SOME RECENT EXPERIENCES ON THE USE OF PENALTY CONSTRAINTS IN AXISSYMETRIC THIN SHELL ANALYSIS

Carlos A. Almeida
Fátima M.N. de Souza

Dept. of Mechanical Engineering, PUC/RJ
Rio de Janeiro - Brasil

ABSTRACT

Penalty functions are used to enforce bending and continuity constraints through modifying the Variational Indicator of thin axissimetric shells. The procedure is found useful to enforce implicity rotational constraints in the formulation to impose interelement continuity and element end constraints. The resulting model procedure allows the removal of element rotation degree-of-freedom with full compatibility between shells of different curvatures at the elements adjoining node. While the process is approximate, good practical results can be obtained provided an appropriate numerical parameter is employed and sufficient accuracy is available in the computer. The formulation has been implemented and test results of some sample problems are compared to other numerical or analytical solutions to show its effectiveness.

RESUMO

Funções de penalização são utilizadas para preescrever vinculações de flexão e de continuidade através da modificação do Indicador Variacional na formulação de cascas axissimétricas finas. O procedimento é útil em garantir vínculos da rotação transversal, obtida de forma implícita em relação aos deslocamentos da superfície média da casca, de forma a satisfazer as condições de continuidade e de engastamento do modelo. A formulação resultante permite a compatibilidade no ponto nodal comum a cascas com diferentes curvaturas. Apesar de ser um processo numérico aproximado, a técnica de penalização fornece bons resultados de ordem prática mas dependentes da escolha do parametro numérico de penalidade e da disponibilidade de precisão do computador. A ilustração da eficiência do procedimento da análise de cascas finas de revolução é apresentada através de exemplos numéricos comparados com outras soluções numéricas ou analíticas.

INTRODUCTION

In previous communications, the basic theory to impose constraint conditions on a Variational Indicator governing problem has been presented for the analysis of pipe, pipe mitred and beam elements [1-3]. Considering the standard finite element formulation two end conditions are, in general, required to the element assemblage: firstly, the continuity condition when elements are joined at a common node and, secondly, the fixity condition when an element is clamped to a rigid flange. If these end conditions can be expressed in terms of derivatives of mid-surface displacements, as it is the case of negligible transverse shear deformations, a set of compatibility constraints can be used within the element formulation employing only translational degrees-of-freedom.

One of the most widely used technique to impose such constraints, the Lagrange multipliers, suffers the inconvenience of increasing the number of unknown degrees-of-freedom to be solved for, as well as, leading to indefinite matrices in linear analysis. An alternative method to handle the problem of imposing constraints by using "penalty functions" originates from literature of optimization [4,5]. Although only recently it has been used explicitly in the finite element process, in the particular case of equality constraint equations the technique has shown to be well suited to avoid both difficulties mentioned in connection with Lagrange multipliers [6]. The basic technique in this method is to add the constraint to be achieved in the solution, say

$$\text{CONSTRAINT} = 0 \tag{1}$$

to the Variational Indicator of the problem in the following penalty -function form,

$$\pi = U - W + (1/2)\ \alpha\ (\text{CONSTRAINT})^2 \tag{2}$$

where U and W are the total strain energy and total potential of external loads, respectively, and α is the penalty parameter to be chosen after some numeric experiments. The solution obtained using Eq. (2), with $\delta\pi = 0$, will satisfy the condition in Eq. (1) to a required accuracy provided α is selected sufficiently large. The penalty contribution in Eq. (2) corresponds to the potential energy in a rotational spring with stiffness parameter α, and the solution obtained depends on the size of α. A large value of α, or a large spring constant, enforces the compatibility between elements or the fixity condition,and the results convey to a solution that verifies the constraint condition.

The displacements used to formulate the axissimetric element are associated to in-plane kinematics of the mid-surface of the shell, with the assumption that straight lines initially perpendicular to the shell surface remain so after deformation, and with the same length,see Ref. [8]. Considering the element in Fig. 1, the isoparametric cubic interpolation for displacements and geometric coordinates are, respectively,

$$u_i(\xi) = \sum_{k=1}^{4} h_k \, u_i^k \quad ; \quad i = 1,2$$

and

$$r(\xi) = \sum_{k=1}^{4} h_k \, r_k \quad ; \quad z(\xi) = \sum_{k=1}^{4} h_k \, z_k \tag{3}$$

where

ξ = isoparametric longitudinal coordinate, $-1 \leq \xi \leq +1$,

u_i = Cartesian displacements of material point (ξ),

r, z = Cartesian coordinates of material point (ξ),

$h_k(\xi)$ = isoparametric interpolation function [7],

u_i^k = Cartesian displacements of nodal point k,

r_k, z_k = Cartesian coordinates of nodal point k

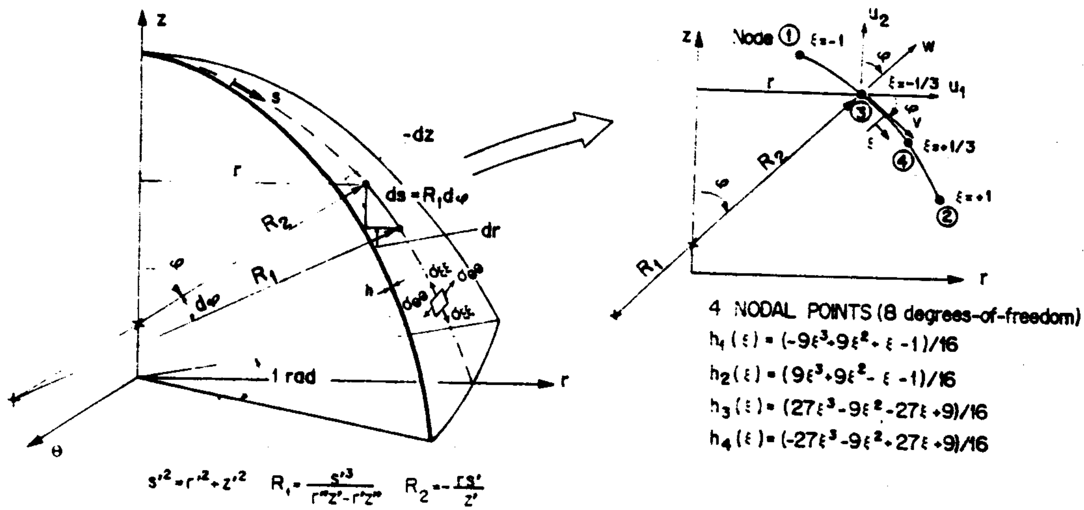

FIGURE 1 - Axissimetric Thin Shell Geometry and Element Model Representation

For the axissimetric thin shell geometry, transverse shear deformations are negligible compared to the element longitudinal and circumferential strain components. These components expressed in terms of the midsurface displacements $v(\xi)$ and $w(\xi)$ in the normal and tangential directions, respectively, reduce to [8]

IMPOSITION OF END CONDITIONS

The objective is to enforce continuity on the midsurface rotation between elements without introducing additional degree-of-freedom. In classical analysis of beam structures, this continuity is achieved by representing the element kinematic with a rotation degree-of-freedom. However, the continuity can be enforced in the formulation by using a penalty procedure. Considering the thin shell element two different constraint conditions shall be imposed: first, the fixity condition when an element is clamped to a rigid flange and, secondy the continuity condition when two elements are adjoined at a common node.

Fixity Condition. When an element is fixed to a rigid flange, the boundary conditions are that at the end node, say for $\xi = -1$, there are no displacements or rotations. Hence, corresponding to equation (4) - TERM 2, the boundary conditions at fixed node i is, see Fig. 2,

$$v_i = w_i = 0 \tag{6}$$

and

$$\left(\frac{v}{R_1} - \frac{1}{s'}\frac{dw}{d\xi}\right)\Big|_{\xi=-1} = 0 \tag{7}$$

where

$$\frac{dw}{d\xi}\Big|_{\xi=-1} = \sum_{k=1}^{4} \frac{dh_k}{d\xi}\Big|_{\xi=-1} w_k \tag{8}$$

The constraint condition in equation (6) simple means that the displacements at node i must be set equal to zero, whereas in equation (7) is imposed with a penalty parameter. In accordance with equation (1), we have

$$\text{CONSTRAINT} = \left[\left(\frac{v}{R_1} - \frac{1}{s'}\frac{dw}{d\zeta}\right)\right]\Big|_{\xi=-1} - 0] \tag{9}$$

that substituing into equation (2) and imposing the stationary condition for π, $\delta\pi = 0$, results into the following penalty matrix, defined to the element degrees-of-freedom,

where

$$K_P^F = \alpha \quad G_F^T \quad G_F$$
$$G_F = [\ \ldots.\ a_1\ b_1\ \ a_2\ b_2\ \ldots.\ ] \tag{10}$$

with

$$a_i = 1/R_1(\xi = -1)$$
$$b_i = -\frac{1}{s'}\frac{dh_i}{d\xi}\Big|_{\xi=-1} \tag{11}$$

The penalty matrix K_P^F with a relative large value of α is added to the element stiffness matrix to enforce the constraint in equation (7), by using the direct stiffness method. In the next section, an example illustrates how the appropriate value of α is chosen numerically in accordance with the equilibrium matrix condition.

$$\varepsilon_{\xi\xi} = \frac{1}{s'}\frac{\partial v}{\partial \xi} + \frac{w}{R_1} + \frac{1}{s'}\frac{\partial}{\partial \xi}\left(\frac{v}{R_1} - \frac{1}{s'}\frac{\partial w}{\partial \xi}\right)\zeta$$

$$\varepsilon_{\theta\theta} = \underbrace{\frac{1}{r}\left(\frac{v\,r'}{s'} - \frac{w\,z'}{s'}\right)}_{\text{TERM 1}} + \underbrace{\frac{1}{r}\frac{r'}{s'}\left(\frac{v}{R_1} - \frac{1}{s'}\frac{dw}{d\xi}\right)\zeta}_{\text{TERM 2}} \quad ;0{,}5h \le \xi \le 0{,}5h \qquad (4)$$

where the upper simbol(') on the variable refers to differentiation with respect to ξ coordinate, R_1 and R_2 are principal radii of curvature of the shell midsurface, see Fig. 1, and ζ is the local coordinate normal to the shell $(h/2 \le \zeta \le h/2)$. In equations (4) the membrane effect is represented by TERM 1 and are due to stretching of the shell longitudinal and transverse fibers, while TERM 2 accounts for bending effects defined by the mindsurface rotation and its local derivate. Combining equations (3), (4) and the rotation transformation $R(\phi)$,

$$\begin{bmatrix} v & (\xi) \\ w & (\xi) \end{bmatrix} = R\ (\phi) \begin{bmatrix} u_1 & (\xi) \\ u_2 & (\xi) \end{bmatrix} \qquad (5)$$

a set of constitutive equations associated to the element global displacements is obtained. Thus, using the stress-strain transformation for the plane strain condition with the above results the element total strain energy U in Eq. (1) is obtained. The equilibrium equations that governs the linear response of the element is derived by invoking the stationarity condition on the π function.

The basic assumption in using the foregoing strain components is that each differential length of the shell can rotate independently although by virtue of using equation (3), the midsurface displacements are continuous within an element and accross the element boundaries. Therefore, the interaction effects in the rotation between elements of different curvatures (continuity condition) or an element and a flange (fixity condition) cannot be properly modeled. Basically, only membrane states of stress can be represented in the formulation, and to render the element applicable to such bending situations, it is necessary to include in the kinematics a procedure to prescribe rotations at the end nodes.

The objective of this paper is to show how to amend the basic finite element formulation in a very simple way to account for the interaction effects. The imposition of these effects allow the model to represent bending continuity and bending contraints in an axissimetric thin shell analysis. This is achieved by using a penalty procedure to enforce the required continuity conditions. In the next Section of this paper we discuss the additional constraints included in the formulation Variational Indicator, as shown in equation (1), so that the element is applicable to the modelling of interaction effects. The formulation has been implemented and in Section 3 the results obtained in the analysis of some sample problems are presented.

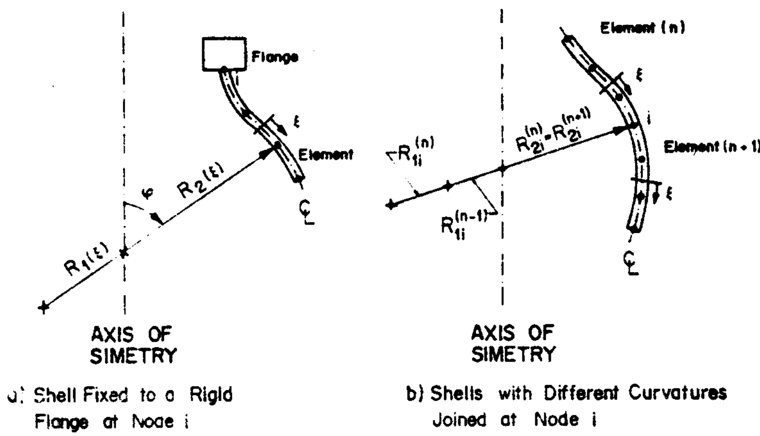

FIGURE 2 - Fixity and Continuity Conditions at Node I for the Axissimetric Shell Formulation

Continuity Condition. At the common node of two shell elements, the displacement are automatically continuous because both elements share the same degrees-of-freedom. In addition the continuity condition,

$$\text{CONSTRAINT} = \left[\left(\frac{v}{R_1} - \frac{1}{s'} \frac{dw}{d\xi} \right) \Big|^{n}_{\xi=+1} - \left(\frac{v}{R_1} - \frac{1}{s'} \frac{dw}{d\xi} \right) \Big|^{n+1}_{\xi=-1} \right. \tag{12}$$

representing the equality of midsurface rotation at the common node is to be imposed. Substituting into equation (2) from equation (12), and invoking $\delta\pi = 0$, the following penalty matrix is obtained

$$\mathbf{K}_P^c = \alpha \ \mathbf{G}_c^T \ \mathbf{G}_c \tag{13}$$

where $G_c = [\ \ldots\ a_1^n\ b_1^n\ a_2^n\ b_2^n\ \ldots\ a_1^{n+1} b_1^{n+1}\ a_2^{n+1}\ b_2^{n+1}\ \ldots\ ]$

with

$$a_i^n = 1/R^n(\xi = +1)$$

$$b_i^n = -\frac{1}{s'} \frac{dh_i}{d\xi} \Big|^{n}_{\xi=+1} \qquad ;\ \text{for } i \neq 4$$

$$a_i^{n+1} = -1/R^{n+1}(\xi = -1) \tag{14}$$

$$b_i^{n+1} = \frac{1}{s_i'} \frac{dh_i}{d\xi} \Big|^{n+1}_{\xi=-1} \qquad ;\ \text{for } i \neq 1$$

and for the common node of the two elements we have

$$a_4^n \equiv a_1^{n+1} = 1/R_1^n(\xi = +1) - 1/R^{n+1}(\xi = -1) \tag{15}$$
$$b_4^n \equiv b_1^{n+1} = -\left(\frac{1}{s_i'} \frac{dh_i}{d\xi} \Big|^n_{\xi=+1} - \frac{dh_i}{s'_i} \Big|^n_{\xi=-1} \right.$$

The resulting penalty matrix associated to the translational degrees--of-freedom of both elements is then added to the structure stiffness matrix.

ANALYSIS RESULTS

The foregoing formulation has been implemented and in this Section the analysis responses predicted using the axissimetric thin shell element are compared to available numerical and analytical results. In all analysis the 4-node element model is employed to accurately represent the shell geometry on the displacement solutions; Gauss integration scheme is used to evaluate the element matrices and forces, at a number of stations specified in each example.

Effect of Penalty Parameter Size-Case Study wity Clamped Ends. Cylinder. The cylinder shown in Fig. 3 was analysed under internal pressure. The purpose of this analysis is to investigate the effects of the element size on the response predicted, and thus arrive at some guidelines for the effectiveness of the penalty procedure on the imposition of the constraints condition. A seven 4-node element model was employed in the geometry representation with a fine mesh near to the clamped end of the cylinder to represent bending effects accurately. Numerical solutions for various size of the penalty parameter was then obtained for the shell maximum radial displacements, as shown in Table 1. The results indicate that for a wide range of α (10^5 to 10^{11}) the

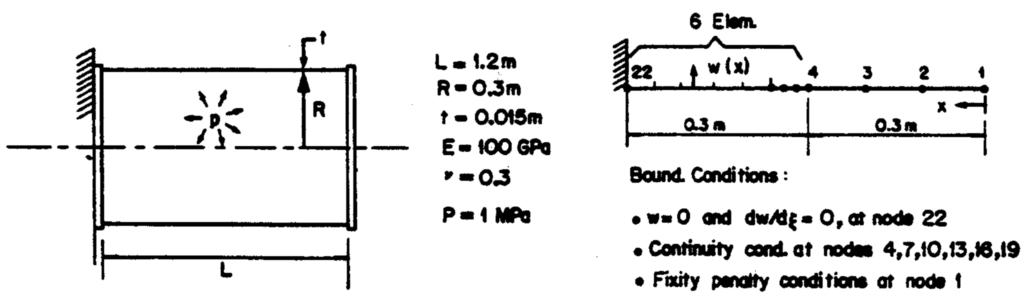

FIGURE 3 - Thin Walled Cylinder Considered and FE Model Representation

Analysis of a Nozzle. The geometry of a bending free nozzle designed for the structural transition between a spherical shell and a pipe run was established in Refs. [11,12]. The objective of this analysis is to verify the numerical solutions obtained with the present

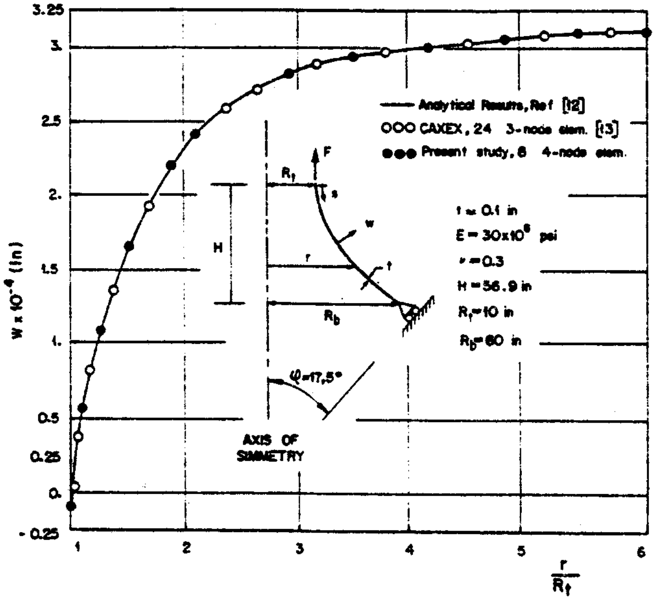

FIGURE 5 - Normal Displacements of the Membrane Nozzle Under Axial Loading

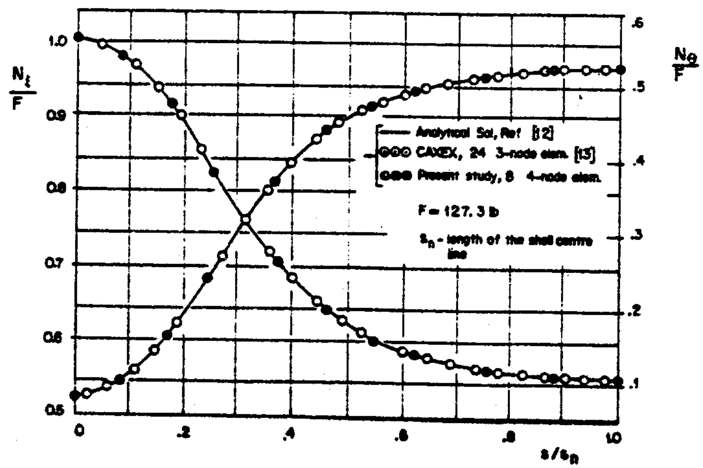

FIGURE 6 — Stress Resultants in the Longitudinal - ξ and Circumferential - θ Directions of the Shell

solutions are in very good agreement with the analytical results, whereas for small or very large values the solutions are in error-a positive indefinite coeficient matrix is obtained for α = 1.E17. In the formulation, the penalty matrix is added to the original stiffness matrix to impose the constraint conditions. Thus, an appropriate value of the penalty parameter is that the elements of both matrices have the same order of magnitude (in the present study, k_{ii}^{max} = 1.1 E8). Figure 4 presents the numerical soltutions for the fixed-ends cylinder radial displacements with a four element model, and good agreement is then noted.

Table 1 - Numerical Solutions for the FE Maximum Radial Displacements for Different Values of α (w_{anal} =0.626x10^{-4} , 2x/L = 0.722)

α	w_{max}x10^4	2x/L	Node
10^3	0.6263	0.750	13
10^5	0.6260	0.722	12
10^7	0.6260	0.722	12
10^9	0.6260	0.722	12
10^{11}	0.6260	0.722	12
10^{13}	0.6361	0.694	11
10^{15}	0.6540	0.333	3
10^{17}	Matrix Not Positive Definite		

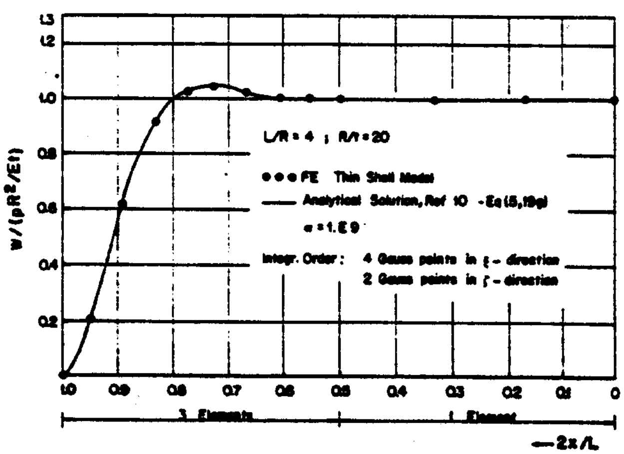

Fig. 4 - Comparison of Finite Element Solutions with Analytical Results for the Radial Displacements of the Cylindrical Shell

formulation for a quite complex geometry shell, but under the membrane state-of-stress only. An eight 4-node equally spaced element model was used for represent the nozzle shown in Fig. 5, under axial loading at one end and simple supported at the base. Analytical results for the shell normal displacements, in Fig. 5, and stress resultants, in Figure 6, are compared to numerical solutions obtained with the present formulation and with the higher-order element formulation implemented in the CAXEX program described in [13]. A very good agreement between numerical and analytical results was then obtained. Moreover, the present model reduced to about one-half the number of degrees-of-freedom required for the shell solution when compared to the numerical solution including rotation degrees-of-freedom, explicity. The analysis study required four integration station in the longitudinal direction and two stations through the shell thickness.

Analysis of a Hiperboloidal Tower. The cooling tower constructed of reinforced concrete was analyzed in [14] with isoparametric axissimetric shell elements obtained from reduction of the 2/D ring model reported in Ref. [15]. The reduced model employed rotation degrees-of-freedom to represent the displacements at points out of the shell midsurface. Prestressed cables apply a ring load P in the radial direction at the top edge of the tower which is simple supported at the base. The axissimetric element model in this study was used to represent the shell at same point location refered in [14]. In this

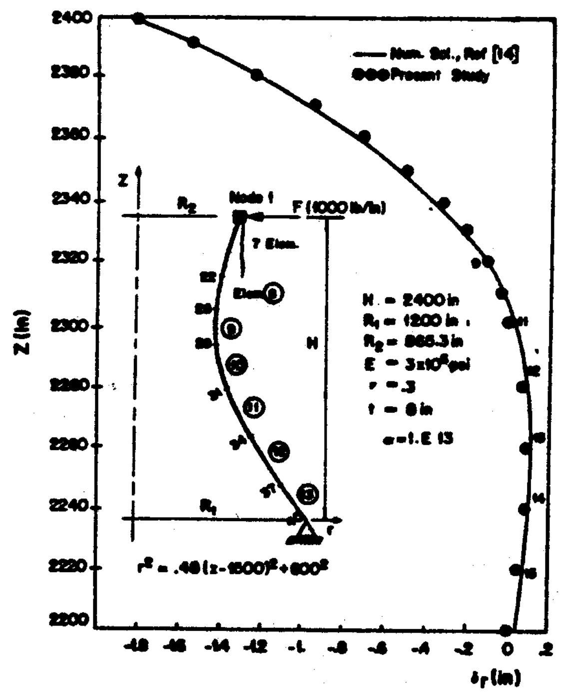

FIGURE 7 - Radial Displacements at the Hyperboloidal Tower Due to Concrete Reinforcement on the Top Ring

analysis, membrane and bending effects are activated in the shell by the applied load and the corresponding radial displacement solutions is presented in Fig. 7, which represents the upper portion of the shell, only. The lower region is essentially under pure membrane state of stress and a good agreement in the results is observed. The numerical integration employed four and two integration stations in the longitudinal and normal directions, respectively.

CONCLUSIONS

Some experiences on the application of the assimetric shell element presented in [9], that accounts for bending and membrane effects, and for interactions with rigid flanges and continuity between elements are discussed. The interaction effects are included in a novel, but very simple and efficient manner using a penalty function formulation. Although the results of some sample solutions indicates the applicability of the element, the total element formulation is based on a number of assumptions and further performance studies are still required to identify the limit range of problems for which the element can be further employed.

REFERENCES

[1] Bathe, K.J., and Almeida, C.A., "A Simple and Effective Pipe Elbow Element-Linear Analysis", J. Appl. Mechanics, vol. 47, nº 1, 1980, pp. 93-100.

[2] Almeida, C.A., and Salles, A.C., "Analysis of Pipe Mitred Bends Through a Modified Beam Model - F E Approach", RCBM, vol. VII, nº 1, Janeiro 1985, pp 53-70.

[3] Almeida, C.A., and Andueza, A., "Analise Tridimensional de Vigas Utilizando Modelo com Número Reduzido de Graus-de-liberdade",RBCM, vol. VII, nº 4, Novembro 1985, pp. 335-352.

[4] Moe, J., "Penalty Function Methods", chapter in Optimum Structural Design-Theory and Applications, Gallagher, R.H. and Zienkiewicz,O. C., Eds., John Wiley, 1973.

[5] Almeida, C.A., "Enforcement of Bending Continuity and Bending Constraint Through the Penalty Function Method", COBEM-85, pp.465-467, Dec. 1985, S.J. Campos.

[6] Zienkiewicz, O.C. "Constrained Variational Principles and Penalty Function Methods in Finite Element Analysis", Conf. on Num. Sol. of Differential Equations, pp. 207-214, Scotland, July, 1973.

[7] Bathe, K.J. Finite Element Procedures in Engineering Analysis, Prentice-Hall Inc., 1982.

[8] Novozhilov, V.V. "Thin Shell Theory", P. Noordhoff Ltd, 1964.

[9] Nogueira de Souza, F.M., "Elemento Unidimensional de Casca Fina Axissimetrica com Numero Reduzido de Graus-de-Liberdade" M.S. Thesis, Dept. de Eng. Mecânica, PUC/RJ, August 1988.

[10] Kraus, H., Thin Elastic Shells, John Wiley & Sons, 1967.

[11] Bevilacqua, L. and Noronha, Jr., D.B., "Determination of Bending Free Geometries for Shells of Revolution Under Axiasymmetric Loads", RBCM, vol. VII, nº 1, 1985.

[12] Noronha, Jr., D.B., "Determinação de uma Forma "Ótima" para Bocais Radiais de Vasos Esféricos", Ms. Thesis, Mech. Eng. Dept. , PUC/RJ, January 1983.

[13] Jospin, R.J., "Analise Dinâmica Linear de Cascas Axissimétricas com um Elemento Curvo", Ms. Thesis, COPPE-UFRJ, 1978.

[14] Weaver, W. and Johnston, P.R., Finite Elements for Structural Analysis, Prentice Hall, inc., Englewood Cliffs, New Jersey, 1984.

[15] Ahamad, S., Irons, B.M., Zienkiewicz, O.C., "Curved Thick Shell and Membrane Elements with Particular Reference to Axissimetric Problems", Proc. 1nd Conf. Math. Meth. Struct. Mech., WPAFB Ohio, 1968, pgs. 539-572.