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LEAST-SQUARES FORMULATIONS APPLIED TO MISCIBLE FLOW PROBLEMS

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Abstract. Incompressible miscible flows in porous media which characterize tertiary recovery process in oil reservoir are mathematically modeled by a coupled non-linear partial differential equation system with appropriate boundary and initial conditions.

This system can be solved by an implicit sequential method breaking it in a elliptic sub-system involving pressure and velocity fields coming from mass conservation equation and Darcy's law together with a transport equation predominantly convective for the concentration, which is the most important variable.

In this work, after rewriting these equations as first order differential equation systems, finite element method, with least-squares variational formulations is applied to solve this elliptic subsystem as well as the transport equation.

We also consider and discuss the approximation improvement for the vector variable when adding to our system the non rotational flux condition.

The formulations here considered besides of being mixed formulations are symmetric and equal order interpolations can be used for the elliptic sub-system involved fields as well as for the concentration and its derivative in the transport equation.

Numerical simulations are presented for the tracer injection problem with varied mobility ratios showing the good stability of the proposed formulations.

1 INTRODUCTION

The development of new numerical methods together with the continuous growth of computational resources, much has contributed to the application of numerical porous media flow simulations. One of these applications, for instance, aims to provide informations needed to increase the production in a oil field reservoir.

Our concerning problem is the simulation of cases of tertiary recovery in oil fields which have its natural conditions altered by fluid injection to increase the internal pressure and consequently the flow in producing wells.

The mathematical model here used is written as a system of non-linear differential equations, for which the resolution, used in this work, is the implicit sequential method described in section 3, that uncouples the problem in an elliptic sub-system, for the determination of the velocity and pressure fields, and a transport equation for the concentration calculation. Several approximations, using finite elements methods based on Galerkin formulations has been employed to solve this problem as, for instance, those mentioned in references Malta (1995), Garcia (1997) and Ney (2002).

In this work we evaluate the numerical results for the problem of miscible flows, when least-squares formulations are used. Velocity and pressure are obtained initially by the least-squares formulation in its usual form, as presented in section 4. Another least-squares formulation, considering null curl that improves the velocity approximation is also presented.

In section 5 is shown how is the concentration calculation when using least-squares semi-discrete formulations, with the transport equation described as a first order equivalent system (Fernandes, 2007; Fernandes and Leal-Toledo, 2008).

Since in our example of interest the source term is represented by *Dirac deltas*, we present in section 6, a singularity removal technique used in (Ewing, 1977; Garcia, 1997; Malta, 1995).

Next, simulations are presented using the proposed formulations comparing their results with some others described in the literature.

2 MATHEMATICAL MODEL

The mathematical model for incompressible miscible fluids flow simulation in a porous media as it takes place in a oil reservoir is defined by a coupled non-linear partial differentials system of equations, together with their respective boundary and initial conditions as described next:

Consider a domain $\Omega \subset R^2$, with border Γ, where gravitational effects are neglected. The governing equations on an interval $t \in [0, T]$, can be written as (Peaceman, 1977; Ewing, 1977):

$$\nabla \cdot \mathbf{u} = f \quad \text{in} \quad \Omega \times (0, T) \tag{1}$$

$$\mathbf{u} = -\frac{\mathbf{k(x)}}{\mu(c)} \nabla p \quad \text{in} \quad \Omega \times (0, T) \tag{2}$$

$$\lambda \frac{\partial c}{\partial t} + \nabla \cdot (c\mathbf{u}) - \nabla \cdot (\mathbf{D}(\mathbf{u}, \mathbf{x}) \nabla c) = f\widehat{c} \quad \text{in} \quad \Omega \times (0, T) \tag{3}$$

with boundary and initial conditions:

$$\mathbf{u} \cdot \mathbf{n} = 0 \quad \text{in} \quad \Gamma \tag{4}$$

$$\mathbf{D}(\mathbf{u}, \mathbf{x}) \nabla c \cdot \mathbf{n} = 0 \quad \text{in} \quad \Gamma \tag{5}$$

$$c(\mathbf{x}, 0) = c_0(\mathbf{x}) \quad \text{in} \quad \Omega \tag{6}$$

being $\mathbf{x} = (x, y)$ the position vector, $p(\mathbf{x}, t)$, the pressure ; $\mathbf{u}(\mathbf{x}, t)$, Darcy's velocity of the mixture with components $u_x(\mathbf{x}, t)$ and $u_y(\mathbf{x}, t)$; $\lambda = \lambda(\mathbf{x})$, the porosity; $\mathbf{k}(\mathbf{x})$, the porous media permeability; $c(\mathbf{x}, t)$, the concentration of the fluids mixture, $f = f(\mathbf{x}, t)$, the source and sink terms and $\widehat{c}$ is the injected concentration at injection wells and the resident concentration at production wells.

The diffusion-dispersion tensor $\mathbf{D}$ in the concentration equation can be expressed by (Peaceman, 1977; Ewing, 1977):

$$\mathbf{D} = \alpha_m I + \mid \mathbf{u} \mid \left\{ \alpha_l E(\mathbf{u}) + \alpha_t E(\mathbf{u})^{\perp} \right\}, \tag{7}$$

where the (i, j) component of tensor $E(\mathbf{u})$ is given by:

$$E(\mathbf{u})_{i,j} = \frac{1}{\mid \mathbf{u} \mid^2} u_i u_j, \tag{8}$$

and the norm $\mid \mathbf{u} \mid$ is defined as:

$$\mid \mathbf{u} \mid = (u_1^2 + u_2^2)^{\frac{1}{2}} \tag{9}$$

and

$$E(\mathbf{u})^{\perp} = \mathbf{I} - E(\mathbf{u}) \tag{10}$$

being $\mathbf{I}$, the identity matrix; α_m, the molecular diffusion coefficient, α_l, the longitudinal dispersion coefficient α_t, the transversal dispersion coefficient.

The unknown of major interest in the system of equations (1-6) is the concentration $c(\mathbf{x}, t)$, that in the numerical simulations of reservoir recovering techniques indicates how much of the production is influenced by the injection of a fluid that is, how much oil it can come to be recovered by this intervention in the reservoir normal production behaviour.

In the equation (2) - Darcy's law - the viscosity μ is given by the empirical relation (Settari et al., 1977):

$$\mu(c) = \mu_{resident} \left[1 - c + M^{\frac{1}{4}} c \right]^{-4}, \quad c \in [0, 1] \tag{11}$$

where

$$M = \frac{\mu_{resident}}{\mu_{injected}} \tag{12}$$

is the viscosities ratio between the resident and injected fluids $(\mu_{resident})$ and $(\mu_{injected})$, respectively, being this usually known as mobility ratio.

3 SOLUTION ALGORITHM

In this section we present the Implicit Sequential Method (IMPES) as the solving algorithm for the described mathematical model. Using the notation $\left.\frac{\partial c}{\partial t}\right|_{t=t_i}$ to designate the approximations on time $t = t_i$.

By this method we expand the term $\nabla \cdot (\mathbf{u}c)$, from the non-linear transport equation (Garcia, 1997; Pinto, 1991; Coutinho and Alves, 1996) as:

$$\nabla \cdot (\mathbf{u}c) = \mathbf{u} \cdot \nabla c + c \nabla \cdot \mathbf{u} \tag{13}$$

Substituting (1) in (13) one has:

$$\nabla \cdot (\mathbf{u}c) = \mathbf{u} \cdot \nabla c + fc \tag{14}$$

the linearizaded implicit sequential method is:

For $n = 0, 1, 2, ..., N$ find $\mathbf{u}^n$, p^n and c^{n+1} in $\Omega \times [0, T]$ satisfying to:

$$\nabla \cdot \mathbf{u}^n = f^n \tag{15}$$

$$\mathbf{u}^n = -\frac{k(\mathbf{x})}{\mu(c^n)} \nabla p^n \tag{16}$$

$$\left.\frac{\partial c}{\partial t}\right|_{t=t_{n+1}} + \mathbf{u}^n \cdot \nabla c^{n+1} - \nabla \cdot D(\mathbf{u}^n) \nabla c^{n+1} + c^{n+1} f^{n+1} = \widehat{c}^{n+1} f^{n+1} \tag{17}$$

with boundary and initial conditions given by Eq. (4), Eq. (5) and Eq. (6).

Noting that in the elliptic sub-system formed by equations Eq. (15) and Eq. (16) there is no boundary condition for $p(\mathbf{x}, t)$. So its solution cannot be determined and in order to raise this indetermination one needs to prescribe the pressure in some point of the domain in such a way that the solution $p(\mathbf{x}, t)$ has null mean, that is:

$$\int_\Omega p(\mathbf{x}, t) dx = 0 \quad t \in (0, T). \tag{18}$$

This algorithm, that has been successfully used in others papers(?) (Garcia, 1997; Ney, 2002), is also adopted in the present work. Hence at each time step the pressure and the velocity are calculated by the elliptic subsystem above described in $t = t_n$, and the velocity results are used in the concentration calculation in the transport equation.

4 APPROXIMATION FOR THE EQUATIONS IN PRESSURE AND VELOCITY

As is the velocity and not the pressure that appears in the concentration equation, special attention must be given to the attainment of accurate approximations for this field in order to minimize the concentration approximation errors so it does not affect his precision. Keeping the attention to this problem we present, in this section, least-squares formulations applied to the elliptic sub-system for the determination of the pressure and for the velocity described in the previous section.

4.1 Problem Statement

Consider for simplicity, $\Omega \subset R^2$ a regular limited domain with borders Γ, such that $\Gamma_u \cup \Gamma_p = \Gamma$ and $\Gamma_u \cap \Gamma_p = \emptyset$ where $\emptyset$ is an empty set.

We describe our problem as:

For a given value of f find the fields $\mathbf{u}$ and p satisfying to:

$$\nabla \cdot \mathbf{u} = f \quad \text{in} \quad \Omega \tag{19}$$

$$\mathbf{u} = -k(\mathbf{x}) \nabla p \quad \text{in} \quad \Omega \tag{20}$$

$$p = p_0 \quad \text{in} \quad \Gamma_p \tag{21}$$

$$\mathbf{u} \cdot \mathbf{n} = 0 \quad \text{in} \quad \Gamma_u, \tag{22}$$

being $\mathbf{u} = (u_1, u_2)$, the velocity vector and p, the scalar value defining the pressure.

The least-squares functional for this system is given by:

$$J_1(u,p) = \frac{1}{2}\int_\Omega (\nabla \cdot \mathbf{u} - f)(\nabla \cdot \mathbf{u} - f) + (\mathbf{u} + k(\mathbf{x})\nabla p)(k^{-1}(\mathbf{x})\mathbf{u} + \nabla p)d\Omega \tag{23}$$

Together with this functional variation we define the following problem:

$Problem\ P_1$: Find $(\mathbf{u}, p)$, such that:

$$\mathbf{B}\left\{(\mathbf{u}, p); (\mathbf{q}, \eta)\right\} = \mathbf{F}(\mathbf{q}) \tag{24}$$

with $\mathbf{B}(\cdot,\cdot)$ and $\mathbf{F}(\cdot)$, defined as follows.

$$\mathbf{B}\left\{(\mathbf{u}, p); (\mathbf{q}, \eta)\right\} = \int_\Omega (\nabla \cdot \mathbf{u})\left(\nabla \cdot \mathbf{q}\right) + \left(\mathbf{u} + k\left(\mathbf{x}\right)\nabla p\right)\left(k^{-1}\left(\mathbf{x}\right)\mathbf{q} + \nabla \eta\right)$$

$$\mathbf{F}(\mathbf{q}) = \int f \nabla \cdot \mathbf{q} d\Omega \tag{25}$$

It is possible to show (Leal-Toledo, 1992; Pehlivanov and Carey, 1994) that, besides this problem be a mixed one there is no need, in this case, of compatibility between the approximation spaces for the $\mathbf{u}$ and p variables. Thus it is possible to accommodate equal order approximations, obtaining, in this case, the following error estimates:

$$\|p - p_h\|_{H_1} + \|\mathbf{u} - \mathbf{u}_h\|_{H_{div}} \leq ch^k \tag{26}$$

where $||\mathbf{u}||^2_{H_{div}}$ e $||p||^2_{H_1}$ are defined as:

$$||\mathbf{u}||^2_{H_{div}} \equiv ||\mathbf{u}||^2_{L_2} + ||div\mathbf{u}||^2_{L_2} \tag{27}$$

and

$$||p||^2_{H_1} \equiv ||p||^2_{L_2} + ||\nabla p||^2_{L_2} \tag{28}$$

being ∇ the gradient operator and $||\cdot||_{L_2}$ the L_2 norm, defined in its usual form.

4.2 Mixed formulation with curl

Looking for better approximations for the velocity field, which is the greatest interest problem variable, a formulation including the null curl equation for $\mathbf{u}$ was proposed in (Leal-Toledo, 1992; Pehlivanov and Carey, 1994)

Thus the elliptic sub-system can be rewritten as:

$$\nabla \cdot \mathbf{u} = f \quad \text{in} \quad \Omega \tag{29}$$

$$\mathbf{u} = -k(\mathbf{x})\nabla p \quad \text{in} \quad \Omega \tag{30}$$

$$k^{-1}(\mathbf{x})\nabla \times \mathbf{u} = 0 \quad \text{in} \quad \Omega \tag{31}$$

$$p = p_0 \quad \text{in} \quad \Gamma_p \tag{32}$$

$$\mathbf{u} \cdot \mathbf{n} = 0 \quad \text{in} \quad \Gamma_u \tag{33}$$

$$\mathbf{u} \times \mathbf{n} = 0 \quad \text{in} \quad \Gamma_p \tag{34}$$

The least-squares functional for the present problem is

$$J(u,p) = \frac{1}{2}\int_\Omega \left\{(\nabla\cdot\mathbf{u} - f)(\nabla\cdot\mathbf{u} - f) + (k^{-1}(\mathbf{x})\nabla\times\mathbf{u})(k^{-1}(\mathbf{x})\nabla\times\mathbf{u})\right\} d\Omega +$$

$$+\frac{1}{2}\int_\Omega \left\{(\mathbf{u} + k(\mathbf{x})\nabla p)(k^{-1}(\mathbf{x})\mathbf{u} + \nabla p)\right\} d\Omega \tag{35}$$

Proceeding to the variation of J, we obtain the folowing variational problem:

Problem P_2: Find $(\mathbf{u}, p)$, such that:

$$\mathbf{B}\left\{(\mathbf{u},p);(\mathbf{q},\eta)\right\} = \mathbf{F}(\mathbf{q}) \tag{36}$$

with $\mathbf{B}(\cdot,\cdot)$ e $\mathbf{F}(\cdot)$, defined as it follows:

$$\mathbf{B}\left\{(\mathbf{u},p);(\mathbf{q},\eta)\right\} = \int_\Omega \left\{(\nabla\cdot\mathbf{u})(\nabla\cdot\mathbf{q}) + (\mathbf{u} + k(\mathbf{x})\nabla p)\left(k^{-1}(\mathbf{x})\mathbf{q} + \nabla\eta\right)\right\} d\Omega +$$

$$+\frac{1}{2}\int_\Omega \left\{\left(k^{-1}(\mathbf{x})\nabla\times\mathbf{u}\right)\left(k^{-1}(\mathbf{x})\nabla\times\mathbf{q}\right)\right\} d\Omega \tag{37}$$

$$\mathbf{F}(\mathbf{q}) = \int f\nabla\cdot\mathbf{q}\, d\Omega \tag{38}$$

We can observe that, in the discretization for this system, the unknown nodal number is kept unchanged respect to the former formulation thus, maintaining the order of the algebraic system of equations to be solved.

For this formulation error estimate (Leal-Toledo, 1992; Pehlivanov and Carey, 1994) for the finite element discretization can be found:

$$\|p - p_h\|_{H_1} + \|\mathbf{u} - \mathbf{u}_h\|_{H_1} \leq ch^k \tag{39}$$

where k is the approximation polymnomium degree booth for p as for $\mathbf{u}$, h is a spatial discretization parameter and c a constant.

In this case it is also possible to obtain the L^2 error estimate (Pehlivanov and Carey, 1994) given by:

$$\|p - p_h\|_{L_2} + \|\mathbf{u} - \mathbf{u}_h\|_{L_2} \leq ch^{k+1} \tag{40}$$

for equal order interpolations, although different interpolations orders can be used to approximate p and $\mathbf{u}$ (Pehlivanov and Carey, 1994).

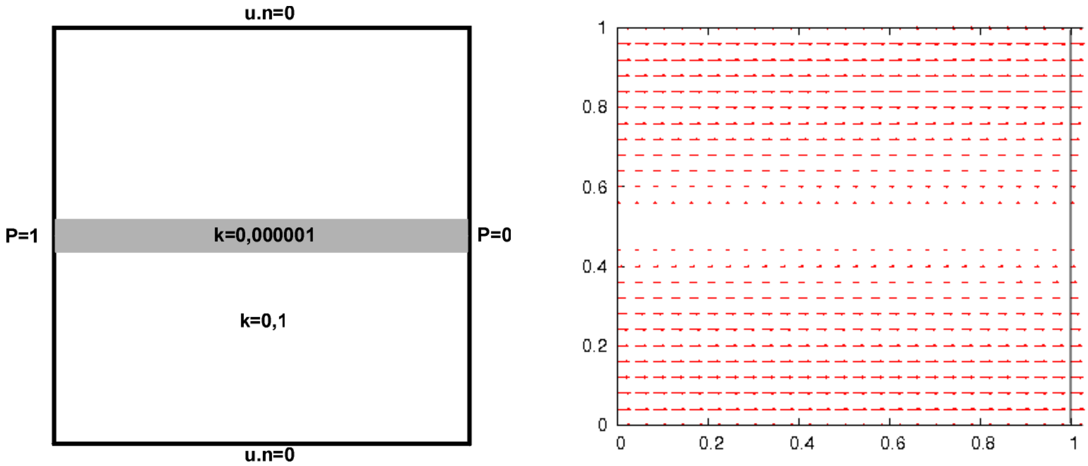

Figure 1: Heterogeneous Permeability — Domain and Results

4.3 Numerical Results

To test the herein proposed formulations we next present results obtained for a heterogenous domain as shown by Figure 1, using the least-squares formulation as described in problem P_2. By this result it is possible to see, in this same figure, that with the inclusion of the equation $\nabla \times \mathbf{u} = 0$ to our formulation there is no oscillations and we can detect the parabolic profile which is not captured in others formulations such as those presented in (Garcia, 1997).

5 TRANSPORT EQUATION

In this section we present the least-squares formulation applied to the transport advective-diffusive equation described as first order equivalent equations system using semi-discrete formulations (Vasconcelos, 2001).

5.1 Problem statement

Let $\Omega \subset R^2$ be a two-dimensional limited domain with regular border Γ such that:

$$\Gamma_d \cup \Gamma_n = \Gamma$$

$$\Gamma_d \cap \Gamma_n = \emptyset \tag{41}$$

The transport advective-diffusive equation is given by:

$$\lambda \frac{\partial c}{\partial t} + \mathbf{u} \cdot \nabla(c) - \nabla \cdot (\mathbf{D}\nabla c) = f \quad \text{em} \quad \Omega \times (0, T) \tag{42}$$

being λ the porous media porosity and boundary and initial conditions given by:

$$c = g \quad \text{in} \quad \Gamma_d \times (0, T) \tag{43}$$

$$\mathbf{D}\nabla c \cdot \mathbf{n} = g \quad \text{in} \quad \Gamma_n \times (0, T) \tag{44}$$

$$c = c_0 \quad \text{in} \quad \Omega \quad \text{for} \quad t = 0, \tag{45}$$

where c is the unknown quantity being transported by the constant advective term $\mathbf{u} = (u_1, u_2)$, f is the source term, $\mathbf{D}$ is the diffusion coefficient and g is the prescribed boundary value of c in Γ_d.
To employ least-squares formulation the equation (42) is rewritten as:

$$\begin{aligned} &\lambda \frac{\partial c}{\partial t} + \mathbf{u} \cdot \mathbf{q} - \nabla \cdot \mathbf{D}\mathbf{q} = f \quad \text{in} \quad \Omega \times (0, T) \\ &\mathbf{q} = \nabla c \quad \text{in} \quad \Omega \times (0, T) \end{aligned} \tag{46}$$

On the next section we present semi-discrete least-squares formulations applied to the transport equation when we rewrite it as a system like described by Eqns. (46).

5.2 Least-squares formulations for transport equation

For comparison purposes we present for equation convection-diffusion, described in (46), three different least-squares formulations suggested in (Vasconcelos, 2001) namely: one implicit formulation, one weighted least-squares formulation and a θ least-squares formulation as next described.

5.3 Totally implicit formulation

We call here of totally implicit formulation when all terms in the system are written in time t_{n+1}. In this case the least-squares functional associated to the equation system (46) is given by:

$$J(c_h, \mathbf{q}_h) = \frac{1}{2}\left[\left(\frac{c_h^{n+1} - c_h^n}{\triangle t} + \mathbf{u} \cdot \mathbf{q}_h^{n+1} - \nabla \cdot \mathbf{D}\mathbf{q}_h - f^{n+1}\right.,\right.$$

$$\left.\left.\frac{c_h^{n+1} - c_h^n}{\triangle t} + \mathbf{u} \cdot \mathbf{q}^{n+1} - \nabla \cdot \mathbf{D}\mathbf{q}_h - f^{n+1}\right) + \left(\mathbf{q}_h^{n+1} + \nabla c_h^{n+1}, \mathbf{q}_h^{n+1} + \nabla c_h^{n+1}\right)\right] \tag{47}$$

The problem of find the solution of system (46) based on the minimization of the functional (47) respect to c_h^{n+1} and to $\mathbf{q}^{n+1}$ can be defined as:
$Problem\ P_3$: Find c_h^{n+1} e $\mathbf{q}_h^{n+1}$, such that:

$$\left(\frac{c_h^{n+1} - c_h^n}{\triangle t} + \mathbf{u} \cdot \mathbf{q}_h^{n+1} - \nabla \cdot \mathbf{D}\mathbf{q}_h, \frac{v_h^{n+1}}{\triangle t} + \mathbf{u} \cdot \mathbf{p}^{n+1} - \nabla \cdot \mathbf{D}\mathbf{p}_h\right) +$$

$$\left(\mathbf{q}_h^{n+1} + \nabla c_h^{n+1}, \mathbf{p}_h^{n+1} + \nabla v_h^{n+1}\right) = \left(f^{n+1}, \frac{v_h^{n+1}}{\triangle t} + \mathbf{u} \cdot \mathbf{p}^{n+1} - \nabla \cdot \mathbf{D}\mathbf{p}_h\right) \tag{48}$$

The discretization for c_h, $\mathbf{q}_h$, v_h, $\mathbf{p}_h$ can be given in its usual way by the finite element method.

5.4 Weighted least-squares formulation

The formulation we call weighted least-squares based on those proposed in (Jiang, 1998; Vasconcelos, 2001) for others equations , consist in to weight the transport equation of the system (46) by a θ factor, to obtain the approximation of this equation on time $t_{n+\theta}$, and to consider the steady equations of this system on time t_{n+1}, that is:

$$\frac{c_h^{n+1} - c_h^n}{\triangle t} + \mathbf{u} \cdot \mathbf{q}_h^{n+\theta} - \nabla \cdot \mathbf{D}\mathbf{q}_h^{n+\theta} = f^{n+\theta}\Omega \tag{49}$$

for $0 \leq \theta \leq 1$. The discrete functional of the da weighted least-squares formulation in this case is then given by:

$$J(c_h, \mathbf{q}_h) = \frac{1}{2}\left[\left(\frac{c_h^{n+1} - q_h^n}{\triangle t} + \mathbf{u}\cdot\mathbf{q}_h^{n+\theta} - \nabla\cdot\mathbf{D}\mathbf{q}_h^{n+\theta} - f^{n+\theta},\right.\right.$$

$$\left.\left.\frac{c_h^{n+1} - c_h^n}{\triangle t} + \mathbf{u}\cdot\mathbf{q}_h^{n+\theta} - \nabla\cdot\mathbf{D}\mathbf{q}_h^{n+\theta} - f^{n+\theta}\right) + \left(\mathbf{q}_h^{n+1} - \nabla c_h^{n+1}, \mathbf{q}_h^{n+1} - \nabla c_h^{n+1}\right)\right] \tag{50}$$

The formulation, how it is proposed, uses to approximate $\mathbf{q}_h^{n+\theta}$ and $f^{n+\theta}$ in the functional (50) the following relations:

$$\begin{aligned} \mathbf{q}_h^{n+\theta} &= \theta\mathbf{q}_h^{n+1} + (1-\theta)\mathbf{q}_h^n \\ f^{n+\theta} &= \theta f^{n+1} + (1-\theta) f^n \end{aligned} \tag{51}$$

Substituting these approximations in (50) we have:

$$J(c_h, \mathbf{q}_h) = \frac{1}{2}\left[\left(\frac{c_h^{n+1} - c_h^n}{\triangle t} + \mathbf{u}\cdot\left(\theta\mathbf{q}_h^{n+1} + (1-\theta)\,\theta\mathbf{q}_h^n\right) + \nabla\cdot\mathbf{D}\left(\theta\mathbf{q}_h^{n+1} + (1-\theta)\,\mathbf{q}_h^n\right) -\right.\right.$$

$$\theta f^{n+1} + (1-\theta)\, f^n, \frac{c_h^{n+1} - c_h^n}{\triangle t} + \mathbf{u}\cdot\left(\theta\mathbf{q}_h^{n+1} + (1-\theta)\,\theta\mathbf{q}_h^n\right) - \nabla\cdot\mathbf{D}\left(\theta\mathbf{q}_h^{n+1} + (1-\theta)\,\mathbf{q}_h^n\right) -$$

$$\left.\left.\theta f^{n+1} + (1-\theta)\, f^n\right) + \left(\mathbf{q}_h^{n+1} - \nabla v_h^{n+1}, \mathbf{q}_h^{n+1} - \nabla v_h^{n+1}\right)\right] \tag{52}$$

Find the solution of system (46) based on the minimization of the functional (52) respect to c_h^{n+1} and to $\mathbf{q}_h^{n+1}$ is:

$Problem\ P_4$: Find $(c_h^{n+1}, \mathbf{q}_h^{n+1})$, such that:

$$\left(\frac{c_h^{n+1} - c_h^n}{\Delta t} + \mathbf{u}\cdot(\theta\mathbf{q}_h^{n+1} + (1-\theta)\mathbf{q}_h^n) + \nabla\cdot\mathbf{D}(\theta\mathbf{q}_h^{n+1} + (1-\theta)\mathbf{q}_h^n),\right.$$

$$\left.\frac{v_h^{n+1}}{\Delta t} + \mathbf{u}\cdot\theta\mathbf{p}_h^{n+1} + \nabla\cdot\mathbf{D}\theta\mathbf{p}_h^{n+1}\right) + (\mathbf{q}_h^{n+1} - \nabla c_h^{n+1}, \mathbf{p}_h^{n+1} - \nabla v_h^{n+1}) =$$

$$\left(\theta f^{n+1} + (1-\theta) f^n, \frac{v_h^{n+1}}{\Delta t} + \mathbf{u}\cdot\theta\mathbf{p}_h^{n+1} + \nabla\cdot\mathbf{D}\theta\mathbf{p}_h^{n+1}\right) \tag{53}$$

5.5 Formulation θ least-squares

As in the previous formulation, in the θ least-squares formulation firstly proposed in (Vasconcelos, 2001), the whole system variables are assumed to be on time $t^{n+\theta}$ and the the θ-weighted approximation is in the scalar variable $c^{n+\theta}$, that is:

$$J(c_h, \mathbf{q}_h) = \frac{1}{2}\left[\left(\frac{c_h^{n+1} - c_h^n}{\Delta t} + \mathbf{u}\cdot\mathbf{q}_h^{n+\theta} + \nabla\cdot\mathbf{D}\mathbf{q}_h^{n+\theta} - f^{n+\theta},\right.\right.$$

$$\left.\left.\frac{c_h^{n+1} - c_h^n}{\Delta t} + \mathbf{u}\cdot\mathbf{q}_h^{n+\theta} + \nabla\cdot\mathbf{D}\mathbf{q}_h^{n+\theta} - f^{n+\theta}\right) + \left(\mathbf{q}_h^{n+\theta} - \nabla c_h^{n+\theta}, \mathbf{q}_h^{n+\theta} - \nabla c_h^{n+\theta}\right)\right] \tag{54}$$

In this formulation we approximate $c_h^{n+\theta}$ as:

$$c_h^{n+\theta} = \theta c_h^{n+1} + (1-\theta)c_h^n. \tag{55}$$

Substituting (55) in (54) we get:

$$J(c_h, \mathbf{q}_h) = \frac{1}{2}\Bigg[\Bigg(\frac{c_h^{n+1} - c_h^n}{\Delta t} + \mathbf{u}\cdot\mathbf{q}_h^{n+\theta} + \nabla\cdot\mathbf{D}\mathbf{q}_h^{n+\theta} - f^{n+\theta},$$

$$\frac{c_h^{n+1} - c_h^n}{\Delta t} + \mathbf{u}\cdot\mathbf{q}_h^{n+\theta} + \nabla\cdot\mathbf{D}\mathbf{q}_h^{n+\theta} - f^{n+\theta}\Bigg) +$$

$$(\mathbf{q}_h^{n+\theta} - \nabla(\theta c_h^{n+1} + (1-\theta)c_h^n), \mathbf{q}_h^{n+\theta} - \nabla(\theta c_h^{n+1} + (1-\theta)c_h^n))\Bigg] \tag{56}$$

To solve the system (46) based on the minimization of this functional respect to c_h^{n+1} and to $\mathbf{q}_h^{n+\theta}$ can be reduced to:

$Problem\ P_5$: Find (c_h^{n+1} and $\mathbf{q}_h^{n+\theta}$), such that:

$$\left(\frac{c_h^{n+1} - c_h^n}{\Delta t} + \mathbf{u}\cdot\mathbf{q}_h^{n+\theta} + \nabla\cdot\mathbf{D}\mathbf{q}_h^{n+\theta}, \frac{v_h^{n+1}}{\Delta t} + \mathbf{u}\cdot\mathbf{p}_h^{n+\theta} + \nabla\cdot\mathbf{D}\mathbf{p}_h^{n+\theta}\right) +$$

$$(\mathbf{q}_h^{n+\theta} + \nabla(\theta c_h^{n+1} + (1-\theta)c_h^{n+1}), \mathbf{p}_h^{n+\theta} + \nabla\theta v_h^{n+1}) =$$

$$\left(f^{n+\theta}, \frac{v_h^{n+1}}{\Delta t} + \mathbf{u}\cdot\mathbf{p}_h^{n+\theta} + \nabla\cdot\mathbf{D}\mathbf{p}_h^{n+\theta}\right) \tag{57}$$

6 SINGULARITIES REMOVAL

The numerical simulation of tertiary oil recovery technique treats of a single-phase flow with two components (solvent and oil) in the porous media constituting a reservoir. For this problem the oil reservoir has injection and production wells and these wells are the sink and source terms of the model described by the equations (1) to (6).

In this case these source terms are *Dirac delta* functions applied in the injection as in the production wells bringing serious implications in the resolution of the elliptic sub-system, like the loss of regularity of the velocity field.

Having in mind this problem we present next how to cope with this difficulty like described in (Ewing, 1977) and used in (Garcia, 1997; Malta, 1995). For a configuration with N_p wells at each time step t_n, decompose velocity field u^n of the elliptic sub-system in:

$$\mathbf{u}^n = \mathbf{u}_s^n + \mathbf{u}_r^n \tag{58}$$

where $\mathbf{u}_r^n$ is the regular part of the velocity field in $t = t_n$, and $\mathbf{u}_s^n$ is the singular part given by:

$$\mathbf{u}_s^n = \sum_{i=1}^{N_p} \mathbf{u}_s^{n,i} \tag{59}$$

Then, as in (Ewing, 1977), the equation $\nabla\cdot\mathbf{u} = f$ is solved as:
i) *Singular part*:

$$\nabla\cdot\mathbf{u}_s^{n,i} = f^n \tag{60}$$

where

$$f^n(\mathbf{x}) = \sum_{i=1}^{N_p} Q_i \delta(x_i, y_i), \tag{61}$$

being $\delta(x_i, y_i)$ a Dirac delta function in (x_i, y_i) and Q_i are the specified flows in the N_p production and injection wells.
ii) *Regular part*:

$$\nabla \cdot \mathbf{u}_r^n = 0. \tag{62}$$

Substituting the equation (58) in the boundary condition $\mathbf{u} \cdot \mathbf{n} = 0$, the regular part of the problem has non-homogeneous boundary condition in $\mathbf{u}_r^n$, that is:

$$\mathbf{u}_r \cdot \mathbf{n} = -\mathbf{u}_s^n \cdot \mathbf{n} \quad \mathrm{em} \quad \Gamma \tag{63}$$

The pressure on time t_n is also decomposed in a singular and a regular part as:

$$p^n = p_s^n + p_r^n \tag{64}$$

The singular part of the problem is given by the solution of the following problem:

$$\nabla \cdot \mathbf{u} = f \cdot \delta(x_i, y_i) \tag{65}$$

$$\mathbf{u} = \nabla p \tag{66}$$

that has exact solution (Ewing, 1977) given by:

$$p_s^n = \sum_{i=1}^{N_p} p_s^{n,i} = \sum_{i=1}^{N_p} \frac{Q_i}{2\pi(k_i/\mu_i)} ln|x - x_i| \tag{67}$$

where $k_i = k(\mathbf{x}_i)$ and $\mu_i = \mu(c^n(\mathbf{x}_i))$ are the values of k and μ in the well i on time t_n.
Then with the velocities decomposition (58) the part regular of system is given by:

$$\mathbf{u}_r^n = -\frac{k(\mathbf{x})}{\mu^n} \nabla p_r^n + \sum_{i=1}^{N_p} \left(\frac{k(\mathbf{x})\mu_i}{\mu(c^n(\mathbf{x}))k_i} - 1 \right) \mathbf{u}_s^{n,i} \tag{68}$$

with $\mathbf{u}_s$ being explicitly calculated from equation (67).

The result of the presented operations in this section give us a system in $\mathbf{u}_r^n$ and p_r^n with more regularity, without the source term (62) and with non-homogeneous boundary conditions (63).

7 NUMERICAL SIMULATIONS

For the simulations here presented a two-dimensional domain where, as usual, an alternate configuration of injection and production wells represented by a distribution known as five spot pattern as shown in Figure 2, where the oil reservoir is composed by several equal blocks.

By symmetry the domain in all simulations here performed we consider only a quarter of this five wells arrangement with side length side of $L = 1000ft$, with a injection well I located in coordinates $(0, 0)$ and a production well P in $(1000, 1000)$ as it is shown by Figure 2.

7.1 Five spot configuration

All simulations here treated considered a homogeneous reservoir, discretized with a 50×50 mesh of bilinear isoparametric elements, with permeability tensor components $k_x = k_y = 100mD$, porosity $\lambda = 0.1$, molecular diffusivity coefficient $\alpha_m = 0.0$, longitudinal and transversal dispersion coefficients $\alpha_l = 1.0$ and $\alpha_t = 0.0$. The injection well operates with a flow rate of 200 $\frac{ft^3}{day}$ by a time period to the filling of 5% of its porous volume which give us a total time of injection of $25days$.

Then the injector well operates with a flow rate of 200 $\frac{ft^3}{day}$ during 25 days, so $\widehat{c} = 1$ until time equal 25 days and $\widehat{c} = 0$ after this time.

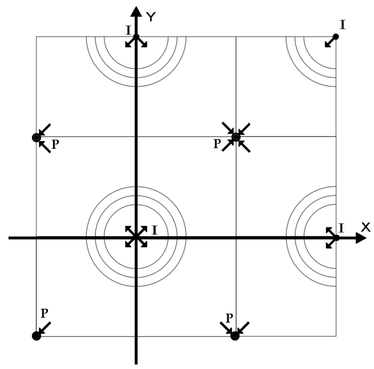

Figure 2: Configuration of a quarter of the five-spot pattern (Castro, 1999)

7.2 Simulation of tracer injection

We begin by considering the tracer injection problem used to give some characteristics of the oil reservoir like the flow direction. The coming time and tracer concentration on the production wells are also important informations to this purpose. Usually the injected tracer does not interferes in the resident fluid properties which allow us to consider the fluids mixture with mobility ratio equal one. In this case $\mu(c)$, given by equation (11) is constant and the elliptic sub-system is solved only once.

Numerical results for the concentration are presented on Figures 3 to 5 for $(M = 1)$ and time step equal $\Delta t = 1$ day.

Although using here a coarse mesh the results obtained by the formulations presented in this work we obtained comparable results which are in good agreement to those found in another works like (Garcia, 1997; Malta, 1995; Ney, 2002) with the use of stabilized Galerkin formulations employing finer grids.

7.3 Adverse mobility ratio simulation

The problem of adverse mobility ratio occurs when the injected fluid is less viscous then the resident fluid. Variations on parameter M and of dispersion parameters are factors directly related to the reservoir recovery rate which tend to reduce with the increase of the mobility ratio M (Ney, 2002). Miscible flows with adverse mobility ratio often result in numerical results with spurious oscillations as it can be seen in Figure 6, besides of have an high computational cost, since in this case the elliptic sub-system has to be solved at each time step. Numerical results obtained by the formulations here studied are presented on Figures 7 to 9 for $(M = 41)$,

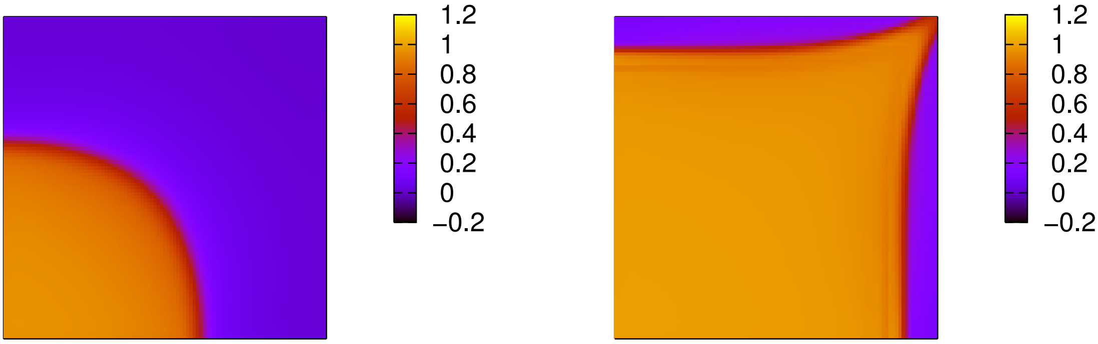

Figure 3: Implicit formulation in $t = 300$ and $t = 1500$ days

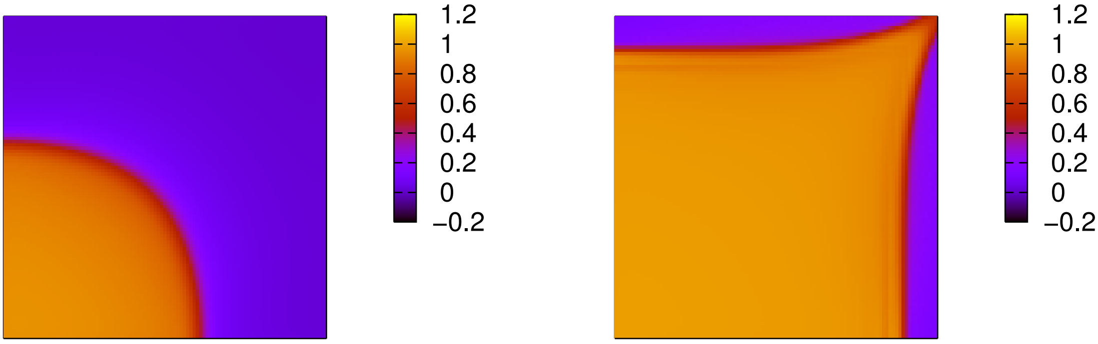

Figure 4: Weighted formulation$(\theta = 0,5)$ with $t = 300$ and $t = 1500$ days

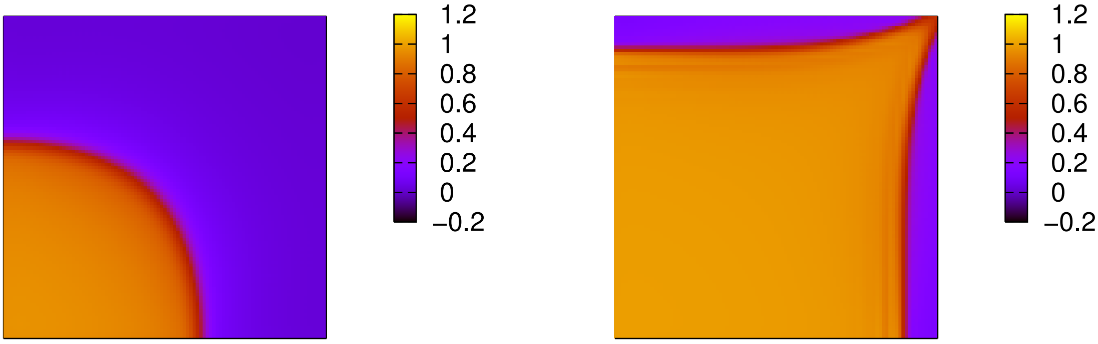

Figure 5: θ Least-squares formulation $(\theta = 0,5)$ with $t = 300$ and $t = 1500$ days

with a time step of $\Delta t = 10$ days.

For all the analyzed examples stable solutions with the predicted physical behaviour were found even with the coarse mesh of e 50×50 bilinear elements.

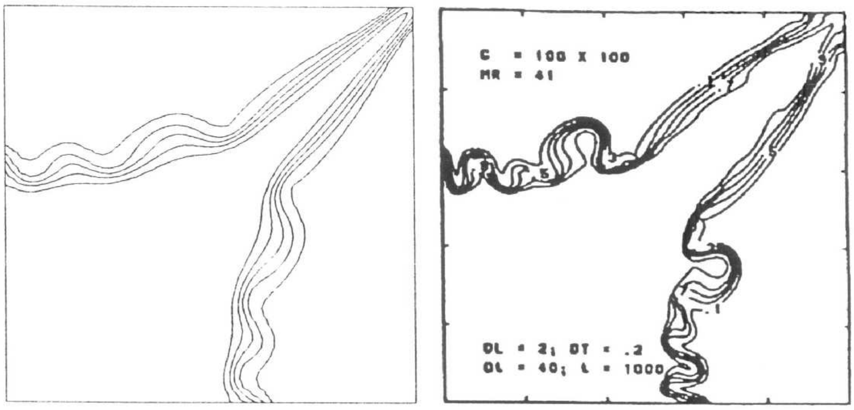

Figure 6: Spurious oscillations (Garcia, 1997)

8 CONCLUSIONS

In this work we presented least-squares formulations applied to the incompressible miscible flow problems. We evaluated the behaviour of those formulations when applied to a elliptic subsystem as well as to a predominantly convective transport equation .

For the examples here studied with the semi-discrete formulation the best results in terms of stability and precision, when compared to the literature, were found by the totally implicit formulation ($\theta = 1$) an by the θ least-squares formulation. For the elliptic sub-system the formulation is of the mixed type allowing equal order interpolations for the involved variables.

Besides of the symmetry of the proposed formulations the inclusion of the null curl equation improves the precision for the vector variable without increasing the number of equations of the discretized system. For the transport equation, described as a first order differential equations system there is a increase of the number of the equations of the discretized system calling attention to the fact that these formulations generate symmetric positive definite matrices with good stability characteristics avoiding the use of adjusting parameters.

Ours results show that least-squares formulations for transient problems are a good option to classical formulations as well to stabilized formulations thus deserving more studies.

For future work it is worth to mention the stability analysis of the proposed formulations and the development of adaptive refinement techniques.

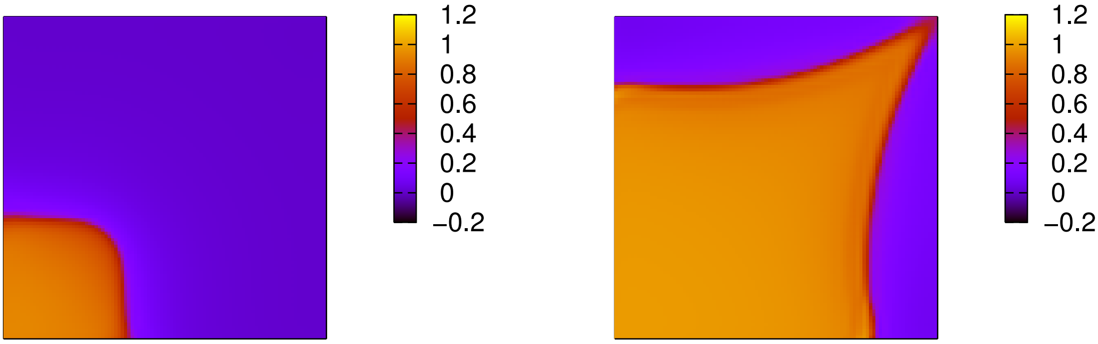

Figure 7: Implicit formulation with $t = 300$ and $t = 1500$ days

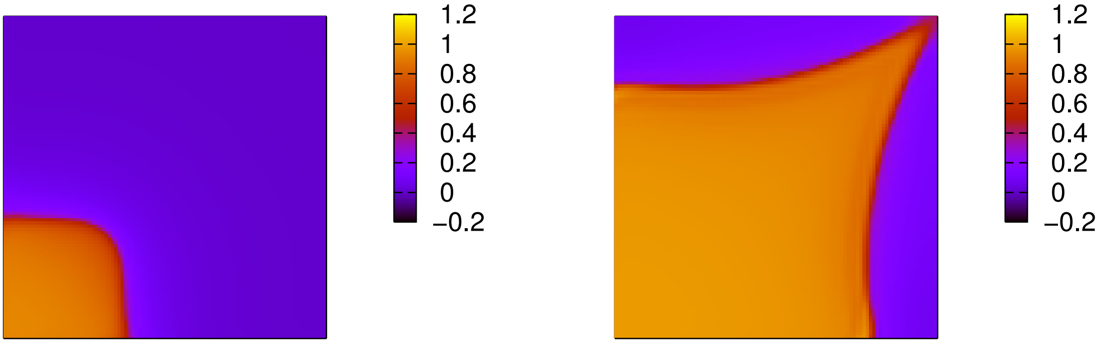

Figure 8: Weighted formulation $(\theta = 0,5)$ with $t = 300$ and $t = 1500$ days

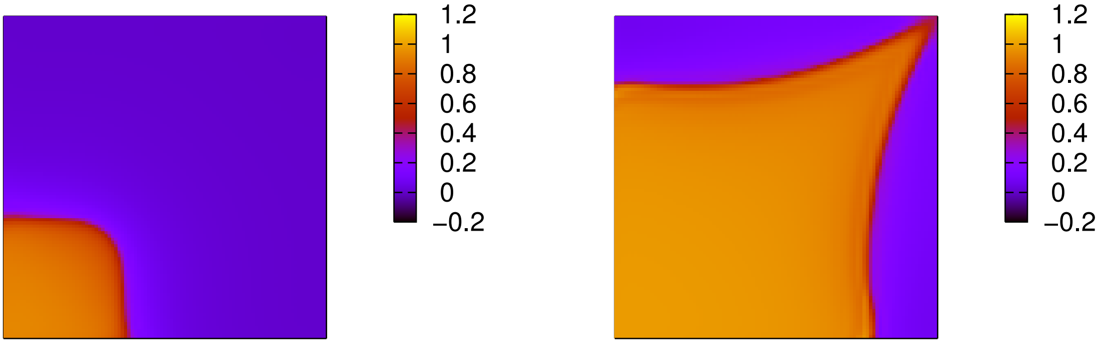

Figure 9: θ least-squares formulation $(\theta = 0,5)$ with $t = 300$ and $t = 1500$ days

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