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MIN-MAX GRADIENT SURFACE RECONSTRUCTION VIA ABSOLUTE MINIMIZATION

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Abstract. We consider a surface reconstruction problem with incomplete data. This problem arises on many applications in topography, bathymetry and 3D objects visualization. We adopt a reconstruction criterium that choose as solution the Lipschitz continuous function that extends the data with minimum Lipschitz constant. This yields a Lipschitzian extension problem (LEP), for which the classical minimization approach has some drawbacks as the lack of canonical solutions. By using the stronger concept of absolute minimizer, it is possible to derive an Euler-Lagrange equation, called Aronsson equation, that in the case of LEP's turns to be the infinity Laplacian partial differential equation (PDE). In particular, the absolute minimizers coincide with the viscosity solutions of the infinity Laplacian and there exists a convergent finite differences scheme to approximate them. We apply this approach on the resolution of practical problems in bathymetry and video images reconstruction. We discuss the implementation and show our numerical results.

1 INTRODUCTION

Surface reconstruction has been an area of intensive research in the last years, related to an increasing set of applications. Among the many different fields in which this problem arise we can found reconstruction models for terrain in topography, for ocean floors and river–beds in bathymetry, for image diagnosis in medicine, for cultural heritage in archeology, or for machine parts in industrial applications. According to each specific problem, the available information may be obtained from height topography measures via satellite triangulation, depth bathymetry measure via echosounder, or from digital scanning devices. This information can be given in different ways such as contour lines (level curves), boundary functions or a discrete set of values. Anyway, in almost all the cases (maybe after some discretization procedure), we have to deal with point cloud data, and it is necessary to adopt some interpolation criterium in order to proceed to the reconstruction process. Usually, the interpolation criteria deal with some kind of minimization problem (maybe related to some PDE), that responds to modelization hypothesis on the behavior of each problem and on the expected properties of the solutions.

A nice approach on the terrain reconstruction problem is given in Hormann et al. (2003) where, starting from C^1 contour lines data, a continuous model is presented in order to obtain a piecewise C^1 surface. The implementation consists in a triangular mesh generation based on Delaunay triangulation, and a surface reconstruction using a Hermite interpolation procedure based on the shortest distances from a interpolation point to the nearest contours and height and derivative information at the contours. The drawback of this approach is that smoothness hypothesis may be too restrictive in order to represent real terrains. The implementation allows to loose the smoothness on certain terrain characteristics such as ridges and valleys which are reconstructed as sharp features but it can not model a continuous sharp ridge or valley across the contours.

An interesting surface reconstruction model from oriented point sets for 3D objects visualization is presented in Schall et al. (2006). This scheme combines an elegant global Fast-Fourier-Transform (FFT) based reconstruction technique from Kazhdan (2005) with an adaptive procedure which employ an error-guided subdivision of the input data and partition of unity blending. This procedure allows to balance global and local approximations in order achieve higher reconstruction resolutions and less memory consumption than the original global approach. This is an appropriate method for the visualization of objects when a high accuracy data set is available, but it is not adequate for sparse data, which is an usual task in industrial quality control procedures.

In the present work, we intend to propose some minimal conditions that interpolation functions should obey in order to obtain an acceptable surface reconstruction. The idea is that a scheme with minimal restrictions should be applicable to a larger set of problems.

The first obvious conditions are the continuity and boundeness of the solutions, and it is also clear that they must be bounded variation functions. So we can ask the interpolation surfaces to be Lipschitz continuous functions. On the other hand, it is reasonable that a reconstruction criterium should not increase the “variations” of the data, which in this case means that the reconstructed surface should not increase the Lipschitz constant of the data. Hence, a good interpolation function must have minimum Lipschitz constant. Finally, our criterium should produce an unique reconstruction surface from each data set, in order to avoid ambiguities.

As we shall see in the sequel, an appropriate approach to this problem, according to the mentioned conditions, is given by the absolute minimization theory. In addition, a convergent finite differences scheme for the numerical approximation of absolute minimizing functions is available. We apply this scheme in two problems related to the obtention of hydrographic charts and the images reconstruction on video sequences.

The paper is organized as follows. In the next section we briefly describe the notion of absolute minimizers and its relation with viscosity solutions theory. In section 3, we discuss some numerical approaches and we show the adopted finite difference scheme. Our study cases with the respective numerical results are shown in section 4. Finally, in section 5, we introduce some conclusions and comments on future work.

2 ABSOLUTE MINIMIZATION PROBLEMS

Consider the classical L^∞ variational problem that consists in the minimization of a functional of the form

$$J(u) = \text{ess sup}_{x\in\Omega} f(x, u, D(u)), \tag{1}$$

where $u \in W^{1,\infty}_g(\Omega) = \{v \in W^{1,\infty}(\Omega) : v = g \text{ on } \mathrm{bd}(\Omega)\}$, Ω bounded. In particular, when $f = \|D(u)\|$ this optimization comprises the problem of finding the Lipschitzian extension to the domain Ω with minimum Lipschitz constant, with the restriction $u = g$ on $\mathrm{bd}(\Omega)$.

The classical minimizer approach for this problem (i.e. u solves (1) if $\forall v \in W^{1,\infty}_g(\Omega)$: $J(u) \le J(v)$) presents some drawbacks. In particular, the obtained solution is not canonical (and hence, not unique), in the sense that it fails to satisfy comparison and stability principles. Specifically, if $u(g_1)$ and $u(g_2)$ are classical solutions of (1) for boundary data g_1 and g_2, respectively, the inequality $g_1 \le g_2$ does not imply, in general, that $u(g_1) \le u(g_2)$ in Ω. Moreover, if O is a subdomain of Ω, and $u(u(g_1))$ solves (1) restricted to O with boundary data $u(g_1)$ on $\mathrm{bd}(O)$, it could happen that $u(u(g_1)) \ne u(g_1)$ on O.

These drawbacks in the classical approach for L^∞ variational problems were first observed in Aronsson (1965, 1966, 1967), who has introduced the notion of *absolute minimizer* and the first existence results.

The mentioned notion can be established as follows.

Definition 2.1 *A function $u^* \in W^{1,\infty}(\Omega)$ is an absolute minimizer for J on Ω if u^* minimize*

$$J(u) = \textit{ess sup}_{x\in O} f(x, u, D(u)),$$

on $W^{1,\infty}_{u^*|_{\mathrm{bd}(O)}}$, for any subdomain $O \subset \Omega$.

Note that the above definition does not depend a priori on the function g of problem (1). Hence, we can consider problems of finding absolute minimizing functions satisfying additional conditions of several types. In particular, the L^∞ variational problem (1) can be seen as an absolute minimization problem *with boundary data* g, but we can consider other kinds of constraints.

This formulation localizes the problem and allows to derive an Euler-Lagrange equation, the so-called Aronsson equation

$$A_f[u] = \sum_{i=1}^{n} f_{p_i}(x, u, Du)\frac{\partial}{\partial x_i}(f(x, u, Du)) = 0, \quad \text{in } \Omega.$$

In the special case $f = \|D(u)\|^2$, the Aronsson equation takes the form of the the infinity Laplacian equation

$$\triangle_\infty u = \sum_{i,j=1}^{n} \frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\frac{\partial^2 u}{\partial x_i x_j} = 0, \quad \text{in } \Omega. \tag{2}$$

For this particular case, it is proven in Jensen (1993) that a function $u \in W_g^{1,\infty}(\Omega)$ is an absolute minimizer for (1) if and only if u is a viscosity solution of (2) (for a detailed study of viscosity solutions theory see Crandall et al. (1992)). Recent studies from different points of view can be found in Barron (2001); Crandall et al. (2001); Manfredi et al. (2002). In particular, the solution existence under strictly quasiconvexity assumptions can be found in Barron (2001). A very comprehensive study on absolute minimizing functions for the LEP can be found in Aronsson et al. (2004).

3 NUMERICAL PROCEDURES

On the other hand, it is of great practical importance the computation of good numerical approximations of the solutions, since this problem appears in several contexts, including two-person game theory with random order of play, rapid switching of states in control problems, and shape metamorphism. For the general problem (1) in relation with the classical approach, there is a number of works that can give acceptable solutions in many practical situations. In particular, in Aragone et al. (2004) and Parente et al. (2007) have been introduced discretization procedures based on finite differences in the one-dimensional and the two-dimensional cases of the general problem, respectively. The first one gives a constrained optimization problem that can be solved with penalized Newton type methods, while the second gives a min-max problem that, with adequate convexity assumptions, yields a nonsmooth convex optimization problem that can be solved by using proximal schemes and bundle methods (see Parente et al. (2008); Bonnans et al. (2003)). The numerical experiments for the last scheme evidence the drawbacks of the classical approach, even though the obtained results may be enough for some practical porpoises.

A very attractive study on the numerical approximation of absolute minimizers for the LEP is given in Oberman (2005), where it is defined a finite differences scheme that converges to the viscosity solution of the infinity Laplacian equation.

The scheme is defined as follows.

First, a finite difference grid (i.e., an N vertices graph embedded in $\mathbb{R}^N$) is defined in order to discretize Ω, so vectors $u = (u_1, ..., u_N) \in \mathbb{R}^N$ are identified with functions on the grid. For $n = 1, ..., N$, $e(n)$ is the set of the indices of the nodes that are neighbors of node n.

The discrete Lipschitz constant is defined as

$$L(u_n) = \max_{k \in e(n)} \frac{|u_n - u_k|}{d_k^n},$$

where d_k^n is the distance between nodes n and k.

Given the values $\{u_k, k \in e(m)\}$, it turns that that the unique solution of the discrete LEP

$$\min_{u_n \in \mathbb{R}} L(u_n)$$

is given by the expression

$$u_n^* = \frac{d_i^n u_j + d_j^n u_i}{d_i^n + d_j^n}, \tag{3}$$

where

$$\frac{|u_j - u_i|}{d_i^n + d_j^n} = \max_{k,l \in e(n)} \left\{ \frac{|u_l - u_k|}{d_l^n + d_k^n} \right\}.$$

Hence, the finite difference scheme is defined as the piecewise affine function $\mathcal{F} : \mathbb{R}^N \rightarrow \mathbb{R}^N$ defined by

$$\mathcal{F}(u)_n = u_n - u_n^*,$$

where u_n^* is defined by (3). A vector $u \in \mathbb{R}^N$ is then a solution of the scheme if $\mathcal{F}(u) = 0$.

This scheme satisfy monotonicity and consistency properties that guarantee its convergence to the viscosity solution of (2) provided that appropriate resolution functions tend to zero. In particular, these resolution functions are bounds on the distance between neighbors nodes and the distance from an arbitrary direction vector (unit vector) to the set of grid direction vectors (see Oberman (2005) for details).

Also, the Euler map $\mathcal{S}_\rho : \mathbb{R}^N \rightarrow \mathbb{R}^N$ given by

$$\mathcal{S}_\rho(u) = u - \rho\mathcal{F}(u),$$

is a contraction for the defined scheme $\mathcal{F}$, provided that $\rho < 1$.

Hence, the numerical approximation of the infinity Laplacian can be obtained by a fixed point algorithm on the map $\mathcal{S}_\rho$, with linear convergence rate. Note that the scheme allows to consider any kind of input data by choosing which grid points are fixed valued and which are variable, which will be very convenient for our applications in the next section.

4 APPLICATION TO RECONSTRUCTION PROBLEMS

4.1 Bathymetry Problem

Bathymetry refers to the study of ocean or lakes floors and river–beds through the measurement of underwater depth. Usually, a discrete set of depth data is obtained from an echosounder device and bathymetric charts are produced by the implementation of some interpolation procedure. A main application of these kind of studies is the obtention of hydrographic charts, in order to provide safe navigation routes. In the case of navigable rivers, this information is subject to continuous changes produced by sedimentation, riverside collapses, dredging, etc. These changes may substantially affect the navigation conditions, so it is crucial to dispose of an accurate updated river–bed description.

On the sequel, we present a river–bed reconstruction problem that corresponds to a section of the Paraná river, in Argentina, with bathymetry and riverside data provided by Prefectura Naval Argentina. This problem have been studied in García Bauza et al. (2008), where a Delaunay triangulation based approach and a classical Lipschitz minimization procedure were presented. The first approach is based on heuristic considerations, with the resulting lack of mathematical rigourousness, and the second one, as we have mentioned before, produce non regular reconstructions according to the comparison and stability principles. Now we are able to avoid that drawbacks by adopting the absolute minimization approach.

Figure 1 presents an usual bathymetry data point cloud, where the measurements are distributed combining cross–sections and longitudinal sections and a depth value is assigned to each point. On a first computational model, the data were presented on a regular triangular grid. This choice simplifies the implementation of the algorithm but it involves a slight loss of information since in some cases it is necessary to introduce perturbations on the data location in order to adjust them to the grid. Figure 2 shows a view of the regular discrete model on a river section (left) and a detail of the lower right part of the same section (right). The bathymetry and riverside data have been assigned to some grid points that are considered as fixed valued points, and the rest of the grid points are considered as variable points with zero initial value. The reconstruction obtained by the absolute minimization approach is shown in figure 3.

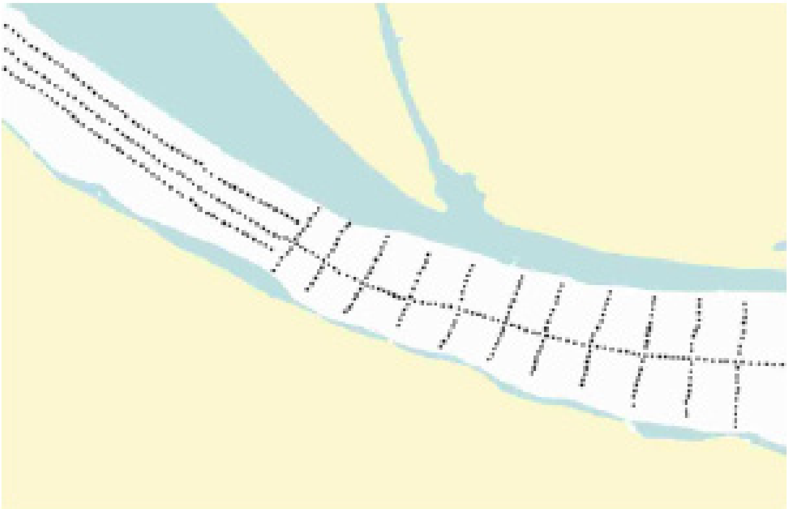

Figure 1: Point cloud from bathymetry measurements

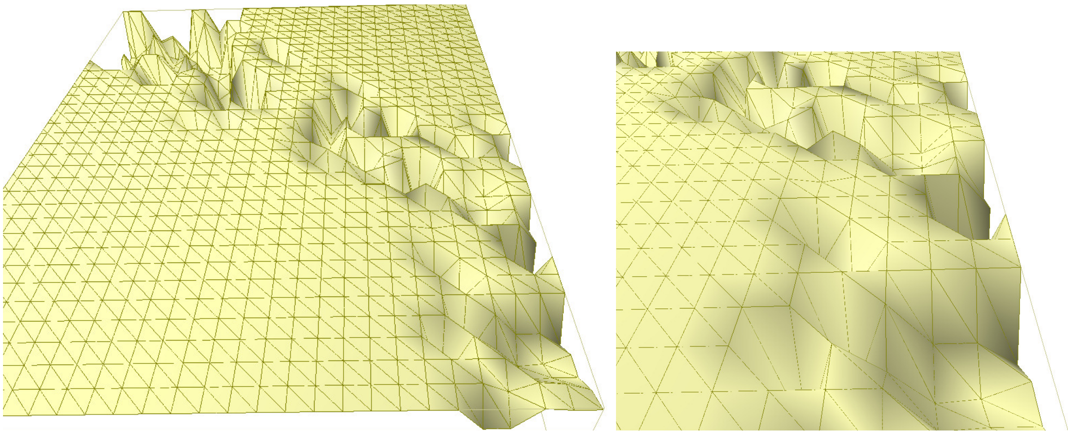

Figure 2: Regular grid data discretization (river section and detail)

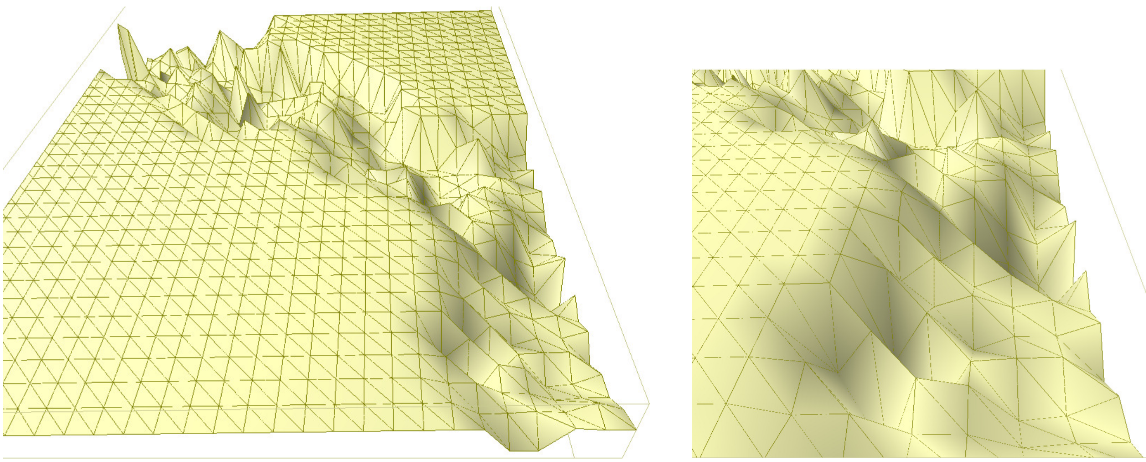

Figure 3: Regular grid solution (river section and detail)

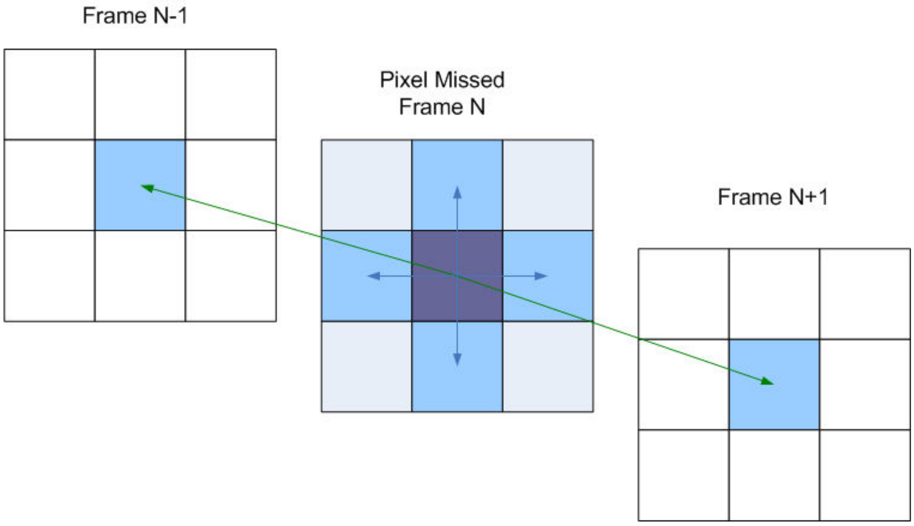

Figure 4: Pixel neighborhood

The obtained chart turns to be a reasonable reconstruction which preserves pics and ridges, so it is acceptable for practical issues. Nevertheless, as the source data are heterogeneously distributed, it should be better to use a non regular triangulations in order to improve the resolution on critical areas. This would be an interesting issue to develop on future studies.

4.2 Video Reconstruction Problem

On this section we deal with the problem of reconstructing missing intermediate images in a video sequence. This problem usually arise on surveillance captures, due to low band–width data transmissions. The remaining information turns to be insufficient for producing an acceptable video sequence, which should display about 20–25 images per second. The available information is given as a set of frames separated by a temporal parameter. Each frame is a grid of pixels with grayscale values. The popular procedures to solve this problem usually deals with simple temporal interpolations (linear or quadratic). The drawback of this approach is that it does not preserve spatial characteristics. In this work, we propose to adopt the absolute minimization approach in order to determine the grayscale value of each pixel in the intermediate frame. On a first computational model, the neighborhood of each pixel is given by the nearest pixels in the same frame and the correspondent location pixels of the previous and next frame in the sequence. This relation is shown on figure 4.

Given two data frames, a first problem consists on obtaining an acceptable reconstruction of an intermediate frame, assuming that its temporal distance to both data frames is the same. Hence, we have a three–dimensional model which intends to preserve some morphological characteristics of the data. The following figures show the implementation of this approach on an intermediate image reconstruction from two surveillance captures. Figure 5 presents the data images, and the obtained intermediate image is shown in figure 6, while figure 7 shows the resulting sequence. As we can see, this reconstruction presents some “phantom images”. This is not necessarily a bad behavior of the method, since in the fast sequencing of the video frames this feature is perceived as a smooth movement effect. Nevertheless, a possible approach in order to enhance the reconstruction is to find appropriate weights to the temporal and spatial parameters and change the set of neighbors of each pixel.

Figure 5: Data frames

Figure 6: Reconstructed intermediate frame

Figure 7: Sequence

5 CONCLUSIONS

On this work we have presented two–dimensional and three–dimensional reconstruction models based on the absolute minimization theory for the obtention of hydrographic charts and the reconstruction of video sequences, respectively. The presented numerical experiences have shown that this approach turns to be an appropriate and effective strategy to face this kind of problems. Our analysis also indicate that irregular grids models should give more accurate reconstructions than regular grids models,so an important task is to determine how to construct the grids. On future works, we expect to obtain better results by studying this topic deeply. Also, we intend to develop a three–dimensional model in order to improve the resolution of 3D medical images visualizations.

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