COMPUTER SOLUTION OF AN INCOMPRESSIBLE VISCOELASTICITY PROBLEM WITH A STRESS - DISPLACEMENT - PRESSURE MIXED METHOD

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RESUMEN

Este trabajo presenta la descripción y los resultados numéricos obtenidos con un nuevo método de elementos finitos mixtos del tipo tensión-desplazamiento-presión. El método está particularmente adaptado para ser usado en conjunto con la formulación de medios viscoelásticos incompresibles en deformaciones planas con los coeficientes físicos dependientes del tiempo. Diferenciándose de la mayor parte de los métodos conocidos, la condición de compatibilidad entre los espacios de elementos finitos es satisfecha tomando el mismo tipo de aproximación para la presión y el tensor desviador de tensiones.

ABSTRACT

This work presents a description and numerical results obtained with a new mixed finite element method of the stress-displacement-pressure type. The method is particularly adapted to be used in connection with an algorithmic formulation for the numerical treatment of incompressible viscoelastic media in plane strains with physical coefficients depending on time. Differently from most of the known methods, the compatibility condition between the three finite element spaces is satisfied, taking the same type of approximation for the pressure and the stress deviator tensor.

Let a viscoelastic medium occupy a bounded open set Ω of $\mathbb{R}^2$, with boundary Γ having a fixed portion Γ_0. We assume that in Ω, the mechanical behavior of the medium at time t, $t > 0$ is described by:

$$\frac{\partial \sigma^D}{\partial t} + \frac{F(\sigma^D)}{\omega} = \alpha\, \varepsilon(\underline{v}) + \beta \frac{\partial \varepsilon(\underline{v})}{\partial t} \qquad (1)$$

with the incompressibility condition

$$\operatorname{div} \underline{v} = 0 \qquad (2)$$

where:

$\underline{v}$ is the velocity (rate of displacement)
σ^D is the stress deviator tensor, 2 x 2 symetric with $\sigma^D_{11} + \sigma^D_{22} = 0$
$\varepsilon(\underline{v})$ is the plane strain tensor given by:

$$\varepsilon_{ij}(\underline{v}) = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)$$

The viscoelastic coefficients α, β and ω, are positive depending on $\underline{x}$ and t, and F is a bijection of $\mathbb{R}^{2\times 2}_{sym}$ in $\mathbb{R}^{2\times 2}_{sym}$, whose Fréchet derivative is bounded over every bounded set, with $F_{11} + F_{22} = 0$. $V^{N\times N}_{sym}$ is the space of symetric tensors N x N, having each component belonging to the same vector space V.

For example, if $\alpha = 2G$, $\beta = 0$ and $F(\sigma^D) = |\sigma^D|^{\lambda-1}\sigma^D$, where λ is real parameter greater or equal to 1, we have a Maxwell-Norton material, G being the shear relaxation modulus. Considering $F \equiv 0$ with $\alpha = 2G + 2\frac{\partial \mu}{\partial t}$ and $\beta = 2\mu$, μ being the cinematic viscosity, then we have a Kelvin-Voigt material, with G independent of time. (See [1] and [2]).

Notice that the stress tensor in this case is the following:

$$\sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} & 0 \\ \sigma_{12} & \sigma_{22} & 0 \\ 0 & 0 & \sigma_{33} \end{bmatrix}$$

where: $\sigma = \sigma^D + pI$, p being the hydrostatic pressure, and I denoting the 3 x 3 identity tensor.

Given body forces $\underline{f}$ and surface forces $\underline{g}$, acting on Γ_1 with $\Gamma = \Gamma_1 \cup \Gamma_0$, the equilibrium equations are the following:

$$\rho_0 \frac{\partial v}{\partial t} - \operatorname{div} \sigma^D + \operatorname{grad} p = f \text{ in } \Omega \tag{3}$$

$$\sigma \cdot n = g \text{ on } \Gamma_1 \tag{4}$$

where ρ_0 is the density of the medium, assumed to be constant and n is the unit normal vector external to Γ_1. Moreover we assume that at time $t = 0$, we have $v = 0$ and $\sigma = 0$.

Given a time interval Δt, for each function $G(x, t)$ we define:

$$G^n(x) = G(x, n\Delta t) \quad n = 1, 2, \ldots$$

As usual, we calculate σ_n^D, p_n and v_n, approximations of σ, p and v by at time $n\Delta t$ by discretization in time of (1) - (2) - (3) with the following implicit scheme:

$$\frac{\sigma_n^D - \sigma_{n-1}^D}{\Delta t} + \frac{F(\sigma_n^D)}{\omega^n} = \alpha^n \mathcal{E}(v_n) + \beta^n \frac{\mathcal{E}(v_n) - \mathcal{E}(v_{n-1})}{\Delta t} \tag{5}$$

$$\operatorname{div} v_n = 0 \tag{6}$$

$$\rho_0 \frac{v_n - v_{n-1}}{\Delta t} - \operatorname{div} \sigma_n^D + \operatorname{grad} p_n = f^n \tag{7}$$

$$\sigma_n \cdot n = g^n \tag{8}$$

$n = 1, 2, \ldots M$, with $v_0 = v^0$, $\sigma_0 = \sigma^0$; M is such that $M\Delta t = T$ where T is an a priori fixed time.

Considering that α, β and ω are time dependent, we do not eliminate σ_n^D from the system (5) - (6) - (7) - (8), and we solve the problem by an iterative algorithm like the one of the augmented Lagrangean type [3] described as follows:

Let τ_{m-1}, u_{m-1} and q_{m-1} be approximations of σ_n^D, v_n and p_n respectively with $\tau_0 = \sigma_{n-1}^D$, $u_0 = v_{n-1}$ and $q_0 = p_{n-1}$ $n \geq 1$. For $m = 1,2,3\ldots$ we calculate the new approximations τ_m, u_m and p_m for these three variables at time $n\Delta t$ by:

$$\frac{\tau_m - \tau_0}{\Delta t} + \frac{F(\tau_m)}{\omega^n} = \alpha\, \mathcal{E}(u_{m-1}) + \beta \frac{\mathcal{E}(u_{m-1}) - \mathcal{E}(u_0)}{\Delta t} \tag{9}$$

$$((u_m, w))_R = ((u_{m-1}, w))_R - s_n H(\tau_m, q_m, u_{m-1}, u_m, v, f, g) \quad \forall w \in [V]^2 \tag{10}$$

where $V = H^1_{0,\Gamma_0}(\Omega) = \left\{ v \mid v \in H^1(\Omega), v = 0 \text{ in } \Gamma_0 \right\}$ and H is defined by

$$H(\tau, q, u, v, w, f, g) = (\tau, \mathcal{E}(w)) + (\lambda \operatorname{div} u - q, \operatorname{div} w) + \frac{\rho_0}{\Delta t}(u - v, w) - (f, w) - \int_{\Gamma_1} g \cdot w \, ds$$

with

$$(r, \operatorname{div} \underset{\sim}{u}_m) = 0 \quad \forall r \in L^2(\Omega) \tag{11}$$

$(\underset{\sim}{u},\underset{\sim}{v})$ is the usual inner product of $(L^2(\Omega))^{N\times N}$, and $((\underset{\sim}{u},\underset{\sim}{v}))_R$ is given by

$$((\underset{\sim}{u},\underset{\sim}{v}))_R = \int_\Omega \mathcal{E}(\underset{\sim}{u}) \cdot \mathcal{E}(\underset{\sim}{v})\, d\underset{\sim}{x} + \frac{\rho_0 C}{\Delta t} \int_\Omega \underset{\sim}{u}\cdot\underset{\sim}{v}\, d\underset{\sim}{x} + R\int_\Omega \operatorname{div} \underset{\sim}{u} \operatorname{div} \underset{\sim}{v}\, d\underset{\sim}{x}$$

where R is a positif real parameter, $R \gg 1$ and C is a constant depending on the maximum value of the function that relates $\mathcal{T}_m$ to $\mathcal{E}(\underset{\sim}{u}_{m-1})$ in (9), for $0 \leq n \leq M$. s_n is a parameter to be determined at each time step in order to ensure the convergence of the algorithm.

Next we determine $\underset{\sim}{u}_m$ and q_m using an algorithm of the Uzawa type [3].

Taking $\underset{\sim}{u}_{m,o} = \underset{\sim}{u}_{m-1}$ and $q_{m,o} = q_{m-1}$ for known values $\underset{\sim}{u}_{m,s-1}$ and $q_{m,s-1}$, we calculate $\underset{\sim}{u}_{m,s}$ and $q_{m,s}$ for s = 1, 2, ... by:

$$((\underset{\sim}{u}_{m,s},\underset{\sim}{v}))_R = ((\underset{\sim}{u}_{m-1},\underset{\sim}{v}))_R - s_n H(\mathcal{T}_m, q_{m,s-1}, \underset{\sim}{u}_{m-1}, \underset{\sim}{u}_o, \underset{\sim}{v}, \underset{\sim}{f}, \underset{\sim}{g}) \quad \forall \underset{\sim}{v} \in [V]^2 \tag{12}$$

$$q_{m,s} = q_{m,s-1} - \rho_n \operatorname{div} \underset{\sim}{u}_{m,s} \tag{13}$$

Letting b be a constant satisfying relation (15) below if one replaces P_h and V_h by $L^2(\Omega)$ and V respectively, then $\underset{\sim}{u}_{m,s} \longrightarrow \underset{\sim}{u}_n$ in $[H^1(\Omega)^2]$ and $p_{m,s} \longrightarrow p_n$ in $L^2(\Omega)$, if $0 < \rho_n < 2R/bs_n$ [4]. Moreover it is possible to prove that $\underset{\sim}{u}_m \longrightarrow \underset{\sim}{v}_n$ in $[H^1(\Omega)]^2$, $\mathcal{T}_m \longrightarrow \sigma_n^D$ in $[L^2(\Omega)]^{2\times 2}$ and $q_m \longrightarrow p_n$ in $L^2(\Omega)$, if the value of C is conveniently fixed in terms of Δt and for s_n sufficiently small. For example if $F(\sigma^D) = \sigma^D$, i.e. for a linear Maxwell model, taking:

$$C = \max_{0 \leq n \leq M} \frac{2G^n \omega^n \Delta t}{\omega^n + \Delta t\, G^n} \quad \text{the algorithm converges if } 0 < s_n < \frac{\omega^n + \Delta t G^n}{G^n \omega^n \Delta t}$$

for an arbitrary Δt.

In order to discretize the problems (9) - (12) - (13) in space, we first partition the domain Ω into finite elements, and then, we define a space $V_h \subset V$ for each velocity component, a subspace Σ_h of $L^2(\Omega)$ for each stress of σ^D and a subspace P_h of $L^2(\Omega)$ for the pressure. These three spaces are associated with a given partition $\mathcal{T}_h$ of Ω into triangles or quadrilaterals with maximum diameter h.

The so generated approximate version of (9) - (13) allows us to solve the problem (12) in velocity with a fixed matrix associated to the inner product $((.,.))_R$ for a great number of time steps, while we only solve local element by element problems for the stress and pressure, i.e. equations (9) and (13).

Clearly, it is necessary to satisfy the compatibility condition between the spaces V_h , Σ_h , and P_h , that is to say, the inf-sup condition for the mixed methods [5].

In this case the conditions are given by:

$$\inf_{\underset{\sim}{v}_h \in [V_h]^2} \ \sup_{\sigma^D \in [\Sigma_h]^{2\times 2}_{sym}} \frac{\int_\Omega \sigma_h^D \, \mathcal{E}(\underset{\sim}{v}_h) \, d\underset{\sim}{x}}{\| \sigma_h^D \|_0 \, \| \underset{\sim}{v}_h \|_1} \geqslant a > 0 \qquad (14)$$

$$\inf_{p_h \in P_h} \ \sup_{\underset{\sim}{v}_h \in [V_h]^2} \frac{\int_\Omega p_h \, \mathrm{div} \, \underset{\sim}{v}_h \, d\underset{\sim}{x}}{\| \underset{\sim}{v}_h \|_1 \, \| p_h \|_0} \geqslant b > 0 \qquad (15)$$

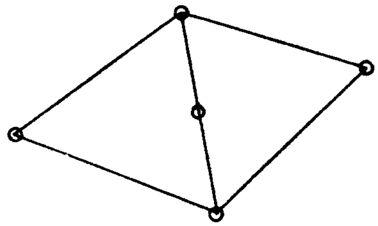
where $\| \cdot \|_0$ and $\| \cdot \|_1$ are the usual norms of $L^2(\Omega)$ and $H^1(\Omega)$ respectively.

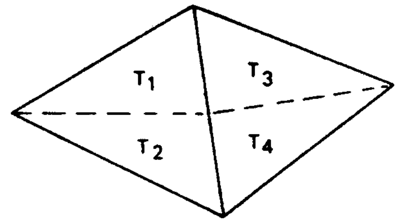
If we take $\Sigma_h = P_h$, which is phisically natural and an optimal choice from the numerical point of view, conditions (14) - (15) turn out to be contradictory.

The following choice gives a very simple alternative to satisfy (14) - (15), while working with stress and pressure discretized in the same way and without introducing any particular restriction to the numerical solution. We observe that this element is similar to the one studied in [6] for another class of problems.

We consider that $\mathcal{C}_h$ is a partition of the domain Ω into convex quadrilaterals, each one of these being subdivided into two triangles by means of an arbitrarily chosen diagonal. V_h is defined as the space of continuous functions that vanish on Γ , whose restriction over each triangle is quadratic, the trace of which over each edge of the quadrilaterals being a linear function. The degrees of freedom of this space are the functional values at the nodes indicated in Figure 1. The space P_h (or Σ_h) is a subspace of the space of functions which are constant over each one of the four triangles obtained by joining the mid-point of the chosen diagonal to the vertices of the quadrilateral not belonging to it. In Figure 2 we illustrate the condition defining such a subspace, namely: for each pair of triangles that do not have a common edge the sum of the constants are equal.

The numerical results given below obtained for a linear Maxwell model confirm the efficiency of the method.





o degree of freedom of the velocity

$p/T_1 + p/T_4 = p/T_2 + p/T_3$ if $p \in P_h$

Figure 1 - element of $\mathcal{T}_h$ and the associate velocity

Figure 2 - element of $\mathcal{T}_h$ and the associate pressure

We consider a viscoelastic medium with unity density (MKS system) occupying a region Ω shown in Figure 3 with the forces $\underset{\sim}{g} = \underset{\sim}{0}$ and $\underset{\sim}{f} = (0,10)$ measured in kN/m^3.

We take $F(\sigma) = \sigma$, $\alpha = 2G$, $\beta = 0$ and $\omega = \mu/G$, that is, a linear Maxwell model, where G and μ are given in Figure 4.

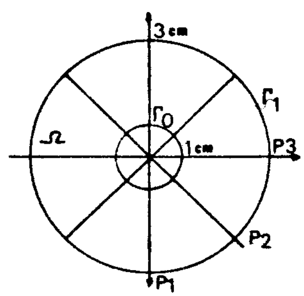


Figure 3

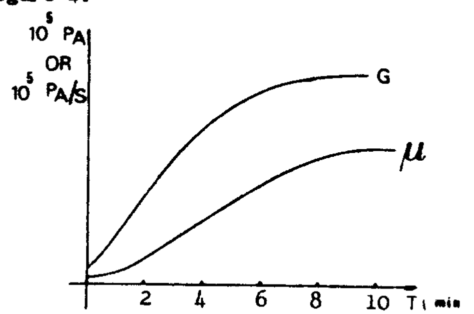


Figure 4

For the discretization of the problem we take $\Delta t = 1/\ell$ where ℓ is an integer parameter describing the mesh (see Figure 5). Considering the symmetry of the problem we take only one quarter of the domain in the computations.

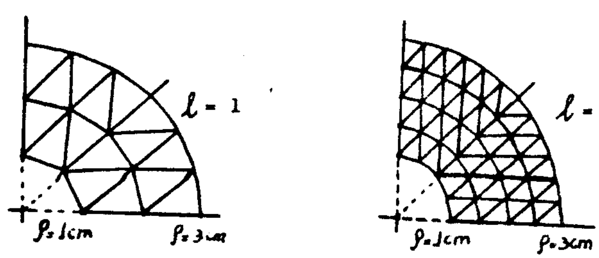


Figure 5

The results obtained for the displacement show a good convergence of the finite element method. In Table I we illustrate this fact by giving the modulus of the displacement in mm at points P_1, P_2, P_3 shown in Figure 3, for times 5 and 10 minutes.

Table I - Computer results for a model problem

ℓ \ P and t	t = 5.0			t = 10.0		
	P_1	P_2	P_3	P_1	P_2	P_3
1	0.1937	0.3202	0.4334	0.3771	0.6238	0.8441
2	0.1985	0.3513	0.4642	0.3802	0.6728	0.8890
4	0.2079	0.3655	0.4763	0.3940	0.6926	0.9026

The convergence of the algorithm is also very fast, as after the first time steps only one or two iterations are needed to attain an accuracy of 10^{-4} and 10^{-5} for the inner and outer loop respectively.

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