A COMPARISON BETWEEN THREE VARIABLE-STEP ALGORITHMS FOR THE INTEGRATION OF THE EQUATIONS OF MOTION IN STRUCTURAL DYNAMICS

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ABSTRACT

Three variable-step algorithms for time integrating the equations of motion in structural and mechanism analysis are compared. The analyzed algorithms are: the Thomas and Gladwell scheme [1,2], which we modified in reference [3] for application to systems of differential/algebraic equations (VSA23); the Hilber, Hughes and Taylor scheme (HHT) [4,5] with a step-size control strategy; and the DASSL code, developed by L. Petzold [6] and distributed through the NETLIB computer library network. A short theory about the error estimation techniques used by the VSA23 and HHT algorithms is presented. Some numerical test examples are shown, representing the kind of problems the user is faced to in practical applications.

RESUMEN

En este trabajo comparamos tres algoritmos de paso variable para la integración temporal de las ecuaciones de movimiento en análisis estructural y de mecanismos. Los algoritmos analizados son: el esquema de Thomas y Gladwell [1,2], el cual modificamos en la referencia [3] para el tratamiento de sistemas de ecuaciones diferenciales/algebraicas (VSA23); el esquema de Hilber, Hughes y Taylor (HHT) [4,5] al cual incorporamos una estrategia de control del tamaño de paso; y el código DASSL, desarrollado por L. Petzold [6] y distribuido a través de la red de computadoras por la biblioteca NETLIB. Presentamos una breve descripción teórica acerca de las técnicas de estimación del error usadas por los algoritmos VSA23 y HHT. Por último, presentamos varios ejemplos que representan el tipo de problemas que enfrenta el usuario en aplicaciones prácticas.

1. INTRODUCTION

In this paper, we compare three variable-step algorithms for time integrating the equations of motion in structural and mechanism dynamics. The objective is to determine which algorithm guarantees the better behavior in a wide spectrum of cases.

Variable-step strategies are specially well suited to deal with problems characterized by high non-linearity and impacts. Both phenomena are typical of mechanism dynamics simulation. Constant step-size strategies do not give a satisfactory answer for these kind of problems, since it is very difficult to the user to find an appropriate time-step that does not lead to divergence nor generates costly computations.

The comparison to be made is based on two numerical examples. The examples are very simple, but at the same time represent clearly the difficulties one is faced to when solving real problems in structural and mechanisms dynamics.

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The first case to be solved is a two degrees-of-freedom linear system, with very distinct eigenfrequencies. It represents the kind of problems we have when treating a multiple degrees-of-freedom structure with a wide eigenspectrum. The second test is a double pendulum, with geometric non-linearity and constraints in the formulation of the equations of motion. The system to time integrate in this case is a system of differential-algebraic equations, typical of mechanism models.

2. ERROR ESTIMATION, SCALING AND STEP-SIZE SELECTION

The time-step varying strategy relies on three aspects: the error estimator, the scaling of the error measure and the algorithm for modifying the time-step based on the alterations of the scaled error.

This section summarizes the techniques used in the Thomas and Gladwell (VSA23) and Hilber, Hughes and Taylor (HHT) schemes. We do not present any theory about Petzold's DASSL code, since it was retrieved from the computer library network NETLIB and used "as is" (we note however, that the techniques used for the error estimation in this code are very similar in nature to those of the VSA23 code, and can be found elsewhere [6]).

In all cases, the error estimator we use is an estimation of the local truncation error. Then, the time-step is modified according to the variations of this measure.

2.1 The VSA23 code

The VSA23 code is a modified version of the Thomas and Gladwell method [1,2]. The procedure is briefly described here; a thorough description is given in references [3,7].

In VSA23, the local truncation error is measured by comparison of the results of the integrator to those given by a higher order algorithm (e.g. a first order accuracy algorithm is used as integrator while a second order accuracy algorithm sets the reference). Let $q^a_{n+\psi}$ be the position vector given by the integrator and let $q^b_{n+\psi}$ be the position vector predicted using the reference algorithm, both evaluated at the equilibrium time instant t_ψ [3,7]. Then, we can estimate the local error $e_{n+\psi}$ in the following form (see figure (1)) :

$$e_{n+\psi} = q^a_{n+\psi} - q^b_{n+\psi} \tag{1}$$

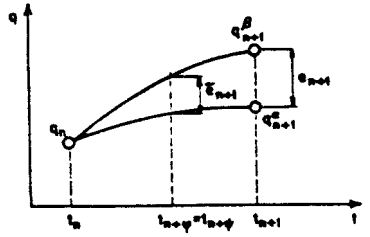

Figure 1 : Error estimation in the VSA23 code

Although the error estimation is actually computed at an intermediate time $t_{n+\psi}$, we use it directly as an estimation of the error at the end of the time step t_{n+1}, i.e.

$$e_{n+1} \cong e_{n+\psi} \tag{2}$$

This measure is afterwards scaled (using a characteristic length of the problem being solved) and used to determine the step-size of the integrator.

The selection of the step-size in this code is practically in the original form proposed by Thomas and Gladwell, with some slight modifications introduced by the authors in [3,7]. The strategy is based on maintaining, at every time

instant, the scaled error measure $\bar{e}_{n+1}$ beneath a user specified tolerance *TOL*:

$$\bar{e}_{n+1} < TOL \tag{3}$$

When the error estimation satisfies the user prescribed tolerance TOL, the step n is accepted and the next step-size h_{n+1} is predicted by

$$h_{n+1} \leftarrow r\, h_n$$
$$r = \left\{ 0.8 \sqrt{\frac{\mathrm{TOL}}{\bar{e}_{n+1}}} \;;\; \text{with } r \text{ s.t. } 1.0 \le r \le 2.0 \right\} \tag{4}$$

where the factor 0.8 is used to assure that the error computed using the new time step estimation will verify the specified tolerance.

On the other hand, if the error estimation is greater than TOL, the step n is rejected and a new try is made using a time step given by

$$h_n \leftarrow r\, h_n$$
$$r = \left\{ 0.8 \sqrt{\frac{\mathrm{TOL}}{\bar{e}_{n+1}}} \;;\; \text{with } r \text{ s.t. } 0.5 \le r \le 0.9 \right\} \tag{5}$$

2.2 The Hilber-Hughes-Taylor algorithm

This algorithm is currently implemented in MECANO [8] as a constant step-size integrator. We adopted a simple strategy, based on estimating the local error by differences between the predicted higher order derivatives, and transformed it into a variable-step one [9]. The resulting scheme preserves the second order accuracy of the HHT for appropriate values of the algorithm parameters.

Let: $q_n, \dot{q}_n, \ddot{q}_n$ be the computed positions, velocities and accelerations at time t_n. Then, it can be easily shown that the positions predicted by the HHT algorithm at t_{n+1} can be expressed in the form:

$$q_{n+1} = q_n + h_n \dot{q}_n + \frac{1}{2} h_n^2 \ddot{q}_n + \beta\, h_n^3\, \delta_n \tag{6}$$

where

$$\delta_n = \frac{\ddot{q}_{n+1} - \ddot{q}_n}{h_n}$$

and where $h_n = t_{n+1} - t_n$ is the current step-size and β is an algorithm parameter influencing its accuracy and dissipation.

Comparing (6) with the Taylor's series expansion around t_n, we can estimate the local error looking at the last "complete" term :

$$e_{n+1} = \frac{h^3\, q^{(3)}(t_n)}{3!} \tag{7}$$

$q^{(3)}(t_n)$ is the third order derivative of $q(t)$ at t_n with respect to time, which can be estimated from the difference between acceleration vectors:

$$\left(\frac{d\ddot{q}}{dt}\right)_n \approx \frac{\ddot{q}_{n+1} - \ddot{q}_n}{h_n} = \delta_n \tag{8}$$

Then, the estimation of the local truncation error is:

$$e_{n+1} = \frac{h_n^2}{6} \left(\ddot{q}_{n+1} - \ddot{q}_n\right) \tag{9}$$

The following strategy has been followed, in order to determine reference values to serve as a basis for comparison. We distinguished between two kinds of degrees of freedom in a mechanism model: the values of positions and of rotations. Following Shampine and Gordon [10], we compute then a reference value at each degree of freedom by adding up two terms: a relative and an absolute part:

$$q_{ref\,i} = |q_{i\,n}| + q_{char\,i} \tag{10}$$

with $q_{char\ i}$, a characteristic value for the considered DOF, evaluated using the rule:

$$q_{char\ i} = \begin{cases} \dfrac{L_d}{\sqrt{NEL}} & \text{for position DOF's} \\ \dfrac{\pi}{10} & \text{for rotation DOF's} \end{cases} \tag{11}$$

L_d is the diagonal of a box containing the entire finite element model and NEL the number of elements.

A scaled measure of the error is computed using the L_∞ norm and compared to a user-specified tolerance TOL:

$$\tilde{e}_{n+1} = \max_{i=1,N} \left(\frac{e_i}{q_{ref\ i}} \right) \leq \text{TOL} \tag{12}$$

This condition is used as criterion of acceptance or rejection of a computed step.

The time step is adjusted trying to maintain the error estimation below the tolerance. The variation is made by following a strategy intended to keep the time step unchanged during long periods, avoiding a continuously time step changing mode of operation that would deteriorate the algorithm performance.

By following ideas presented in reference [10], we tried to keep the error value equal to $TOL/2$ at any time instant and distinguished between four different conditions to select the time step:

i. If the computed error is greater than the tolerance, we reject the previous step and recompute it using an increment equal to half the previous value.

ii. If the error is less than the tolerance, but greater than half its value, we accept the computed time step but decrease the next time increment trying to make the error equal to half the tolerance (see figure 2).

iii. If the error is less than half the tolerance, but greater than one eighth $TOL/2$, we accept the computed time step and keep the value of increment at its current value. This criterion is based on considering that, if we double the time increment, the error measure would be greater than the desired limit (one half of the tolerance).

iv. If the error is less than one sixteenth of the tolerance, we accept the computed step and double the time increment.

Figure 2 displays the different actions to follow during the time step selection process.

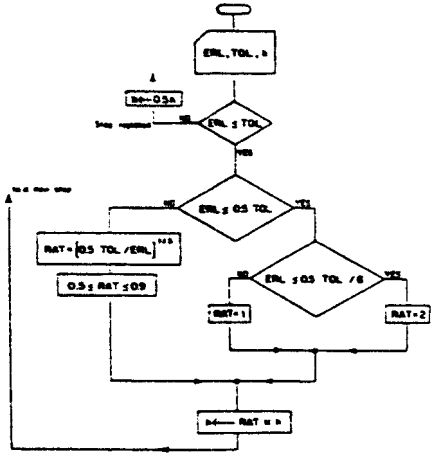

Figure 2 : Time step selection process in the HHT algorithm

3. NUMERICAL EXAMPLES

A very important property of a time integration algorithm for structural dynamics, is that it should be capable of performing the analysis without introducing too much spurious dissipation (i.e. highly excited modes should not be damped out).

In a user-selected step-size strategy, the user himself chooses the components to be retained in the response (and those to be damped out) by selecting an appropriate time step h. A well-known rule is to make $h = T_K/10$, where T_K is the smallest period of the components he wants to accurately time integrate.

In a variable-step strategy, the user does not control the time step h directly. The method automatically adjusts it based on error monitoring considerations. The step-size selection strategy should be such that when, for instance, high frequency modes are strongly excited, the algorithm integrates them accurately (i.e. by respecting the $h = T_K/10$ rule)

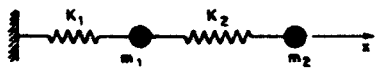

Figure 3 : linear 2 DOF's system

In order to test the performance of the VSA23, DASSL and HHT algorithms, we first analyse a 2-DOF linear oscillator (figure 3). The system parameters will be varied to test two conditions:

i. a first test in which the energy of the high frequency mode is equal to the energy of the low frequency one;
ii. a second test in which the energy of the high frequency mode is much lower than that of the low frequency one.

These tests clearly point out some qualities and deficiencies of the algorithms when dealing with structural dynamics problems, and are fully described in section 3.1

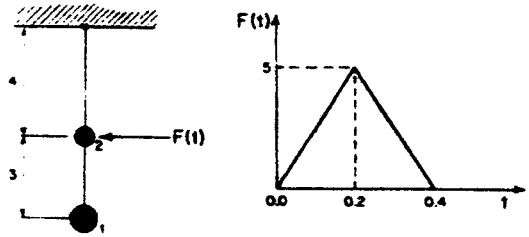

Figure 4 : double pendulum

A second example we made is that of a double-pendulum system. The test shows the performance of the algorithms when dealing with a differential-algebraic system with many degrees of freedom and several constraints in the formulation. The initial configuration is at rest and a time varying horizontal force acts at node 2 as described in figure 4

3.1 Linear oscillator

The equations of motion corresponding to this example are of the form:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{Bmatrix} + \begin{bmatrix} 10001 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} = 0 \tag{13}$$

and after solving the associated eigenproblem we found the two eigenfrequencies: $\omega_1 = 100.005$; $\omega_2 = 0.999949$. If $\Phi = [\phi_1 , \phi_2]$ is the matrix of eigenvectors, we can project the system (13) onto its modal base, $y = \Phi^T q$, with y being the modal displacements vector. It results in an uncoupled system of the form:

$$\begin{cases} \ddot{y}_1 + \omega_1^2 y_1 = 0 \\ \ddot{y}_2 + \omega_2^2 y_2 = 0 \end{cases} \tag{14}$$

We impose initial conditions to (13) such that

$$q(0) = q_0 \quad ; \quad \dot{q}(0) = 0 \tag{15}$$

Then, the initial system energy will be of the strain type. We compute then modal energies in the form

$$E_i = \frac{1}{2} \omega_i^2 y_{i0}^2 \tag{16}$$

where E_i is the total initial energy associated to mode i. This energy distribution will remain unchanged during the integration, since the system is linear and undamped. As we have said earlier, we test two cases that differ in the way the modal energies are excited:

i. $E_1 = 1$; $E_2 = 1$. This case represents a structure in which high frequency modes are strongly excited. The initial conditions (in order to have the above distribution of initial energy) are: $q_{01} = 0.01428285$; $q_{02} = 1.41428285$.

ii. $E_1 = 1$; $E_2 = 10000$. Now, the high frequency modes have a small spurious excitation and they do not participate in the global response. The initial configuration for this case is: $q_{01} = 0.000282843$; $q_{02} = 1.41428425$.

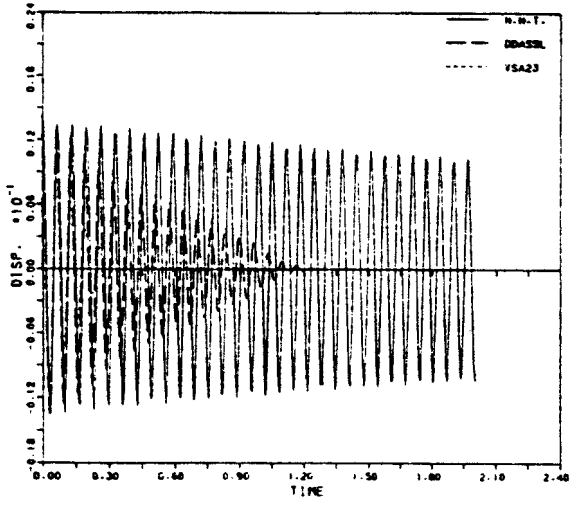

Figure 5 : high frequency mode behavior for the linear system when $E_1 = E_2$

The results of case (i) are shown in figure 5. As it may be seen, the Thomas and Gladwell modified method (VSA23) and the Petzold one (DASSL) have a large amount of numerical dissipation in the high frequency mode. This is not desirable, since in this case the high frequency mode has an important part of the total energy.

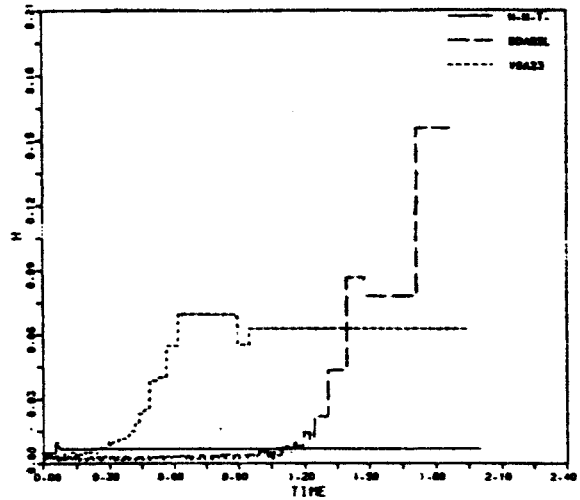

Figure 6 : time step evolution when $E_1 = E_2$

On the other hand, the HHT scheme has a good behavior, without introducing too much dissipation in the high frequency mode. In figure 6, the time step evolution for the three methods is presented. Note that the VSA23 and DASSL algorithms increase the step-size by filtering out the high frequency mode. They lead to an economic, but wrong, integration of the equations of motion: in figure 5 we can see that at $t = 1$, both VSA23 and DASSL have completely filtered out the high frequency mode, which had one half of total system energy. We next test the case in which the high frequency mode is excited by a small spurious amount of energy. Now, it would be desirable that the algorithms be capable of detecting this condition and filter the high frequency modes oscillations.

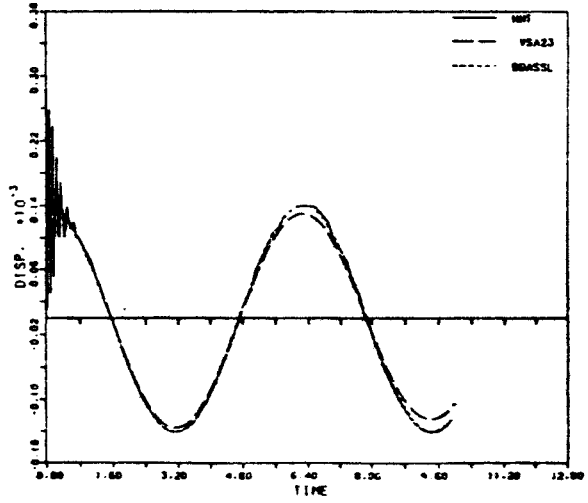

Figure 7 : high frequency mode behavior for the linear system when $E_1 = E_2/10000$

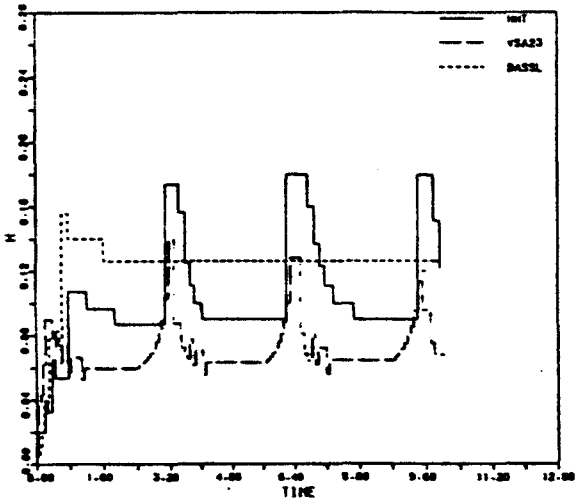

Figure 8 : time step evolution when $E_1 = E_2/10000$

The results of this case are shown in figures 7 and 8, where the evolution of the computed displacements and of the time step for the three methods are compared. We can see that, in this case, the time step is of the same order for all methods, and that the high frequency mode is filtered out as expected.

3.2 Double pendulum

We must first point out that the initial conditions shown in figure 4 were chosen because they did not evidence numerical problems. Other configurations were tested, but we had many troubles to start the integration when using them with the DASSL and VSA23 algorithms (no problem was detected with the HHT scheme).

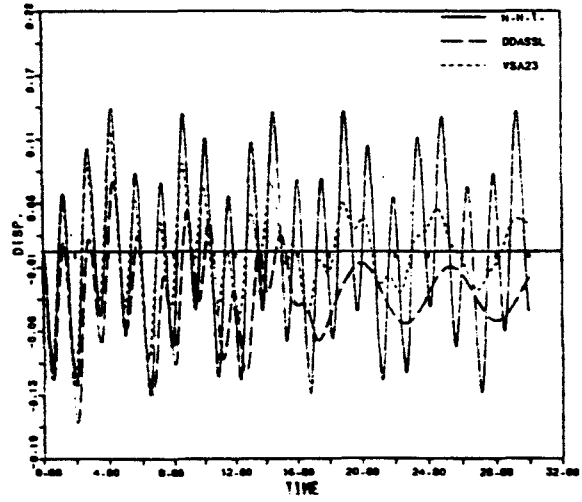

Figure 9 : horizontal position of node 2 for the double pendulum model

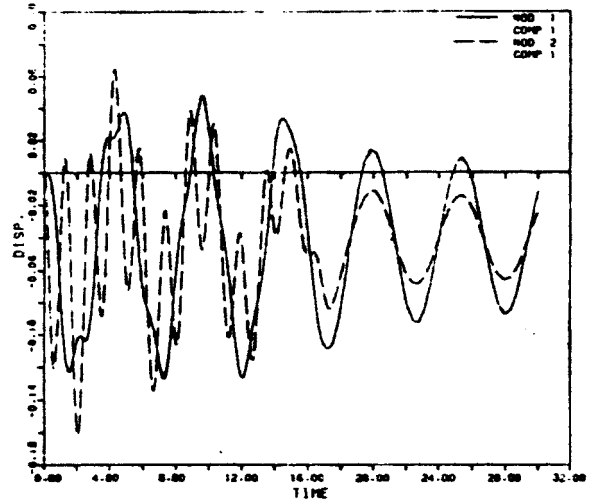

Figure 10 : horizontal position of both nodes for the double pendulum model using DASSL

In figure 9, the evolution of the horizontal displacement at node 2 is shown. This degree of freedom oscillates with a smaller period than that corresponding to node 1. Nevertheless, it is very important to the system response and it would be desirable that the algorithm do not filter it. However, the VSA23 and DASSL algorithms are seen to introduce a strong spurious damping, affecting it significantly. In figure 10 are plotted the horizontal displacement of nodes 1 and 2 as results from DASSL. In that figure, we can appreciate a pseudo-blocking of the hinge at node 2. On the other hand, in figure 11 we present the same curves, now due to the HHT algorithm and they are seen to be undamped and uncoupled.

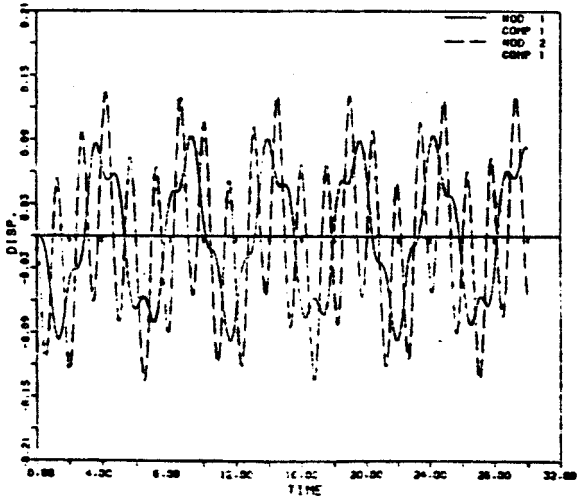

Figure 11 : horizontal position of both nodes for the double pendulum model using HHT

The HHT algorithm gives accurate results, but also its step-size has a good behavior from an stability point of view (i.e. it remains practically constant during the analysis). In figure 12, the evolution of h is presented for the three methods.

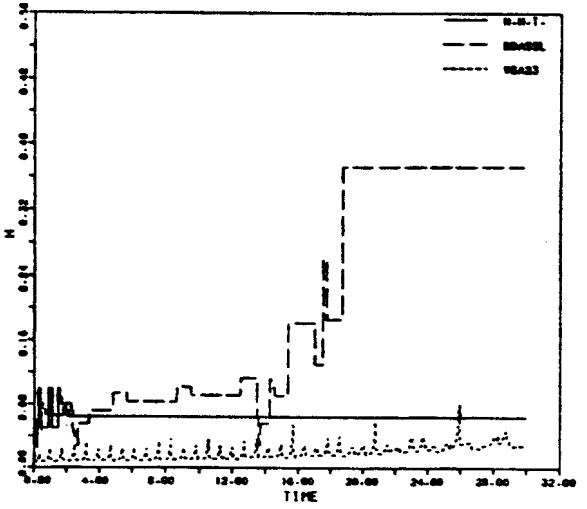

Figure 12 : time step evolution for the double pendulum model

4. CONCLUSIONS

A comparison of three variable-step algorithms has been performed. The HHT algorithm, in conjunction with our step-selection strategy has appeared to be more effective since:

i. When high frequency modes are strongly excited, the algorithm does not dissipate them. However, it correctly dissipates spurious excitations of high frequency modes.

ii. Low frequency modes are correctly integrated.

iii. The time step does not change too much during the analysis, thus improving the algorithm performance.

iv. It has second order accuracy, leading to sensibly larger time steps than the other algorithms for the same accuracy of results.

v. The computational effort is smaller than in the other algorithms, since it estimates the local error by comparing higher order derivatives which are already present in computations.

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