THE TWO-STAGE FEASIBLE DIRECTIONS METHOD FOR NON-LINEAR PROGRAMMING PROBLEMS

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RESUMEN

Presentamos un algoritmo de direcciones factibles basado en conceptos de dualidad, para la resolución de problemas de programación no lineal con restricciones de igualdad y desigualdad. En cada iteración se define una dirección de descenso, la que modificada da origen a una dirección de descenso factible. El esquema de busqueda lineal asegura la convergencia global del método y la factibilidad de todas las configuraciones. Se prueba la convergencia global del proceso y se describen los resultados obtenidos con vários ejemplos numéricos.

ABSTRACT

We present a feasible direction algorithm, based on duality concepts, for the solution of the non-linear programming problem with equality and inequality constraints. At each iteration a descent direction is defined, by modifying it, a feasible and descent direction is obtained. The linear search procedure assures the global convergence of the method, and the feasibility of all the iterates. We prove the global convergence of the algorithm, and show the results obtained in the resolution of some test problems.

INTRODUCTION AND PRELIMINARIES

The general non-linear constrained optimization problem can be defined as follows:

$$\min_{x} f(x) \tag{1.1}$$

$$\text{subject to } g_i(x) \leq 0 \quad ; \quad i = 1,\ldots,m$$

$$\text{and} \quad g_i(x) = 0 \quad ; \quad i = m+1,\ldots,m+p$$

where $f(x)$ and $g_i(x)$ denote real valued functions of a vector x in the n-dimensional Euclidian space R^n.

A considerable research effort has been done to obtain efficient and reliable methods for the solution of this problem. Without trying to make a survey of this area, we can mention different approaches concerning each of the components of the problem. That is, minimizing the function, "solving" the equality constraints and verifying the inequalities. A very interesting survey in this sense, has been done by Fletcher [9]. The best known methods of unconstrained optimization are concerned with the minimization of the function. We shall mention steepest-descent, quasi-Newton, and conjugate gradient methods, when only first derivative information is considered.

Equality constraints may be eliminated, linearized, or penalized. Methods using simple penalty functions are robust, but when a good precision is required the may give ill-conditioning.

Augmented Lagrangians [3,18] avoid in general the ill-conditioning, but they are less robust and their precision is not very good.

When the constraints are linearized, the first idea is to project the steepest descent or the quasi-Newton direction on the tangent subspace. In that case we have the projected gradient method, or the reduced gradient on [1,2,19]. When the constraints are not linear, they need some feasibility restoring scheme. It is also possible to combine the projected gradient direction with a feasibility improvement step [10].

The combination of a linear approximation of the constraints with a quadratic approximation of the objetive function, is the basis of a family of methods which solve a quadratic programming sub-problem in each iteration. They give a direction tangent to the active equality constraints and improve feasibility automatically.

It can be proved that if all the constraints are active, directions given by projected gradient, augmented Lagrangian and quadratic programming subproblem methods are similar.

Inequality constraints may be treated as equalities, if the set of active constraints at the optimum is known. In practice, it is very difficult to make a good prediction of the active set and the methods that use this approach are subject to the so-called "jamming" problem.

Barrier and simple penalty methods search automatically the active inequality constraints but they may give ill-conditioning, and inexact penalty functions produce no feasible points.

The quadratic programming subproblem methods are naturally extended to inequality constraints. They identify very efficiently the active set [11,18,20], but the feasible region of the subproblem may be empty or unbounded.

Duality is at the origin of a family of approaches in the treatment of equality and inequality constraints. Minimax and augmented Lagrangian methods are natural applications of this theory.

If some proper update rule for the Lagrange multipliers is stated, the exact minimization of the intermediate unconstrained problem can be replaced by a single minization technique included in it.

In this work, we present a strong and efficient method for the solution of problem (1.1), with good global convergence qualities.

This is obtained by establishing an update rule for the Lagrange multipliers, without employing active set strategies. Quadratic programming subproblems and ill-conditioning given by penalty functions are also avoided. The method gives a feasible direction for the inequality constraints and, in consequence, feasible intermediate points. A superlinear local convergence may be obtained by obtained by including an approximation of the Hessian of the Lagrangian.

An algorithm is given in Section 2, and some numerical examples are considered in Section 3.

Notations

All vector spaces are finite dimensional, the space of all n x m matrices is denoted by R^{nxm} and the transpose of M by M^T. If ϕ is a real valued function in R^n, then

$$\nabla\phi(x) \equiv \left(\frac{\partial\phi(x)}{\partial x_1}, \frac{\partial\phi(x)}{\partial x_2}, \ldots \frac{\partial\phi(x)}{\partial x_n}\right)^T .$$

We call Ω the feasible region for the inequality constraints, that is

$$\Omega \equiv \{x \in R^n; g_i \leq 0 \ , \ i = 1,\ldots,m\} ,$$

and denote by

$$g(x) \equiv [g_1(x), g_2(x), \dots, g_{m+p}(x)]^T$$

Definitions

Definition 1.1: A point $\bar{x}$ is a "stationary point" of problem (1.1) if there exists a vector $\bar{\lambda}$ in R^{m+p} such that the following requirements are simultaneously satisfied:

$$g_i(\bar{x}) \leq 0 \quad ; \quad i = 1,\dots,m$$

$$g_i(\bar{x}) = 0 \quad ; \quad i = m+1,\dots,m+p$$

$$\bar{\lambda}_i g_i(\bar{x}) = 0 \; ; \quad i = 1,\dots,m$$

and

$$\nabla f(\bar{x}) + \sum_{i=1,m+p} \bar{\lambda}_i \nabla g_i(\bar{x}) = 0.$$

Definition 1.2: A "Kuhn-Tucker point" of the problem (1.1) is a stationary point associated to a vector $\bar{\lambda}$ verifying

$$\bar{\lambda}_i \geq 0 \quad ; \quad i = 1,\dots,m$$

Definition 1.3: $d \in R^n$ is a "descent direction" at point x of a real continuously differentiable function ϕ, if

$$d^T \nabla\phi(x) < 0.$$

Definition 1.4: $d \in R^n$ is a "feasible direction" [21] of problem (1.1) at $x \in \Omega$ if for some $\tau > 0$ we have

$$x + td \in \Omega \text{ for all } t \in [0,\tau].$$

Definition 1.5: A point $\bar{x}$ is a "regular point" of the problem (1.1) if the set of vectors

$$\nabla g_i(\bar{x}) \quad ; \quad i = 1,\dots,m \text{ so that } g_i(\bar{x}) = 0$$

and

$$\nabla g_i(\bar{x}) \quad ; \quad i = m+1, \ldots, m+p$$

is linearly independent

STATEMENT OF THE ALGORITHM

We shall define a feasible direction algorithm based on Lagrangian concepts, for the solution of problem (1.1). The method constructs a sequence $\{x^k\}$, starting from an initial strictly feasible point, verifying

$$g_i(x^k) < 0 \quad ; \quad i = 1, \ldots, m \text{ and } k = 0, 1, 2, \ldots,$$

and converging to a Kuhn-Tucker point of the problem.

Consider

$$\theta_c(x) = f(x) - \sum_{i=m+1}^{m+p} c_i g_i(x) \tag{2.1}$$

where $c \equiv (c_{m+1}, \ldots, c_{m+p})$ is given by the algorithm. A search direction is computed in two stages. First a descent direction of $\theta_c(x)$ is defined; by modifying it, a feasible descent direction in obtained.

A linear search is stated, in order to guarantee the global convergence of the method and the strict feasibility of all the iterates.

An important feature, is that all the constraints are considered in each iteration, and it is not necessary to perform any active set strategy.

The iterative algorithm for solving the general non-linear programming problem (1.1) is stated as follows:

Let $\rho_0 >$, $\alpha \in (0,1)$, $\gamma_0 \in (0,1)$, $\sigma \in (0,1)$, $c_i \geq 0$;

$i = m+1, \ldots, m+p$, and $r_i(x) > \theta$; $i = 1, \ldots, m$ continuous in Ω_a.

Step 0 : Select a strictly feasible point for the inequality constraints

$x^0 \in \Omega$, and the values of α, γ_0, ρ_0 and c_i ;

$i = m+1, \ldots, m+p$.

If necessary, redefine the inequality constraints in a way that

$$g_i(x^0) \leq 0 \quad ; \quad i = m+1,\ldots,m+p.$$

Set $\rho = \rho_0$.

Step 1 : Compute $\lambda_0 \in R^{m+p}$ and the descent direction $d_0 \in R^n$ by solving the linear system of equations

$$d_0 = -\left[\nabla f(x) + \sum_{i=1}^{m+p} \lambda_{0i} \nabla g_i(x)\right] \tag{2.2}$$

$$d_0^T \nabla g_i(x) = -r_i(x)\,\lambda_{0i}\, g_i(x);\; i = 1,\ldots,m \tag{2.3}$$

$$d_0^T \nabla g_i(x) = -g_i(x) \; ; \; i = m+1,\ldots,m+p \tag{2.4}$$

If $d_0 = 0$, Stop

Step 2 : If $c_i < -1.2\,\lambda_{01}$, set $c_i = -2\lambda_{0i}$;

$$i = m+1,\ldots,m+p.$$

Compute $Z = \sum_{i+1}^{m} \lambda_{0i} + \sum_{i=m+1}^{m+p} (\lambda_{0i} + c_i)$

and $\rho_1 = (1-\alpha)Z$.

If $0 < \rho_1 < \rho$, set

$\rho = \frac{1}{2}\rho_1$.

Compute $\lambda \in R^{m+p}$ and the search direction $d \in R^n$ by solving the linear system of equations

$$d = -\left[\nabla f(x) + \sum_{i=1}^{m+p} \lambda_i \nabla g_i(x)\right] \tag{2.5}$$

$$d^T \nabla g_i = -\left[r_i(x)\lambda_i\, g_i(x) + \rho\, |d_0|^2\right];$$

$$i = 1,\ldots,m \tag{2.6}$$

$$d^T \nabla g_i = -\left[g_i(x) + \rho\, |d_0|^2\right];\; i = m+1,\ldots,m+p \tag{2.7}$$

Step 3: Set $\gamma_i = \gamma_0$ if $\lambda_i \geq 0$ or $\gamma_i = 1$ if $\lambda_i < 0$ for constraints, and $\gamma_i = 0$ for the equalities. Call τ the greatest t_i such that

$$g_i\,(x + t_i d) \leq \gamma_i g_i(x)\ ;\quad i = 1,\ldots,m+p\ .$$

Find $\bar{t}\varepsilon\ (0,\tau)$, the first value of the sequence $\{1, 1/\nu,\ 1/\nu^3,\ldots,\}$ with $\nu > 1$, such that

$$\theta_c(x + \bar{t}d) \leq \theta_c(x) + \bar{t}\ \sigma\ d^T\ \nabla\ \theta_c(x)$$

Step 4: Set the new iterate

$$\bar{x} = x + \bar{t}d$$

Step 5: Go to step 1.

We shall make some remarks explaining the behaviour of the algorithm and the ideas behind its construction. Suppose first that there are not equality constraints, in this case:

- At the stationary points of the problem λ and λ_0 are equal to the Lagrange multipliers, according to definition 1.1. At these points, d and d_0 are zero, then they are fixed points of the algorithm.

- d_0 is the steepest descent direction for the function

$$L(x) = f(x) + \sum_{i=1,m} \lambda_{0i}\ g_i(x)$$

- Equalities (2.3) can be considered as an updating rule for λ_0.
  They force d_0 to point to the constraints associated to a positive λ_{0i} - parameter.
  This fact, which performs an automatic selection of the active set of constraints, is confirmed in the statement of the linear search scheme.

- If the minimum of the problem is an interior point, d_0 approaches the steepest descent direction.

- d_0 is contained in the subspace tangent to the active constraints.
  Then, when there are active constraints, d_0 may point towards the exterior of the feasible region.
- It was shown in ref. [13] that d_0 becomes the projected gradient method when all the constraints are active.
- In order to get a fesible direction when there are active constraints, the tangent direction d_0 is modified to obtain a secant direction d. This is done by adding a positive element in the right hand side of (2.3), getting (2.6). ρ is computed in a way to assure that d is a descent direction also (see [13]).
- Strict feasibility is needed to avoid stataionary points which are not Kuhn-Tucker points.
- The stepsize procedure maintains a monotone decrease of the function and acts as a barrier, in order to escape from the constraints with negative λ. It also guarantees strict feasibility at each iteration. We used an Armijo type procedure, wich showed to be very simple and efficient.
  When equality constraints are also considered, we shall mention that:
- Condition (2.4) can be considered as an updating rule for λ_{0i}, wich forces the new configuration to meet the equality constraints.
- Condition (2.7) and the linear search procedure assure that all the iterates are on the same side of the equality constraints.
  Otherwise, it would be necessary to use a nondifferentiable function instead of θ_c defined in (2.1).
  The proof. of global convergence of this method was developed by the author in ref [13]. It was assumed that f is C^1, g_i are C^2, and that all the iterations give regular points of the problem.

NUMERICAL RESULTS

The given algorithm has been applied to several test problems. We report here our experience with six problems, described in a work by Hock et al [14].

We shall identify them with the same number as in the mentioned work.

Problem 35 - (Beale's problem) has 3 design variables and 4 linear inequality constraints.

Problem 43 - (Rosen - Suzuki, [7] has 4 design variables and 3 non-linear inequality constraints.

Problem 78 - [3,15] has 5 design variables and 3 non-linear equality constraints.

Problem 80 - (Powell, [18] is a modification of problem 78; it has 5 design variables, 3 non-linear equality constraints and 10 linear inequality constraints.

Problem 86 - (Colville nº1, [8] has 5 design variables and 15 linear inequality constraints.

Problem 117 - (Colville nº2, [8] has 15 design variables, 5 non-linear inequality constraints and 15 linear inequality constraints.

In all of themm the initial point is feasible for the inequality constraints and non feasible for the equalities. The iterative process was stopped with a value of the function correct to five significant digits, the inequalities verified, and the equalities verified with an error less than 10^{-5}.

The tests were performed on a HB-68 DPS/Multics computer. All the calculations were carried out in single precision (27 bit mantissa), except problem 117, for which double precision was used.

In table 3.1 we give our final results and also intermediate results in which the objetive function value is correct up to two significative digits.

Even if the purpose of the work of Hock et al, was not to study the efficiency of the tested non-linear programming methods, it is very convenient to compare our results with those that they obtained with six different methods. In their work, the best performances are given by VFØ2AD and OPRQP programs.

VFØ2AD was developed by Powell; it is an implementation of Wilson, Han and Powell's method [11,12,16,17].

OPRQP was developed by Biggs, based on his own method described in refs. [5,6]. Note VFØ2AD solves a quadratic programming subproblem at each iteration, and OPRQP needs an active set strategy. Both programs approximate the Hessian of the Lagrangian of the problem, by means of a quasi-Newton method.

In the numerical tests shown in [14], in general VFØ2AD needed a less number of functions and gradient evaluations than OPRQP; but in counterpart, OPRQP used less calculation time.

In the examples considered here, the number of evaluations with our method, generally goes between the number of VFØ2AD and OPRQP. We estimate that computation time per iteration used by our method is similar to that used by Bigg's approach.

Table 3.1 shows that the final convergence of the present method is slow. It seems that this may be improved with the use of quasi-Newton technique.

Problem	Iteration	Func. and grad. evaluations	Function value
35	5	6	0.1123447
	9	11	0.1111125
43	5	9	-43.81453
	13	18	-43.99907
78	5	5	-2.959694
	12	12	-2.919709
80	3	4	0.05478925
	15	18	0.05394989
86	6	6	-32.03453
	9	9	-32.34851
117	34	38	32.81567
	49	64	32.34897

Table 3.1

Considering that the present is a feasible method and that it doesn't make use of quasi-Newton techniques, we conclude that the numerical results are very satisfatory.

The method proved also to be very reliable. This is due to the fact that active set strategies are unnecessary, and that the linear search scheme doesn't introduce discontinuities.

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