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COMPARATIVE ANALYSIS OF THE ADAPTIVE GENERALIZED FINITE ELEMENT METHOD AND THE FINITE ELEMENT METHOD FOR FREE VIBRATION OF BARS

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Abstract. In this paper, the results of the adaptive Generalized Finite Element Method to free longitudinal vibration analysis of straight bars are compared with the results of the *h*-versions and the hierarchical *p*-version of the Finite Element Method. The Generalized Finite Element Method is developed by enriching the standard Finite Element Method space, whose basis performs a partition of unity, with knowledge about the differential equation being solved. The enrichment functions used are dependent on the geometric and mechanical properties of the element. The proposed approach converges very fast and is able to approximate the frequency related to any vibration mode. The main aspects of the adaptive Generalized Finite Element Method are presented and discussed. The efficiency and convergence of the proposed method in vibration analysis of uniform and non-uniform straight bars are checked. The frequencies obtained by the adaptive Generalized Finite Element Method are also compared with those obtained by the analytical solution.

1 INTRODUCTION

The vibration analysis is an important step in design process of structures subject to dynamic loads. The main dynamic properties of these structures are obtained by a free vibration analysis.

The Finite Element Method (FEM) is commonly used in vibration analysis and its approximated solution can be improved using two refinement techniques: *h* and *p*-versions. The *h*-version consists of the refinement of element mesh, on the other side; the *p*-version may be understood as the increase in the number of form functions in the element domain without any change in the mesh (Ribeiro, 2001; Campion and Jarvis, 1996). The *h*-version of FEM gives good results for the lowest frequencies but demands great computational cost to work up the accuracy for the higher frequencies. The accuracy of the FEM can be improved applying the conventional *p* refinement that consists of increasing the polynomial degree in the solution. However Leung and Chan (1998) note that the use of high-order polynomials as shape functions can result in ill conditioning of the solution.

Some enriched methods based on the FEM have been developed in last 20 years. Engels (1992) and, Ganesan and Engels (1992) presented the Assumed Mode Method (AMM) which is obtained adding to the FEM form functions set some interface restrained assumed modes. The Composite Element Method (CEM) (Zeng, 1998a,b,c) is obtained by enrichment of the conventional FEM local solution space with non-polynomial functions obtained from analytical solutions of simple vibration problems. This approach results in a hierarchical refinement called *c*-version. A modified CEM applied to analysis of beams was proposed by Lu and Law (2007). The use of products between polynomials and Fourier series instead of polynomials alone in the element shape functions is recommended by Leung and Chan (1998). They developed the Fourier *p*-element applied to the vibration analysis of bars, beams and plates.

The Generalized Finite Element Method (GFEM) was independently proposed by Babuska and others (Melenk and Babuska, 1996; Babuska, Banerjee and Osborn, 2004; Duarte, Babuska and Oden, 2000) and by Duarte and Oden (Duarte and Oden, 1996; Oden, Duarte and Zienkiewicz, 1998) under the names: special finite element method, generalized finite element method, finite element partition unity method, *hp* clouds and cloud-based *hp* finite element method. In the GFEM, the local solution spaces are formed for functions, not necessarily polynomial, that permits inclusion of a priori knowledge about fundamental solution of the governing differential equation. The local spaces are grouped in the approximated solution space by the Partition of Unity Method that ensures good global approximation.

The application of the Partition of Unity Method in dynamic analysis of structures is not new, although there are few studies in this area. De Bel, Villon and Bouillard (2005) presented a new technique based on the partition of unity to forced vibration analysis of thin plates in medium frequency range. The Partition of Unity Method with interface elements was applied by Hazard and Bouillard (2007) to numerical vibration analysis of sandwich plates equipped with passive damping layers.

One of the main challenges in developing the GFEM to a specific problem is the imposition of essential boundary conditions, since the degrees of freedom used in GFEM generally do not correspond to the nodal degrees of freedom. In most cases the imposition of boundary conditions is achieved by the degeneration of the approximation space or applying penalty methods or Lagrange multipliers.

Arndt, Machado and Scremin (2010) proposed an adaptive method based on the Partition of Unity Method to free vibration analysis of bars and trusses, which was called adaptive Generalized Finite Element Method. In this adaptive method, trigonometric enrichment functions depending on geometric and mechanical properties of the elements are added to the linear FEM shape functions by the partition of unity approach. This technique allows an accurate adaptive process that converges very fast and is able to refine the frequency related to a specific vibration mode. In addition the enrichment functions are easily obtained and the introduction of boundary conditions follows the standard finite element procedure.

The use of trigonometric functions in the construction of form functions is not new. This technique was used in several works as Weaver and Low (1985), Lages and Silva (1992 and 1993), Engels (1992), Zeng (1998a) and Leung and Chan (1998). The innovation in the work of Arndt, Machado and Scremin (2010) is the application of the Partition of Unity Method with a specific set of trigonometric approximation functions producing an adaptive method in which the introduction of boundary conditions follows the standard finite element procedure.

In Arndt, Machado and Scremin (2010) the adaptive GFEM was compared to linear *h*-version of FEM and *c*-version of CEM. In this work the efficiency and convergence of the adaptive GFEM in vibration analysis of uniform and non-uniform straight bars are checked and compared to linear and cubic *h* and hierarchical *p* versions of FEM.

2 VARIATIONAL FORM OF THE AXIAL FREE VIBRATION OF BARS

The bar consists of a straight rod with axial strain (Figure 1). The basic hypotheses are (Craig, 1981): (a) The cross sections which are straight and normal to the axis of the bar before deformation remain straight and normal after deformation; and (b) The material is elastic, linear and homogeneous.

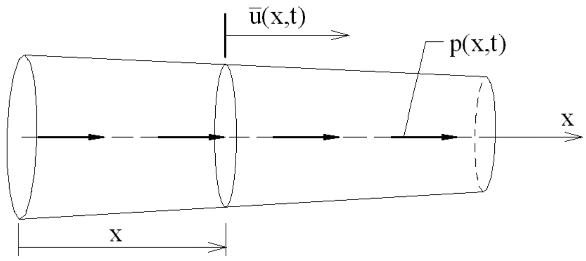

Figure 1: Straight Bar

The vibration of the bar is a time dependent problem. The momentum equation that governs this problem is the partial differential equation

$$\rho A(x)\frac{\partial^2 \bar{u}}{\partial t^2}-\frac{\partial}{\partial x}\left(EA(x)\frac{\partial \bar{u}}{\partial x}\right)=p(x,t)\,, \tag{1}$$

where $A(x)$ is the cross section area, E is the Young modulus, ρ is the specific mass, p is the externally applied axial force per unit length and t is the time. The problem of free vibration consists in finding the axial displacement $\bar{u}=\bar{u}(x,t)$ which satisfies Eq. (1) when $p(x,t)=0$. The solution $\bar{u}=\bar{u}(x,t)$ must satisfy the boundary and initial conditions defined in the problem.

Assuming periodic solutions $\bar{u}(x,t)=e^{i\omega t}u(x)$, where ω is the natural frequency, the free vibration of a bar becomes an eigenvalue problem with variational statement: find a pair (λ,u), with $u\in H^1(0,L)$ and $\lambda\in\mathbf{R}$, so that

$$B(u,v)=\lambda F(u,v)\,, \tag{2}$$

for all admissible test functions $v\in H^1(0,L)$, where $\lambda=\omega^2$ and, $B:H^1\times H^1\mapsto\mathbf{R}$ and $F:H^1\times H^1\mapsto\mathbf{R}$ are bilinear forms.

The bilinear forms for classical boundary conditions are

$$B(u,v)=\int_0^L EA\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}dx, \tag{3}$$

$$F(u,v)=\int_0^L \rho Auvdx. \tag{4}$$

Similarly the bilinear forms for non-classical boundary conditions are

$$B(u,v)=\int_0^L EA\frac{du}{dx}\frac{dv}{dx}dx+k_E u(0)v(0)+k_D u(L)v(L)\,, \tag{5}$$

$$F(u,v)=\int_0^L \rho Auvdx+m_E u(0)v(0)+m_D u(L)v(L)\,, \tag{6}$$

where k_E and k_D are the spring stiffness at left and right bar ends, respectively, and m_E and m_D are the masses at left and right bar ends respectively.

In numerical methods, finite dimensional subspaces of approximation $H^h\subset H^1(0,L)$ are chosen and the variational statement becomes: find $\lambda_h\in\mathbf{R}$ and $u_h\in H^h(0,L)$ so that

$$B(u_h,v)=\lambda_h F(u_h,v)\,,\quad \forall v\in H^h. \tag{7}$$

The approximated solution $u_h(x)$ can be written, for a discrete system with N degrees of freedom, in the following form:

$$u_h(x)=\sum_{j=1}^{N}u_j\phi_j(x)\,, \tag{8}$$

where ϕ_j are the basis functions of the subspace of approximation H^h and u_j are the corresponding degrees of freedom.

3 ADAPTIVE GENERALIZED FINITE ELEMENT METHOD

The Generalized Finite Element Method (GFEM) is a Galerkin method whose main goal is the construction of a finite dimensional subspace of approximating functions using local knowledge about the solution that ensures accurate local and global results. The GFEM was initially named Partition of Unity Finite Element Method (PUFEM) by Melenk and Babuska (1996), and the local enrichment in the approximation subspace is incorporated by the partition of unity approach.

Let $u \in \mathrm{H}^1(\Omega)$ be the function of interest and $\{\Omega_i\}$ be an open cover of domain Ω satisfying an overlap condition:

$$\exists M_S \in \mathbf{N} \text{ so that } \forall x \in \Omega \quad card\{i | x \in \Omega_i\} \le M_S \tag{9}$$

Let $\{\eta_i\}$ be a Lipschitz partition of unity subordinate to the cover $\{\Omega_i\}$ satisfying the conditions:

$$\mathrm{supp}(\eta_i) = \{x \in \Omega \ | \eta_i(x) \neq 0\} \subset [\Omega_i], \quad \forall i \tag{10}$$

$$\sum_i \eta_i \equiv 1 \text{ on } \Omega \tag{11}$$

$$\|\eta_i\|_{L^\infty(R^n)} \le C_\infty \tag{12}$$

$$\|\nabla \eta_i\|_{L^\infty(R^n)} \le C_G / \mathrm{diam}\,\Omega_i \tag{13}$$

where $\mathrm{supp}(\eta_i)$ denotes the support of definition of the function η_i, $[\Omega_i]$ is the closure of the patch Ω_i and, C_∞ and C_G are constants.

If on each sub domain $\Omega_i \cap \Omega$ there is a function space $S_i \subset \mathrm{H}^1(\Omega_i \cap \Omega)$ so that u can be locally well approximated then the global space S taken to approximate u on Ω is obtained by:

$$S := \sum_i \eta_i \ S_i = \left\{ \sum_i \eta_i \ s_i^j \ | s_i^j \in S_i \right\} \subset H^1(\Omega), \tag{14}$$

in other words, the approximated solution at point *x* of the domain is:

$$u_h(x) = \sum_i \sum_{s_i^j \in S_i} \eta_i s_i^j(x) \ a_{ij} \tag{15}$$

where a_{ij} are the degrees of freedom.

In adaptive GFEM (Arndt, Machado and Scremin, 2010) the conventional linear FEM form functions are used as partition of unity functions, so they can be written by

$$\eta_i = \begin{cases} 1 + \dfrac{x - x_i}{x_i - x_{i-1}} & if \ x \in (x_{i-1}, x_i) \\ 1 - \dfrac{x - x_i}{x_{i+1} - x_i} & if \ x \in (x_i, x_{i+1}) \end{cases} \tag{16}$$

in the patch $\Omega_i = (x_{i-1}, x_{i+1})$.

The local approximation space in the patch $\Omega_i = (x_{i-1}, x_{i+1})$ proposed by Arndt, Machado and Scremin (2010) takes the form:

$$S_i = span\{ 1 \quad \gamma_{1j} \quad \gamma_{2j} \quad \varphi_{1j} \quad \varphi_{2j} \quad \ldots \}, \ j = 1, 2, \ldots, n_l \tag{17}$$

$$\gamma_{1j} = \begin{cases} 0 \ \ if \quad x \in (x_{i-1}, x_i) \\ \sin[\beta_{dj}(x - x_i)] \ \ if \quad x \in (x_i, x_{i+1}) \end{cases} \tag{18}$$

$$\gamma_{2j} = \begin{cases} \sin[\beta_{ej}(x - x_i)] \ \ if \quad x \in (x_{i-1}, x_i) \\ 0 \ \ if \quad x \in (x_i, x_{i+1}) \end{cases} \tag{19}$$

$$\varphi_{1j} = \begin{cases} 0 \ \ if \quad x \in (x_{i-1}, x_i) \\ \cos[\beta_{dj}(x - x_i)] - 1 \ \ if \quad x \in (x_i, x_{i+1}) \end{cases} \tag{20}$$

$$\varphi_{2j} = \begin{cases} \cos[\beta_{ej}(x - x_i)] - 1 \ \ if \quad x \in (x_{i-1}, x_i) \\ 0 \ \ if \quad x \in (x_i, x_{i+1}) \end{cases} \tag{21}$$

$$\beta_{dj} = \sqrt{\frac{\rho_d}{E_d}} \mu_j \tag{22}$$

$$\beta_{ej} = \sqrt{\frac{\rho_e}{E_e}} \mu_j \tag{23}$$

where E_d and ρ_d are the bar Young modulus and specific mass on sub domain (x_i, x_{i+1}), E_e and ρ_e are the bar Young modulus and specific mass on sub domain (x_{i-1}, x_i), and μ_j is a frequency related to the enrichment level j.

The adaptive GFEM is an iterative approach whose main goal is to increase the accuracy of the frequency (eigenvalue) related to a chosen vibration mode with order denoted by "target order". The flowchart with blocks A to H presented in Figure 2 represents the adaptive process. In this flowchart, ω_{target} corresponds to the frequency related to the target mode. The first step of the adaptive GFEM process (blocks A to C) consists in obtaining an approximation of the target frequency by the standard FEM (GFEM with n_l = 0) with a coarse mesh. The finite element mesh used in the analysis has to be as coarse as is necessary to capture a first approximation of the target frequency. The subsequent steps (blocks D to G) consist in applying the GFEM with just one enrichment level (n_l = 1) to the same finite element mesh assuming the frequency μ_j (j = 1, blocks D and E) of the enrichment functions (Eq. (18)-(21)) as the target frequency obtained in the last step. Thus, no mesh refinement is necessary along the iterative process.

Both the standard FEM and the adaptive GFEM allow as many frequencies as the total number of degrees of freedom to be obtained. However, in this approach just the precision of the target frequency is effectively improved by the iterative process. The other frequencies present errors similar to those obtained by the standard FEM with the same mesh. In order to improve the precision of another frequency, it is necessary to perform a new analysis by the adaptive GFEM, taking this new one as the target frequency. Few steps are necessary for the method to converge, for each target frequency, and the number of degrees of freedom is smaller than those of the standard FEM to achieve a similar precision, resulting in an appreciative global performance.

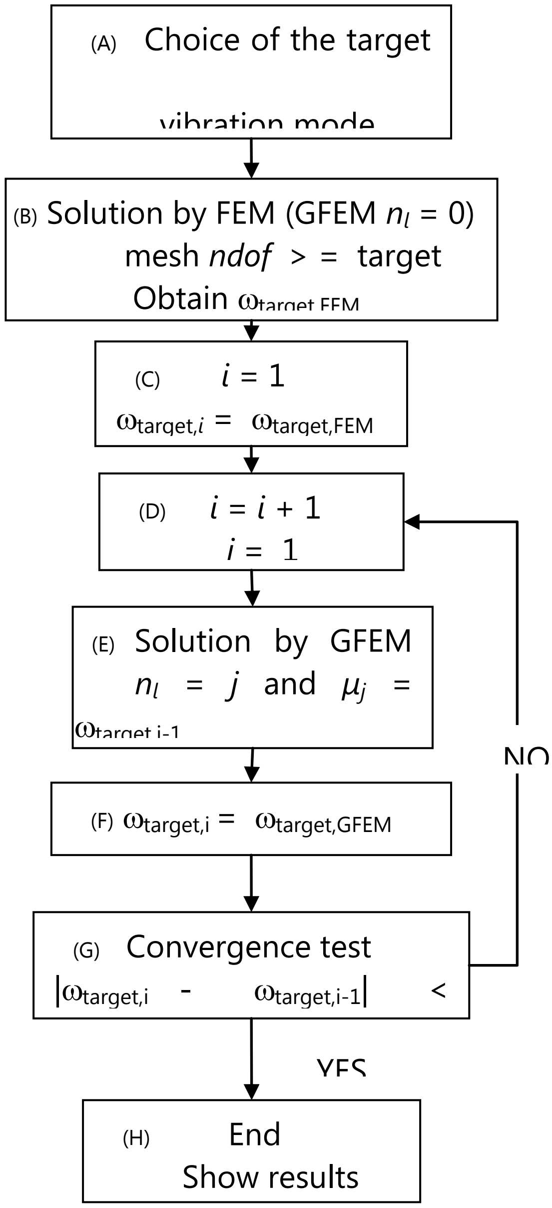

Figure 2: Flowchart of the adaptive Generalized Finite Element Method

4 APPLICATIONS

Numerical solutions for uniform and non-uniform straight bars are given below to demonstrate the application of the adaptive GFEM. To check the efficiency of this adaptive method the results were compared to those obtained by *h* and *p*-versions of FEM.

The number of degrees of freedom (ndof) considered in each analysis is the total number of effective degrees of freedom after introduction of boundary conditions. As an intrinsic imposition of the adaptive method, each target frequency is obtained by different iterative analyses. The mesh used in each analysis is the coarser one, that is, just as coarse as is necessary to capture a first approximation of the target frequency.

4.1 Uniform fixed-fixed bar

The free axial vibration of a fixed-fixed bar (Figure 3) with length L, elasticity modulus E, mass density ρ and uniform cross section area A, has exact natural frequencies (ω_r) given by:

$$\omega_r = \frac{r\pi}{L}\sqrt{\frac{E}{\rho}} \quad , \; r = 1, 2, \ldots \tag{24}$$

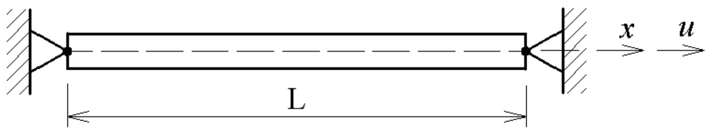

Figure 3: Uniform fixed-fixed bar

In order to compare the exact solution with the approximated ones, in this example a non-dimensional eigenvalue χ_r given by:

$$\chi_r = \frac{\rho L^2 \omega_r^{\,2}}{E} \tag{25}$$

will be used.

Four different adaptive GFEM analyses are performed in order to obtain the first four frequencies. The behavior of relative error for the target eigenvalues in each analysis is presented in Figure 4. In order to capture a first approximation of the target vibration frequency, for the first frequency, the finite element mesh must have at least two elements (one effective degree of freedom), for the second frequency, it must have at least three elements (two effective degrees of freedom), and so on.

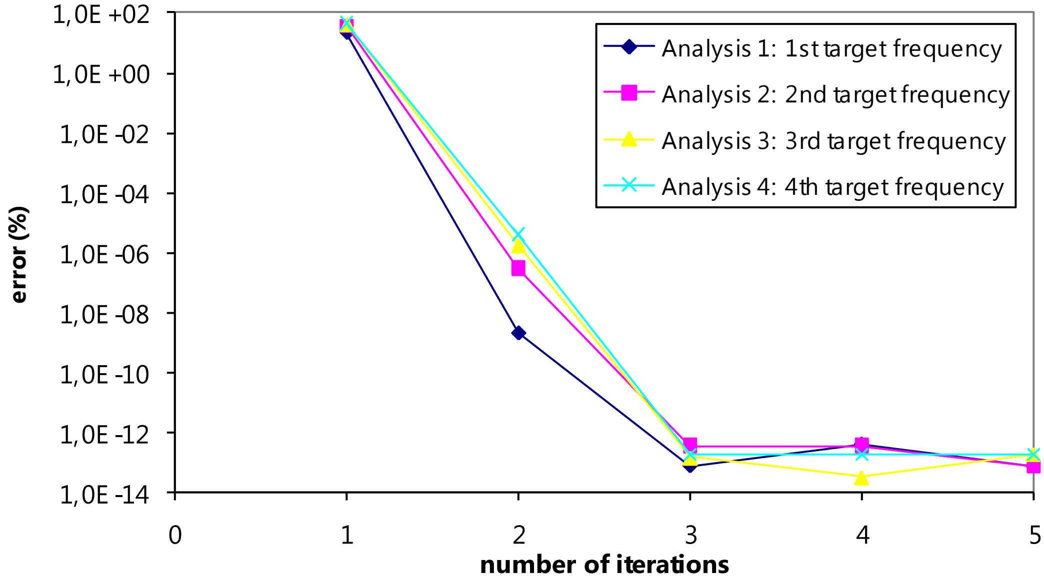

Figure 4: Relative error for the target eigenvalues – Uniform fixed-fixed bar

Table 1 presents the relative errors obtained by the numerical methods. The linear FEM solution is obtained with 100 elements, that is, 99 effective degrees of freedom. The cubic FEM solution is obtained with 33 cubic elements, that is, 98 effective degrees of freedom. The *p* FEM solution is obtained with just one hierarchical 33 node element (polynomial form functions until order 32), that is, 31 degrees of freedom. The analyses by the adaptive GFEM have no more than 24 degrees of freedom in each iteration. For example, the fourth frequency is obtained taking 4 degrees of freedom in the first iteration and 24 degrees of freedom in the two subsequent ones.

r	*h* linear FEM (100e) ndof[(a)] = 99	*h* cubic FEM (33e) ndof = 98	*p* FEM (1e 33n) ndof = 31	Adaptive GFEM (after 3 iterations)	
	error (%)	error (%)	error (%)	error (%)	ndof in iterations[(b)]
1	8,225 e-3	5,432 e-10	1,800 e-13	7,199 e-14	1x 1 dof + 2x 9 dof
2	3,290 e-2	4,715 e-8	1,655 e-8	3,420 e-13	1x 2 dof + 2x 14 dof
3	7,404 e-2	5,368 e-7	3,504 e-8	1,440 e-13	1x 3 dof + 2x 19 dof
4	1,317 e-1	3,010 e-6	1,689 e-8	1,800 e-13	1x 4 dof + 2x 24 dof

Notes: (a) ndof = effective number of degrees of freedom after introduction of boundary conditions;
(b) 1x *n* dof + 2x *m* dof indicates first iteration (FEM) with *n* degrees of freedom and the other two iterations (GFEM) with *m* degrees of freedom in each analysis.

Table 1: Results to free vibration of uniform fixed-fixed bar.

For the uniform fixed-fixed bar, one notes that the adaptive GFEM reaches greater precision than the *h* and *p* versions of FEM. The adaptive process converges rapidly requiring three iterations in order to achieve each target frequency with precision of the 10^{-13} order.

4.2 Uniform fixed-free bar carrying a concentrated mass

In this topic the free axial vibration of a uniform fixed-free bar with a concentrated mass attached to the free end (Figure 5) proposed by Tongue (2002) is analyzed. The properties of this bar are: length L = 1 m, axial stiffness EA = 10 N, linear mass ρA = 1 kg/m and concentrated mass m = 10 kg.

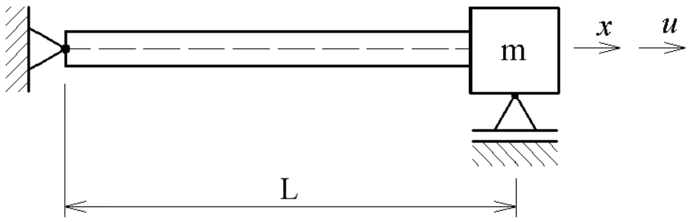

Figure 5: Uniform fixed-free bar carrying a concentrated mass

The natural frequencies of this bar are obtained by the solution of

$$\kappa_r^{\,2} m \sin(\kappa_r L) - \kappa_r \rho A \cos(\kappa_r L) = 0 \tag{26}$$

$$\omega_r = \sqrt{\frac{E}{\rho}}\kappa_r \tag{27}$$

Four different adaptive GFEM analyses are performed in order to obtain the first four frequencies. In order to capture a first approximation of the target vibration frequency, for the first frequency, the finite element mesh must have at least one element (one effective degree of freedom), for the second frequency, it must have at least two elements (two effective degrees of freedom), and so on.

Table 2 presents the relative errors obtained by the numerical methods. The linear FEM solution is obtained with 100 elements, that is, 100 effective degrees of freedom. The cubic FEM solution is obtained with 33 cubic elements, that is, 99 effective degrees of freedom. The p FEM solution is obtained with just one hierarchical 33 node element (polynomial form functions until order 32), that is, 32 degrees of freedom. The analyses by the adaptive GFEM have no more than 20 degrees of freedom in each iteration. For example, the fourth frequency is obtained taking 4 degrees of freedom in the first iteration and 20 degrees of freedom in the two subsequent ones.

r	*h* linear FEM (100e) ndof[(a)] = 100	*h* cubic FEM (33e) ndof = 99	*p* FEM (1e 33n) ndof = 32	Adaptive GFEM (after 3 iterations)	
	error (%)	error (%)	error (%)	error (%)	ndof in iterations[(b)]
1	1,344 e-6	7,865 e-10	6,095 e-13	9,000 e-14	1x 1 dof + 2x 5 dof
2	4,196 e-3	1,575 e-10	9,914 e-13	9,000 e-13	1x 2 dof + 2x 10 dof
3	1,678 e-2	2,381 e-08	6,628 e-11	4,500 e-13	1x 3 dof + 2x 15 dof
4	3,777 e-2	2,696 e-07	2,012 e-12	3,000 e-13	1x 4 dof + 2x 20 dof

Notes: (a) ndof = effective number of degrees of freedom after introduction of boundary conditions;

(b) 1x n dof + 2x m dof indicates first iteration (FEM) with n degrees of freedom and the other two iterations (GFEM) with m degrees of freedom in each analysis.

Table 2: Results to free vibration of uniform fixed-free bar carrying a concentrated mass.

Again the precision reached by the adaptive GFEM exceeds the precision obtained by h and p versions of FEM with greater number of degrees of freedom.

4.3 Fixed-fixed bar with sinusoidal variation of cross section area

In this topic, the longitudinal free vibration of a fixed-fixed non-uniform bar with sinusoidal variation of cross section area, length L, elasticity modulus E and mass density ρ is analyzed. The boundary conditions are $\bar{u}(0,t)=0$ and $\bar{u}(L,t)=0$, and the cross section area varies as

$$A(x) = A_0 \sin^2\left(\frac{x}{L}+1\right), \tag{28}$$

where A_0 is a reference cross section area

Kumar and Sujith (1997) have presented exact analytical solutions for longitudinal free vibration of bars with sinusoidal and polynomial area variations. The equation of motion of axial vibration is reduced to analytically solvable differential equations using appropriate transformations.

This problem is analyzed by the h and p versions of FEM and the adaptive GFEM. Six adaptive analyses are performed in order to obtain each of the first six frequencies.

Table 3 shows the first six non-dimensional eigenvalues ($\kappa_r = \omega_r L\sqrt{\rho/E}$) and their relative errors obtained by these methods. The linear FEM solution is obtained with 100 elements, that is, 99 effective degrees of freedom after introduction of boundary conditions. The cubic FEM solution is obtained with 12 elements, that is, 35 effective degrees of freedom. And the hierarchical p-version of FEM is obtained with one 33 node element, that is, 31 effective degrees of freedom. The analyses by the adaptive GFEM have maximum number of degrees of freedom in each iteration ranging from 9 to 34.

r	Exact solution[(a)]	h linear FEM (100e) ndof = 99	h cubic FEM (12e) ndof = 35	hierarchical p FEM (1e 33n) ndof[(b)] = 31	Adaptive GFEM (after 3 iterations)		
	χ_r	error (%)	error (%)	error (%)	χ_r	error (%)	ndof in iterations[(c)]
1	2,978189	4,737 e-3	2,577 e-5	2,998 e-5	2,978188	2,997 e-5	1x 1 dof + 2x 9 dof
2	6,203097	1,699 e-2	1,901 e-4	6,774 e-6	6,203097	6,871 e-6	1x 2 dof + 2x 14 dof
3	9,371576	3,753 e-2	3,065 e-4	1,643 e-6	9,371576	1,731 e-6	1x 3 dof + 2x 19 dof
4	12,526519	6,632 e-2	7,312 e-4	2,498 e-6	12,526519	2,441 e-6	1x 4 dof + 2x 24 dof

5	15,676100	1,033 e-1	2,332 e-3	2,407 e-7	15,676100	2,044 e-7	1x 5 dof + 2x 29 dof
6	18,823011	1,486 e-1	6,787 e-3	2,163 e-6	18,823011	2,187 e-6	1x 6 dof + 2x 34 dof

Notes: (a) Results from Kumar and Sujith (1997);
(b) ndof = effective number of degrees of freedom after introduction of boundary conditions;
(c) 1x *n* dof + 2x *m* dof indicates first iteration (FEM) with *n* degrees of freedom and the other two iterations (GFEM) with *m* degrees of freedom.

Table 3: Results to free vibration of fixed-fixed bar with sinusoidal variation of cross section area.

One notes that the adaptive GFEM reaches more precise values than the *h*-versions of FEM with even less degrees of freedom. The precision reached for all calculated frequencies by the adaptive process is similar to the one reached by *p*-version of FEM with 31 degrees of freedom. The errors are greater than those from the uniform bars because the exact vibration modes of non-uniform bars cannot be exactly represented by the trigonometric functions used as enrichment functions; however, the precision is acceptable for engineering applications. Each analysis by the adaptive GFEM is able to refine the target frequency until the exhaustion of the approximation capacity of the enriched subspace. Thus the precision can be improved by using a more refined mesh in the adaptive process.

5 CONCLUSIONS

In the adaptive GFEM, trigonometric enrichment functions depending on geometric and mechanical properties of the elements were added to the linear FEM shape functions by the partition of unity approach. This technique allows an accurate adaptive process that converges very fast and is able to refine the frequency related to a specific vibration mode. In addition the enrichment functions are easily obtained and the introduction of boundary conditions follows the standard finite element procedure.

The results have shown that the adaptive GFEM achieves narrower precision than linear and cubic *h*-versions of FEM in free longitudinal vibration analysis of uniform and non-uniform straight bars analyzed for the same number of degrees of freedom. It has been observed that even for free vibration problems of non-uniform bars; the results from the adaptive method are accurate with relatively few degrees of freedom.

In these examples, the adaptive GFEM reached at least similar precision that the *p*-version of FEM with same number of degrees of freedom. For uniform bars the adaptive GFEM results were better than those obtained by *p*-version of FEM.

Future research will extend this adaptive method to other structural elements like beams, plates and shells.

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