of the fact that they are satisfied only in weak form, their drift remains extremely small ($\leq 2.E-7$), observation which is a further consequence of energy conservation.

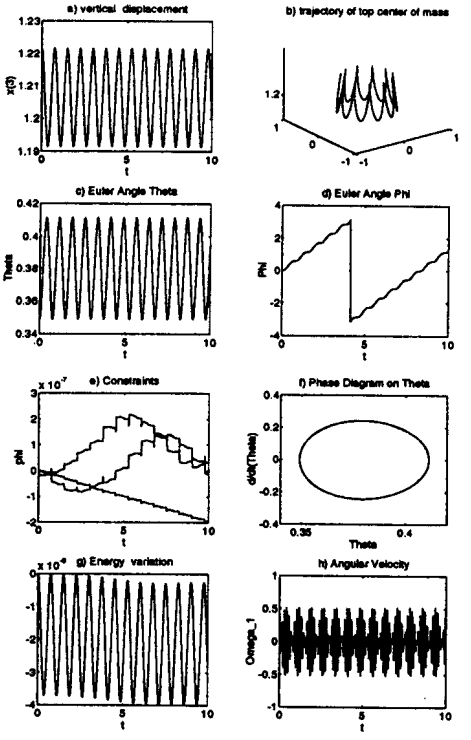

Figure 1 : Response of symmetrical top

Case 1 : $\dot{\psi}_0 = \dot{\phi}_0 = 0\ rad/s$

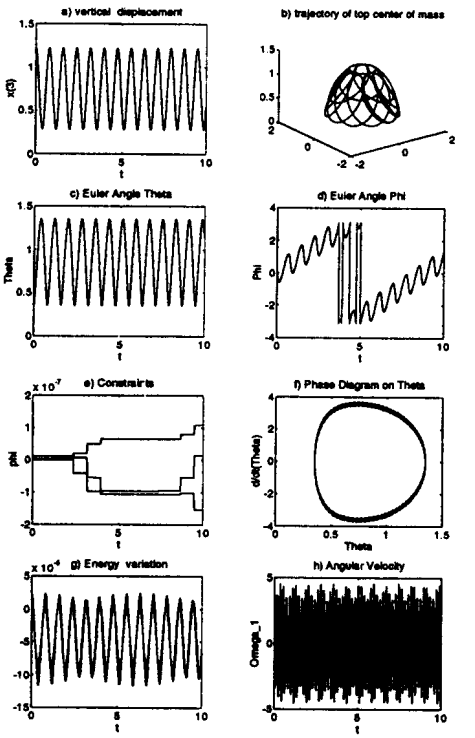

Figure 2 : Response of symmetrical top

Case 2 : $\dot{\psi}_0 = \dot{\phi}_0 = -10\ rad/s$

CONCLUSION

The energy conserving method is a very effective approach to time-integrate the second-order DAE equations arising in multibody dynamics since it provides a natural way to control the instability induced by the kinematic constraints. When formulated in updated lagrangian form in terms of the Euler parameters of the half rotation, it leads to a very simple form of the discretized equations.

Further aspects of the method such as automatic time stepping, the extension to flexible systems and possibly the symmetrization of the linearized equations still have to be investigated in view of the implementation in an industrial code.

Of further interest also is the generalization in the form of an energy decaying scheme as proposed in [11].

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