APPROXIMATE, NUMERICAL AND THEORETICAL METHODS TO SOLVE STEFAN-LIKE PROBLEMS

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RESUMEN

Se presenta una corta revisión sobre métodos aproximados, numéricos y teóricos para resolver problemas de frontera libre para la ecuación del calor de tipo Stefan.

ABSTRACT

We present a short review on the approximate, numerical and theoretical methods to solve free boundary problems for the heat equation of Stefan type.

The present paper presents a summarized version of the lecture delivered on occasion of the Sixth American Congress on Computational Methods held in Paraná and Santa Fe (Argentina) on October 15-18, 1985.

The free boundary problems that belong to Mathematical Physics, particularly those related to the heat and/or diffusion equation, have been of great concern for mathematicians, physicists and engineers due to the wide variety of different processes which involve a mathematical model of this kind; some problems that deserve special consideration are those where a phase-change problem occurs, which are better known as Stefan problem.

The free boundary problems of Mathematical Physics are connected with various branches of Mathematics, Physics and Engineering, in particular, with continuous mechanics, heat transfer, ordinary and partial differential equations, functional analysis, elliptic and evolution variational inequalities, numerical analysis, etc. Among other free boundary problems, we may mention: dam problem [Bai, BaCa], semiconductors problem [HuNa], obstacle problem [BaCa, KiSt], Bingham fluid [Gl], semi-permeable wall problem or "black body" [Duli], Stefan problem [Ru]. A survey of all these problems with a mathematical analysis through variational inequalities was done in [Ta1]; see also [BaCa, Cr2, ElOc, F_r2, K_iSt, Lun, Ma1, Oz, Ru1, Sa, Tay].

Free boundary problems for the heat and/or diffusion equation may be classified into:

i) of explicit type: when $\dot{S}$ appears explicitly in, at least, one of the two conditions on the free boundary, e.g., the Stefan problem [LaCl].

ii) of implicit type: when $\dot{S}$ does not appear explicitly on neither condition on the free boundary, e.g., the problem of diffusion-consumption of oxygen in living tissues [CrGu].

In general, the free boundary problems of explicit and implicit type are mutually related [Fa, Sc].

A discussion on fixed, moving and free boundary problems for the heat or diffusion equation of the explicit or implicit type is given in [Ta4] and several long bibliographies on moving or free boundary problems are given in [Cr2, Pr, Ru1, Ta2, WiSoTr].

Among the free boundary problems for the heat or diffusion equation we may cite: Stefan problem [LaCl, St1, St2, We], diffusion-consumption of oxygen in a living tissue [CrGu], noncatalytic gas-solid diffusion-reaction problem [We], continuous casting problem [Rod], optimal control [Sa], solidification processing [Fl], metals solidification [Bil], solidification of roads [AgFr, AgFrCo], ablation by melting [SzTh], the welding of two steel plates [Tay], the shape of laser melt-pools [Tay], ablation by a high power laser [AnAt], and several other applications in [Cr2, FaPr1, Lun, Ma2, Oc tto, WiSoBo], i.e., electro-mechanical machining, Hele-Shaw flow, solidification of binary alloys, storage of solar energy, etc. A relation among different free boundary problems is analyzed in [Ro].

A discussion on the conflict among physical reality, mathematical rigour and engineering applications is analyzed in [Sz].

In order to have an idea of the importance of the methods and applications related to free and moving boundary problems for the heat and/or diffusion equation, we may mention:

i) Meetings, conferences or seminars completely devoted to the subject [AlCoHo, BoDaFr, FaPr1, Fu1, Ho, Ma2, OcHo, Ta3, WiSoBo]. In addition, there exist several other meetings where papers on this subject, which are not specified here, are presented.

ii) Books completely devoted to this subject [Ca, Cr2, Da, ElOc, Fr2, Ru1, Ta1].

iii) Review papers on the subject, both from the theoretical and/or numerical point of view, such as [Ban, Cr3, Cr4, CrId, Du4, Fo, Fu2, GaSa, Ma1, Me3, MuSu, Niel, No1, Pr1, Pr2, Se2, SoBi, Ta2].

iv) Books that devote several chapters to the subject, for example [CaJa, Cr1, EcDr, Fr1, Go, Je, KiSt, Lui, Lun, Oz, SzTh]. In addition, there are various other books where the subject is treated to a lesser extent which are not specified here.

Some of these papers have been used as background to the present lecture thus citing, in the majority of the cases, the original reference or review paper on the subject.

The outline of what will be further given consists in a selection of theoretical, numerical and approximate methods that have been used in free boundary problems of the Stefan type, such as:

I) Exact solutions
II) Approximate methods or models
III) Integral formulation methods
IV) Front-tracking methods
V) Front-fixing methods
VI) Fixed-domain methods.

We suggest readers to refer to the original papers and the references thereby mentioned in order to have a more extensive information on the different methodologies used, in particular, considering the shortness of the present classification.

I. EXACT SOLUTIONS

i) The mathematical model for the fusion of a semi-infinite material $x > 0$, initially in the solid phase at the melting temperature $T_m = 0°C$, is given by [LaCl]: Find the functions $T = T(x,t)$ and $S = S(t)$ such that

i) $\rho c\, T_t - k\, T_{xx} = 0 \,,\; 0 < x < S(t) \,,\; t > 0$

ii) $T(S(t),t) = 0 \,,\; t > 0$

iii) $k\, T_x(S(t),t) = -\rho l\, \dot{S}(t) \,,\; t > 0$

iv) $T(o,t) = B > 0 \,,\; t > 0$

v) $S(0) = 0$

where k is the thermal conductivity coefficient, ρ is the mass density, c is the specific heat, l is the latent heat of fusion and $B > 0$ is the temperature on the fixed face $x = 0$. The condition (liii) is known as the Stefan condition.

The solution of (1) if given by (Lamé-Clapeyron solution):

$$T(x,t) = B - \frac{B}{\operatorname{erf}(\xi)} \operatorname{erf}\left(\frac{x}{2a\sqrt{t}}\right), \quad 0 \le x \le s(t), \quad t \ge 0$$
$$s(t) = 2a\,\xi\sqrt{t} \tag{2}$$

where $a^2 = k/\rho c$ is the thermal diffusivity coefficient and ξ is the unique solution of the equation:

$$x \exp(x^2) \operatorname{erf}(x) = \frac{Ste}{\sqrt{\pi}}, \quad x > 0 \tag{3}$$

where $Ste = \frac{c\,B}{l} > 0$ is the Stefan number.

ii) A mathematical model for the solidification of a semi-infinite material $x > 0$ (initially in the liquid phase at the temperature $C > 0$ and for $t > 0$ with a temperature $-B < 0$ on the fixed face $x = 0$, considering the jump density at the solid-liquid interphase) is given by [CaJa, We]: Find the functions $T_1 = T_1(x,t)$, $T_2 = T_2(x,t)$ and $S = S(t)$ such that

$$\rho_1 c_1 T_{2_t} - k_1 T_{2_{xx}} = 0, \quad 0 < x < S(t), \quad t > 0$$

$$\rho_2 c_2 T_{2_t} + c_2(\rho_1 - \rho_2)\dot{S}(t) T_{2_x} - k_2 T_{2_{xx}} = 0, \quad S(t) < x, t > 0$$

$$T_1(S(t),t) = T_2(S(t),t) = 0, t > 0$$

$$k_1 T_{1_x}(S(t),t) - k_2 T_{2_x}(S(t),t) = \rho_1\, l\, \dot{x}(t), \quad t > 0 \tag{4}$$

$$T_2(x,0) = T_2(+\infty, t) = C > 0, \quad x > 0, t > 0$$

$$T_1(0,t) = -B > 0, \quad t > 0$$

$$S(0) = 0$$

where $i = 1$ represents the solid phase and $i = 2$ the liquid phase.

The solution of (4) is given by (Neumann solution):

$$T_1(x,t) = -B + \frac{B}{\operatorname{erf}(\sigma/a_1)} \operatorname{erf}\left(\frac{x}{2a_1\sqrt{t}}\right)$$

$$T_2(x,t) = \frac{-C}{\operatorname{erfc}(\sigma/a_o)} \left[\operatorname{erf}(\sigma/a_o) - \operatorname{erf}\left(\delta + \frac{x}{2a_2\sqrt{t}}\right)\right] \tag{5}$$

$$S(t) = 2\sigma\sqrt{t}$$

where

$$a_i = \left(\frac{k_i}{\rho_i\, c_i}\right)^{1/2} \quad (i = 1,2)\ , \qquad \delta = \frac{\sigma}{a_2}\,|\varepsilon|$$

$$a_o = \frac{a_2}{1 + |\varepsilon|}\quad , \quad \varepsilon = \frac{\rho_1 - \rho_2}{\rho_2} \tag{6}$$

$$\frac{k_1\, B}{l\, \rho_1\, a_1\, \sqrt{\pi}}\ \frac{\exp(-\sigma^2/a_1^2)}{\operatorname{erf}(\sigma/a_1)} - \frac{k_2\, C}{l\, \rho_1\, a_2\, \sqrt{\pi}}\ \frac{\exp(-\sigma^2/a_o^2)}{\operatorname{erfc}(\sigma/a_o)} = \sigma \tag{7}$$

$\sigma > 0$

iii) The solutions previously presented are included within a more general methodology known as the similarity method which consists in finding the solution of the form $T(x,t) = u(\eta)$ where $\eta = x/\sqrt{t}$ is the similarity variable [Bou].

Other exact solutions are given in [CaJa, ChSu, Rog, Ta5, Ti].

II. APPROXIMATE METHODS OR MODELS

i) Quasi-stationary method.

It consists in replacing the heat equation (1i) by [St1, St2]:

$$T_{xx} = 0\ ,\ 0 < x < S(t)\ ,\ t > 0 \tag{1ibis}$$

If the boundary condition (1iv) on the fixed face $x=0$ varies with time, i.e., $B=B(t)$, then the solution of problem (1ibis, 1ii-v) is given by:

$$T(x,t) = B(t) - \frac{B(t)}{S(t)}\,x\ ,\ 0 \le x \le S(t)\ ,\ t > 0$$

$$S(t) = \left[\ \frac{2k}{\rho l}\int_0^t B(C)\ dc\right]^{1/2}\quad ,\quad t \ge 0 \tag{8}$$

If the data $B(t)=B>0$ is constant and the Stefan number $Ste = BC/l \ll 1$, then the Lamé-Clapeyron and the Quasi-stationary solutions are very close. A recent analysis with a convection-type condition:

$$-\,k\ T_x(0,t) = h\,[T_L - T(0,t)]\ ,\ t > 0 \tag{9}$$

on the fixed face $x=0$, is given in [SoWiAl].

ii) Balance integral method.

It is based on the physical concept of the thermal layer. It is assumed that the temperature is propagated in a bounded interval $[0,\delta(t)]$ ($\delta(t)$ represents the thermal layer) and that outside that interval the temperature remains equal to its initial value. The method consists in assuming that $\delta(t)$ coincides with the free boundary $S(t)$ and in replacing conditions (1i) and (1iii) by (1itris) and (1iiibis) respectively, where [Goo]:

(1itris) $$\frac{d}{dt}\int_o^{S(t)} T(x,t)\,dx = \frac{k}{\rho c}\,[T_x\,(S(t),t) - T_x\,(o,t)]\ ,\ t>0,$$

(1iiibis) $$k\,T_x^2\,(S(t),t) = \rho l\,T_t\,(S(t),t)\ ,\ t > 0\ .$$

Then, we suggest a polynomial distribution in the x variable for the temperature $T(x,t)$ of the form

$$T(x,t) = \alpha(t)\,(x-S(t)) + \beta(t)\,(x - S(t))^2\ , \tag{10}$$

α and β being in function of S and then S being a solution of a Cauchy problem.

iii) Biot's variational methods.

We introduce the vectorial heat displacement field $H = H(x,t)$ such that:

$$\dot{H} = -\,k\,\nabla T\ ,\ \operatorname{div} H = -\,\rho c T\ . \tag{11}$$

If we consider that H is also a function of a given number of generalized coordinates, $H = H\,(x,t,q_1,\ldots,q_n)$ suitably chosen, we have the following Lagrange equations (similar to those of analytical mechanics) [Bio]:

$$\frac{\partial V}{\partial q_i} + \frac{\partial D}{\partial \dot{q}_i} = Q_i\quad ,\quad i = 1,\ldots,n \tag{12}$$

where

$V = \frac{1}{2}\int\int\int c\,T^2\,dx$: thermal potential

$D = \frac{1}{2k}\int\int\int |\dot{H}|^2\,dx$: dissipation function

$Q_i = -\int\int T\,\frac{\partial H}{\partial q_i}\cdot n\,d\gamma$: generalized thermal forces

In general, for $n = 1$, we take $q_1 = S$ (free boundary) and we suggest for the temperature a polynomial distribution, then finding for s an ordinary differential equation.

iv) Series expansion.

If the initial and boundary data can be developed in a series, then the temperature (of both phases) and the free boundary can be suggested also as a series [RuFaPr, RuSh, Tao].

v) Perturbation series expansion.

We choose a significant parametre of the system (e.g., the Stefan number) and we do a perturbation series expansion with respect to that parametre [Ji, PeDo].

III. INTEGRAL FORMULATIONS METHODS

i) Rubinstein-Friedman Method.

It consists in finding an equivalent integral formulation to the problem under study. For example [Fr3]: $\{u,s\}$ is the solution of the problem:

$$
\begin{aligned}
&\text{i)} \quad u_{xx} - u_t = 0 \ , \ 0 < x < S(t) \ , \ 0 < t < T \\
&\text{ii)} \quad u(0,t) = f(t) \ , \ 0 < t < T \\
&\text{iii)} \quad u(x,0) = h(x) \ , \ 0 \leq x \leq b \\
&\text{iv)} \quad S(0) = b \geq 0 \\
&\text{v)} \quad u(S(t),t) = 0 \ , \ 0 < t < T \\
&\text{iv)} \quad u_x(S(t),t) = -\dot{S}(t) \ , \ 0 < t < T \ ,
\end{aligned} \tag{14}
$$

if the function $v(t) = u_x(S(t),t)$ satisfies the integral equation

$$v = R(v) \tag{15}$$

with

$$
\begin{aligned}
R(v)(t) = {} & 2\,[h(0) - f(0)]\, N(S(t) \ , \ t \ ; \ 0,0) + \\
& + 2 \int_0^b h'(\xi)\, N(S(t) \ , \ t \ ; \ \xi,0)\, d\xi - \\
& - 2 \int_0^t \dot{f}(c)\, N(S(t) \ , \ t \ ; \ 0,c)\, dc + \\
& + 2 \int_0^t v(c)\, G_x(S(t) \ , \ t \ ; \ S(c),c)\, dc
\end{aligned} \tag{16}
$$

$$S(t) = b - \int_0^t v(c)\, dc \tag{17}$$

where

$K(x,t;\xi,c) = \dfrac{1}{2\sqrt{\pi(t-c)}} \exp\left[-\dfrac{(x-\xi)^2}{4(t-c)}\right]$: fundamental solution

$N(x,t;\xi,c) = K(x,t;\xi,c) + K(-x,t;\xi,c)$: Neumann function

$G(x,t;\xi,c) = K(x,t;\xi,c) - K(-x,t;\xi,c)$: Green function .

The operator R is a contraction application on the Banach space

$$V = \{v \in C^{\circ}[0,c] \ / \ \| v \| = \max_{t \in [0,c]} |v(t)| \leq M\} \tag{19}$$

for a happy choice of c , $M > 0$. Moreover, the procedure can be extended in time, thus obtaining the solution for each arbitrary

$T > 0$, and so the unique fixed point can be obtained as the limit of the succession v_n , defined by:

a) $v_o \in V$, arbitrarily chosen.

b) If v_n is known, then $v_{n+1} = R(v_n)$. (20)

Other general methods are given in [Ru1, Ru2].

ii) Other methods.

We can cite:

a) Green function [ChSz, Ko].
b) Embedding technique [Bol1, Bol2].
c) Boundary element method [BrFuTa, BrTeWr, HoUmKi, TaOnKuHa, Wr].

IV. FRONT-TRACKING METHODS

These methods compute, at each time step, the position of the free boundary.

i) Fixed finite-difference grid.

When the grid is fixed in the domain space-time the free boundary will be, in general, between two points of the grid at each time step and thus we shall need special formulas to compute T_x and $\dot{S}$ in a neighbour of the free boundary [C_r2, Fu 2].

ii) Modified grids.

The grid is modified with the passing of time, e.g.;

a) Finite-difference grid with variable time step: We determine as part of the solution, a variable time step such that the free boundary coincides with a grid line in space at each time level. To do that we use for $S(t)$ an integral equation equivalent to the Stefan condition [DoGa].

b) Finite difference grid with variable space step: Since the number of space intervals between the fixed boundary $x = 0$ and the free boundary $S(t)$ are taken constant for all time, and the free boundary is always on the same grid line, then the space step is different in each time step [MuLa].

c) Space-time finite elements: We use a spatial discretization which we adapt in each time step for building quadrilateral finite elements in space-time [BoJa, Ja].

d) Moving finite elements: We use a finite difference procedure in time with finite elements in space which are adapted at each time step to fit the new position of the free boundary [AlMaMi, MiMoBa].

iii) Method of lines.

The time variable is discretized and the partial differential equation is replaced by a sequence of ordinary differential equations at discrete time levels. The position of the free boundary is calculated at each time step [Me1, Me2].

iv) The Polygonal Method.

To solve problem (14) we divide the interval $[0,T]$ in n subintervals of $\theta = T/n$ length. For $t\varepsilon[0,\theta]$ we put $S_\theta(t) = b - h'(b)t$ and we determine $u_\theta = u_\theta(x,t)$, defined in $0 < x < S_\theta(t)$ and $0 < t < \theta$, as the solution to problem (14i–v). Then, we calculate $a_{1_\theta} = u_{\theta_x}(S_\theta(\theta),\theta)$. For $t\varepsilon[\theta,2\theta]$ we put:

$$\dot{S}_\theta(t) = S_\theta(\theta) - a_{1_\theta} \cdot (t-\theta) \tag{21}$$

and we determine $u_\theta = u_\theta(x,t)$ for $0 < x < S_\theta(t)$, $\theta \leq t < 2\theta$, and so forth. Thus we obtain a polygonal $S_\theta = S_\theta(t)$, defined for $t\varepsilon[0,T]$, and a function $u_\theta = u_\theta(x,t)$, defined in $0 < x < S_\theta(t)$ and $0 \leq t \leq T$. When $\theta \to 0^+$ we obtain as limit the solution $\{s,u\}$ for the problem (14).

v) Retarded argument method.

To solve problem (14) for $b > 0$ we built, for each $\theta\varepsilon(0,b)$, a succession $S_\theta(t)$ and $u_\theta = u_\theta(x,t)$ defined in the following way: For $t\varepsilon[0,\theta]$ we put $S_\theta(t) = b$ and we determine $u_\theta = u_\theta(x,t)$ in $0 < x < S_\theta(t)$ and $0 \leq t < \theta$ as the solution of the problem (14i–v). Then, for $\theta \leq t \leq 2\theta$, we calculate

$$S_\theta(t) = b - \int_o^t u_{\theta_x}(S_\theta(\eta-\theta), \eta-\theta)\, d\eta \tag{22}$$

and we determine $u_\theta = u_\theta(x,t)$ in $0 < x < S_\theta(t)$ and $\theta \leq t < 2\theta$ as the solution of problem (14i–v), and so forth. When $\theta \to 0$ we obtain as limit the solution $\{s,u\}$ of problem (14) [Ca, CaHi].

vi) Equivalent Stefan Condition Method.

For the case $b=0$ and condition

$$u_x(0,t) = -g(t)\ ,\ 0 < t < T \tag{23}$$

on the fixed face $x=0$ instead of (14ii), the Stefan condition (14vi) is equivalent to:

$$S(t) = \int_o^t g(c)\, dc - \int_o^{S(t)} u(x,t)\, dx\ ,\ 0 \leq t < T \tag{24}$$

Thus, we can define the operator

$$R_1 : S(t) \to r(t) \tag{25}$$

where $r(t)$ is defined as the second member of (24) with u the solution of problem (14i,v), (23) with $b=0$.

Then, we can define a succession $\{S_n,u_n\}$ $(n \geq 0)$ in the following way: If S_n is known, we calculate u_n as the solution of problem (14i,ii,v) with $b=0$ and then we compute $S_{n+1} = R_1(S_n)$. In [Ev, Sel], we study the case $g(t) \equiv 1$ with $S_o(t) \equiv t^1$, thus obtaining the following results:

a) $\dot{S}_n \geq 0$, $S_{2n-1} \leq S_{2n+1} \leq S_{2n} \leq S_{2n-2}$ (26)

b) S_n and u_n are convergent to the solution of problem (14i,v, vi), (23) and $S(0) = 0$.

With this methodology we can see that the free boundary problem can be interpreted as the limit to a succession of moving boundary problems.

V. FRONT-FIXING METHODS

A Method to track the free boundary is to fix it by a suitable choice of new variables:

i) Inmobilization method.

The Landau transformation

$$\xi = \frac{x}{S(t)} \tag{27}$$

fixes the free boundary $x = S(t)$ at $\xi = 1$ for all time $t > 0$ [La, Cr5]. By using this methodology, in [Co, FaPr3, FaPr4] we prove the general existence and unicity results. Some complementary variants have been done, e.g.:

a) In [Nit], we use the double transformation

$$\xi = \frac{x}{S(t)} \quad , \quad c = \int_0^t \frac{d\eta}{S^2(\eta)} \tag{28}$$

for a one-phase Stefan problem.

b) In [Fu2], we use the transformation

$$\xi_1 = \frac{x - l_1}{S(t) - l_1} \quad , \quad \xi_2 = \frac{l_2 - x}{l_2 - S(t)} \tag{29}$$

for a two-phase Stefan problem in the interval $[l_1, l_2]$.

ii) Isotherm migration method.

It is a curvilinear transformation in which the dependent temperature variable u is exchanged with one of the space variables. In the one dimensional case, $u = u(x,t)$ becomes $x = x(u,t)$. The expression $x = x(u,t)$ indicates how a specific temperature u moves through the medium, i.e., how isotherms migrate (in particular, the free boundary) [Ch].

VI. FIXED-DOMAIN METHODS

It consists in reformulating the problem, over the whole fixed domain occupied by the two phases, such that the Stefan condition is included within the new formulation of the equations and the initial and boundary conditions of the problem. Moreover, the position of the free boundary will later reappear, as a consequence of our knowledge of the solution of the new problem posed.

i) **Enthalphy method.**

The enthalpy function is defined by:

$$H(T) = \int_{T_o}^{T} [\rho(\xi)\, c(\xi) + l\, \rho(\xi)\, \delta(\xi - T_m)]\, d\xi \tag{30}$$

where T_o $(< T_m)$ is a fixed temperature, T_m is the melting temperature and δ is the Dirac distribution. The enthalpy function has incorporated the heat jump ρl at the free boundary.

For the two-phase multidimensional Stefan problem the problem is reduced to the following differential equation in the distribution al sense:

$$\frac{\partial H(T)}{\partial t} = \nabla \cdot (K\, \nabla T) \tag{31}$$

with the corresponding initial and fixed boundary conditions, where

$$K(T) = \begin{cases} K_1(T) & \text{if } T < T_m \\ K_2(T) & \text{if } T > T_m \end{cases} \tag{32}$$

With the Kirchoff transformation

$$v = \int_{T_o}^{T} K(t)\, dt \tag{33}$$

equation (31) is reduced to

$$\frac{\partial H(v)}{\partial t} = \Delta v \tag{34}$$

In trying to avoid the difficulties that the H jump produces, we have the following possibilities:

a) Weak solution: It satisfies a suitable integral form of the differential equation in which the derivatives of H and T (or v) do not appear [At, Cro, ElOc, Fr4, Ka, Ol, Ros].

b) Regularization enthalpy method:

bi) Due to the discontinuous jump of $H = H(T)$ at the melting temperature $T = T_m$, we suggest to make it regular over a small temperature zone $T_m - \varepsilon \leq T \leq T_m + \varepsilon$ where $\varepsilon > 0$, i.e. $H_\varepsilon = H_\varepsilon(T)$ is a continuous function. In [Je, JeRo, No2] we find error estimates $\| T - T_{\varepsilon hc} \|$ in terms of the regularization parametre ε, time step c and spatial step h.

bii) We replace the enthalpy jump at the phase-change interphase by an equivalent heat capacity $\tilde{C}(T)$, expressing the problem in the following [BoCoFaPr]:

$$\tilde{C}(T)\, T_t = \nabla \cdot (K(T)\, \nabla T) \tag{31bis}$$

c) Various numerical methods are used, e.g., implicit and explicit finite-differences, implicit, explicit and space-time finite elements, regularization method with implicit finite-differences or finite-elements, for example [At, BuSoUs, Ci, VoCr]. For numerous references on this subject see [Cr2, ElOc, Lun, Ma3, Ta2].

d) In [At] we describe an enthalpy method to solve a welding problem in which a mushy region appears.

e) The convergence for the approximate free boundary is given in [No3].

f) A semigroup approach is given in [BeBrRo, MaVeVi].

ii) Variational inequalities.

We consider the following one-phase fusion problem:

$$
\begin{aligned}
&\frac{\partial\theta}{\partial t} - \Delta\theta = 0 \ , \ t > l(x) \\
&\theta \equiv 0 \quad , \quad t \leq l(x) \\
&- \nabla\theta \cdot \nabla l = 1 \quad \text{on} \quad t = l(x) \\
&\left.\frac{\partial\theta}{\partial n}\right|_{\Gamma_2} = 0 \quad , \quad \left.\theta\right|_{\Gamma_3} = 0 \\
&- \left.\frac{\partial\theta}{\partial n}\right|_{\Gamma_1} = b(\theta-\theta_1) \\
&\theta(x,0) = 0
\end{aligned}
\tag{35}
$$

where $b > 0$, $\theta_1 > 0$, $t = l(x)$ represents the solid-liquid interphase and $\Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ is the boundary of the phase-change material Ω . In [Du1, Du2] we realize the change of the function unknown (similar to [Bai] for the dam problem):

$$
u(x,t) = \begin{cases} \int_{l(x)}^{t} \theta(x,c)\, dc & \text{if} \quad t > l(x) \\ 0 & \text{if} \quad t \leq l(x) \end{cases}
\tag{36}
$$

which satisfies the following variational inequality (see also [FrKi]):

$$
\int_\Omega u_t \ (v-u) \ dx + \int_\Omega \nabla u \cdot \nabla \ (v-u) \ dx + b \int_{\Gamma_1} (u-\theta_1 t) \ (v-u) \ d\gamma \geq
$$

$$
\geq - \int_\Omega^\Omega (v-u) \ dx \quad , \quad \forall v \in K \quad ,
\tag{37}
$$

$u(t) \in K$, $u(0) = 0$

where

$$V = \{v \in H^1(\Omega)/v/\Gamma_3 = 0\} \quad ,$$
$$K = \{v \in V/v \geq 0 \quad \text{in} \quad \Omega\} \quad . \tag{38}$$

a) Other problems have been studied with an analogous formulation, for example: electrochemical machining [El], diffusion-consumption of oxygen in a living tissue [Du5], injection of fluid into a laminar cell under the assumption of Hele-Shaw flow [ElJa], simulation and control in a Stefan problem [Sa].

b) For the two-phase Stefan problem, the new function [AgFr, Du3, Fr, Pa, Ta6]:

$$u(x,t) = \int_o^t [k_2 \ \theta^+(x,c) - k_1 \ \theta^-(x,c)] \ dc \tag{39}$$

is introduced to obtain a variational inequality formulation.

c) A numerical analysis corresponding to the variational inequality formulation is given in [Fe, IcKi, KaSa, KiIc, Nie2, PiVe, TiTi].

d) In [Fr5], a quasi-variational inequality formulation is given for the unidimensional one-phase Stefan problem.

iii) Alternating phase truncation method.

It consists in solving a heat conduction problem in either the liquid or the solid phase alternatively in successive time steps [BeCiRo, RoBeCi].

Note: Numerous references (other than the ones hereby cited) can be found in recent publications and in the references within them, e.g., [BoDaFr, Ca, Cr2, Ta2, WiSoTr].

NOMENCLATURE

$a^2 = \frac{k}{\rho c}$: thermal diffusivity

a_o : coefficient defined in (6)

$-B < 0$: temperature on the fixed face

c : specific heat

$\tilde{c}$: equivalent heat capacity defined in (31 bis)

$C > 0$: initial temperature

D : dissipation function defined in (12)

f : function defined in (14ii)

g : function defined in (23)

G : Green function defined in (18)

h : function defined in (14iii)

h : coefficient defined in (9)

H : enthalpy function defined by (30)

$\mathbf{H}$: heat displacement field defined in (11)

k : thermal conductivity

K : fundamental solution defined in (18)

K : convex set defined in (38)

ℓ : latent heat of fusion

N : Neumann function defined in (18)

q_i : generalized coordinates

Q_i : generalized thermal forces defined in (12)

R : operator defined by (16)

R_1 : operator defined by (25)

S : solid-liquid interface

Ste : Stefan number

t : time variable

T : temperature

T_L : coefficient defined in (9)

T_m : melting temperature

u : transformation defined by (36) and (39)

u : function defined by problem (14)

v : Kirchoff transformation defined by (33)

v : function defined by (15)

V : thermal potential defined in (12)

V : Banach space defined by (19)

V : Hilbert space defined in (38)

x : spatial variable

Greek Letters

α,β : coefficients defined in (10)

ε : dimensionless parametre defined in (6)

δ : dimensionless parametre defined in (6)

σ : coefficient which characterizes the boundary S by (5)

ρ : mass density

θ : function defined by problem (35)

ξ : dimensionless parametre defined in (2)

ξ : dimensionless parametre defined by the Landau transformation in (27).

ξ_1,ξ_2 : dimensionless parametres defined in (29)

Subscripts

$i=1$: solid phase

$i=2$: liquid phase

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