

GEOMETRICAL NON-LINEAR DYNAMIC ANALYSIS OF PLATES AND SHELLS BY A FEM BASED ON POSITIONS

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Abstract: In this work it is presented a Finite Element (FEM) formulation based on positions to analyze plates and shells exhibiting large deformations under dynamic loads. The objective is to apply a geometrically exact total Lagrangian description associated with Reissner kinematics capable of accurately solving non linear dynamics using a simple time integration procedure, i.e., the Newmark β . In order to make it possible co-rotational formulations must be avoided and the positional formulation, based on unconstrained vectors, is applied. This formulation does not apply the Euler-Rodrigues formula for finite rotations, resulting into a constant mass matrix. A simple proof of the momentum conserving property for rigid bodies is provided and high order curved elements are used to avoid locking. For flexible structures the conserving property is checked by selected examples.

1 INTRODUCTION

An accurate analysis of structures that exhibit large deflections is of great importance for structural design. The increasing search for economy and optimal material application leads to the conception of very flexible structures. As a consequence, the equilibrium analysis in the non-deformed position is no more acceptable for most applications. This affirmation is confirmed by the large amount of research regarding this subject. One can see pioneering studies related to nonlinear analysis of structures as the works of Bisshopp and Drucker (1945); Mattiasson (1981); Goto et al. (1985); Jenkins et al. (1966); Kerr, (1964); Mondkar and Powell (1977); Belytschko et al. (1977); among others.

Moreover, some structures are naturally geometrically nonlinear as, balloons, airbags, cables, membranes etc. The design of this kind of structures requires more sophisticated theories than the linear ones. One can see, for instance, the works of Stein and Hedgepath (1961); Baginski and Brakke (2006); Pipkin (1994); Bonet et al. (2000) among others.

This study is concerned with the development of a new Finite Element methodology to solve geometrical non-linear dynamics of shells. In order to achieve a robust formulation, the resulting element should be free of shear and volumetric locking. This problem is solved here by the natural presence of the transverse shear strain in the proposed kinematics. The novelty of the proposed formulation is the use of positions and generalized unconstrained vector mapping, resulting in a naturally objective continuum representation of the shell, free of large rotation descriptions and locking. There exists other kind of rotation free elements as proposed in the works of Oñate and Zerate (2003) and Flores and Oñate (2007), developed specifically for thin shells and based on curvature considerations.

There are different approaches regarding time integration for transient non-linear FEM dynamics, one can mention as an example, three different approaches. The first is the explicit time integration for fast solutions, analysed in details by Argyris et al. (2003) and works cited therein, where small time steps are adopted and an indirect control of errors is made. The second is the so called variational energy conserving algorithms, necessary for co-rotational like formulations, see for instance Simo et al. (1992). And finally, the implicit time integration for total Lagrangian formulations as described by Lopez (2007).

The formulation proposed here is total Lagrangian and, due to its unconstrained vector mapping, it presents constant mass matrix. It is possible to apply the Newmark β integrator as a momentum conserving algorithm. A simple proof of the momentum conserving property of the Newmark β method is given in this paper. The proof is restricted to total Lagrangian formulation (not extended to co-rotational formulations) and trivially fulfills the energy conserving property for rigid bodies. All required features of the formulation as: locking free, frame invariance and momentum conserving (linear and angular) are checked in the numerical example section.

2 STRAIN MEASURE AND SPECIFIC STRAIN ENERGY POTENTIAL

This section summarizes simple concepts used to derive the proposed formulation. The Green strain tensor is derived directly from the gradient of the change of configuration function, represented by letter \mathbf{A} , given as follows:

$$A_{ij} = \frac{\partial f_i}{\partial x_j} \quad (1)$$

where f_i is the change of configuration function, as depicted in Fig. 1, and x_j represents variation regarding initial position.

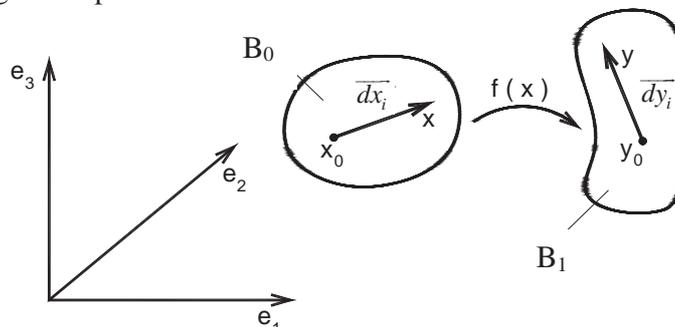


Fig. 1 Change of configuration

In Fig. 1 dx_i and dy_i represent an infinitesimal fiber in the initial and current continuum configurations, respectively. Following Ogden (1984), the Green strain tensor can be written as:

$$E_{ij} = \frac{1}{2} [A_{ki}A_{kj} - \delta_{ij}] = \frac{1}{2} [C_{ij} - \delta_{ij}] \quad (2)$$

In which index notation is adopted. The variables C_{ij} and δ_{ij} are the right Cauchy-Green stretch tensor and the Kroenecker delta, respectively. The following quadratic strain energy per unit of initial volume is adopted,

$$u_e = \frac{1}{2} E_{ij} C_{ijkl} E_{kl} \quad (3)$$

resulting into a linear elastic constitutive law relating second Piola-Kirchhoff stress and Green strain, usually called Saint-Venant–Kirchhoff elastic law, i.e.:

$$S_{ij} = \frac{\partial u_e}{\partial E_{ij}} = C_{ijkl} E_{kl} \quad (4)$$

The elastic tensor is given by

$$C_{ijkl} = \frac{2G\nu}{1-2\nu} \delta_{ij}\delta_{kl} + G(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \quad (5)$$

Where G is the shear modulus, given by,

$$G = \frac{E}{2(1+\nu)} \quad (6)$$

with E being the well known Young modulus and ν the Poisson’s ratio.

The true stress (Cauchy stress) is achieved directly from the Second Piolla-Kirchhoff stress following simple expressions given by Ogden (1984) or Ciarlet (1993), for instance. For the sake of completeness it is necessary to recall that the right Cauchy-Green stretch tensor is positive definite, symmetric and has six independent values.

3 EQUIVALENCE OF CLASSICAL AND GENERALIZED UNCONSTRAINED MAPPING OF SOLIDS:

This section presents the equivalence of a classical finite element solid mapping and its counterpart, the generalized vector mapping. Fig. 2 shows a solid (following plane stress or plane strain condition) element of quadrangular shape classically mapped from a usual non-

dimensional coordinate system. Fig. 2 also shows the same solid mapped from the same non-dimensional coordinate system, this time using generalized unconstrained vector parameters.

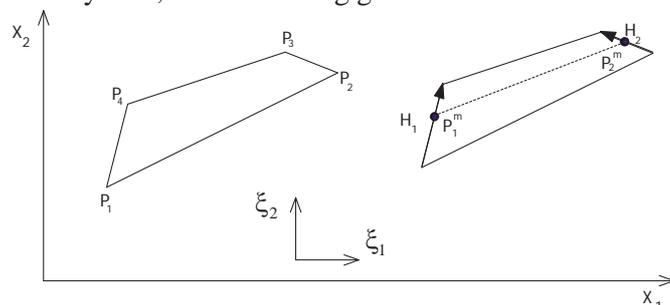


Fig. 2 Classical and generalized unconstrained vector discretizations

Without loss of generality the equivalence is shown for a low order mapping and in the next section extended to consider high order interpolations. In order to be complete, let's take

the following shape functions: $\phi_1 = \frac{1}{4}(1 - \xi_1)(1 - \xi_2)$, $\phi_2 = \frac{1}{4}(1 + \xi_1)(1 - \xi_2)$,

$$\phi_3 = \frac{1}{4}(1 + \xi_1)(1 + \xi_2),$$

$$\phi_4 = \frac{1}{4}(1 - \xi_1)(1 + \xi_2) \text{ and } \varphi_1 = \frac{1}{2}(1 - \xi_1) \text{ and } \varphi_2 = \frac{1}{2}(1 + \xi_1).$$

where ξ_i are non-dimensional coordinates. The classical continuum mapping is written as:

$$x_i = \phi_\ell(\xi_1, \xi_2) X_{\ell i} \quad \text{for } i=1,2 \quad \text{and } \ell=1,2,3,4 \quad (7)$$

where x_i are the coordinates of any point of the mapped continuum, ϕ_ℓ are the shape functions and $X_{\ell i}$ are the coordinates of nodes P_ℓ named position parameters.

A totally similar mapping is given by:

$$x_i = \varphi_\ell X_{\ell i}^m + \frac{H_{\ell i} \varphi_\ell}{2} \xi_2 \quad \text{for } i=1,2 \quad \text{and } \ell=1,2 \quad (8)$$

where the mid line nodal coordinates $X_{\ell i}^m$ and nodal generalized vectors $H_{\ell i}$ are given by:

$$X_{1i}^m = \frac{X_{1i} + X_{4i}}{2}; \quad X_{2i}^m = \frac{X_{2i} + X_{3i}}{2}; \quad H_{1i} = X_{4i} - X_{1i} \text{ and } H_{2i} = X_{3i} - X_{2i} \quad (9)$$

Expression (8) shows that the equivalent vector mapping is done using non unitary vectors $H_{\ell i}$ as parameters. In order to generalize the formulation one writes the nodal vectors H_{1i} and H_{2i} as functions of their lengths H_1 and H_2 resulting:

$$x_i = \varphi_\ell X_{\ell i}^m + \frac{H^{(\ell)}}{2} \xi_2 V_{i\ell} \varphi_\ell \quad \text{for } i=1,2 \quad \text{and } \ell=1,2 \quad (10)$$

where brackets mean that summation is not implied. The first term of Eq. (10) describes the reference line approximation. The values $V_{i\ell}$ are the generalized nodal vectors. These vectors are not orthogonal to the reference line and could not be unitary, if desired, see Eq. (8). However, for the initial configuration they are made unitary, but in the current configuration these vectors assume non unitary values. This feature is the original of the generalized

unconstrained vector mapping of solids. Finally, if one adopts constant height (h_0), a usual assumption for bars and shells, Eq. (10) turns into the desired continuum mapping, i.e.,

$$x_i = \phi_\ell X_{\ell i}^m + \frac{h_0}{2} \xi_2 V_{i\ell} \phi_\ell \quad \text{for } i=1,2 \quad \text{and} \quad \ell=1,2 \quad (11)$$

This mapping is totally similar to the classical one, given in Eq. (7) and, consequently, has the same objectiveness, Jelenic and Crisfield (1999).

4 IMPROVED FINITE ELEMENT KINEMATICS:

Improving the solid description of Fig. 2 by a three dimensional representation of a shell. One can approximate the mid-surface positions of a shell, see Fig. 3, by the following mapping.

$$f_i^{m0} = x_i^m = \phi_\ell X_{\ell i}^m \quad (12)$$

$$f_i^{m1} = y_i^m = \phi_\ell Y_{\ell i}^m \quad (13)$$

where x_i^m is the i th coordinate of a generic point in the mid surface of the shell at initial configuration, $X_{\ell i}$ is the i th coordinate of node ℓ , y_i^m is the i th coordinate of a generic point in the mid surface of the shell at current configuration, $Y_{\ell i}$ is the i th coordinate of node ℓ at current configuration.

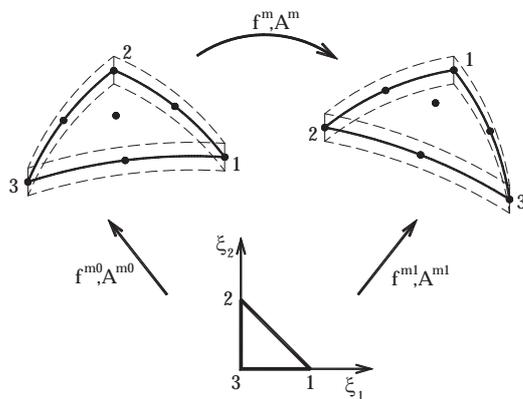


Fig. 3 Mid-surface mapping

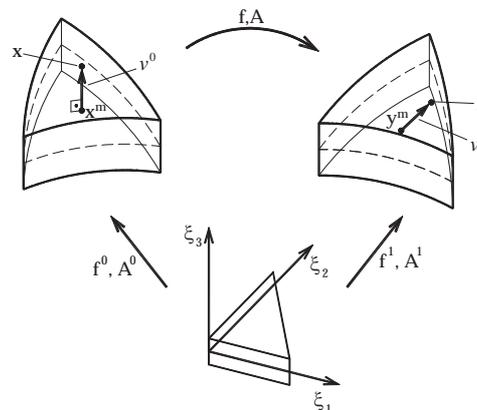


Fig. 4 Position vectors

One can see in Fig. 3 that f^{m0} is the positional mapping from the auxiliary coordinate to the initial mid surface configuration, f^{m1} is the positional mapping from the auxiliary coordinate to the current mid surface configuration, f^m is the positional mapping from the initial configuration to the current one (not to be written) and the values A^{m0} , A^{m1} , A^m are their respective gradients. Expression (12) is totally similar to the reference line description of the first term of Eq. (11). This time, high order approximations for a surface are used.

To complete the shell kinematic description for both initial and current configurations, one realizes that the difference between a point out of the mid-surface and its corresponding belonging to the mid-surface generates position vectors \vec{v}^0 or \vec{v}^1 , see Fig. 4.

A general point of the shell can be defined by adding the position vectors to the corresponding mid-surface point, i.e.,

$$x_i = x_i^m + v_i^0 \quad (14)$$

$$y_i = y_i^m + v_i^1 \quad (15)$$

Following what has been described in section 3, for constant strain regarding ξ_3 , one writes v_i^0 and v_i^1 as functions of non-dimensional coordinates, as

$$v_i^0 = \frac{h_0}{2} \phi_\ell V_{\ell i}^0 \xi_3 \quad (16)$$

$$v_i^1 = \frac{h_0}{2} \phi_\ell V_{\ell i}^1 \xi_3 \quad (17)$$

where h_0 , v_i^0 , v_i^1 are, respectively, the initial constant thickness of the shell, the normal unit vector to the initial mid-surface and the current generalized vector (not necessarily normal to the mid-surface).

To consider linear strain variation regarding ξ_3 an improved kinematics is required. This is done adopting a new scalar variable called here the linear rate of thickness variation and denoted by letter a . It is not necessary to introduce this new variable for initial configuration, so expression (16) does not change, however expression (17) turns into:

$$v_i^1 = \frac{h_0}{2} \phi_\ell V_{\ell i}^1 [\xi_3 + a \xi_3^2] \quad (18)$$

The introduced quadratic term allows linear strain variation along the transverse direction of the shell. The final generalized nodal vectors $V_{\ell i}^1$ are not constrained, so the final thickness of the shell is not the same as the initial one and can be recovered as:

$$v_i^1 = \phi_\ell V_{\ell i}^1 \quad (19)$$

$$h = h_0 \sqrt{v_i^1 v_i^1} \quad (20)$$

The final mapping is generalized and unconstrained, like Eq. (11), improved in the membrane and thickness sense and is written as:

$$f_i^0 = x_i = \phi_\ell X_{\ell i} + \frac{h_0}{2} \xi_3 \phi_\ell V_{\ell i}^0 \quad (21)$$

$$f_i^1 = y_i = \phi_\ell Y_{\ell i} + \frac{h_0}{2} [\xi_3 + \phi_\ell A_\ell \xi_3^2] \phi_k V_{ik}^1 \quad (22)$$

In which the linear rate of thickness variation (scalar) is parameterized by its nodal values A_ℓ , as follows:

$$a = \phi_\ell A_\ell \quad (23)$$

The current position has seven unknown parameters per each node ℓ , i.e., three positions $Y_{\ell i}$, three generalized nodal vectors $V_{\ell i}$ and the nodal rate of thickness variation A_ℓ . Function f_i^0 is used to find A^0 while function f_i^1 is used to find A^1 (trial). The composition of these two values for each integration point gives the numerical value of the gradient of the change of configuration for any initial geometry (curved), i.e.,

$$A = A^1 (A^0)^{-1} \quad (24)$$

It is worth to show the derivatives of f_i^1 regarding the non-dimensional variables ξ_j , constituting the gradient A_{ij}^1 as follows:

$$A_{i1}^1 = \phi_{\ell,1} Y_{\ell i} + \frac{h_0}{2} \left\{ \left[\xi_3 + \phi_\ell A_\ell \xi_3^2 \right] \phi_{k,1} V_{ik} + \left[\phi_{\ell,1} A_\ell \xi_3^2 \right] \phi_k V_{ik} \right\} \quad (25)$$

$$A_{i2}^1 = \phi_{\ell,2} Y_{\ell i} + \frac{h_0}{2} \left\{ \left[\xi_3 + \phi_{\ell} A_{\ell} \xi_3^2 \right] \phi_k V_{ik} + \left[\phi_{\ell,2} A_{\ell} \xi_3^2 \right] \phi_k V_{ik} \right\} \quad (26)$$

$$A_{i3}^1 = \frac{h_0}{2} \left[1 + 2\phi_{\ell} A_{\ell} \xi_3 \right] \phi_k V_{ik} \quad (27)$$

5 DYNAMIC NON LINEAR EQUATION:

From this section on, a unified notation will be adopted for nodal parameters, they will be called simply $Y_{\ell i}$ for each element ℓ and for i varying from 1 to 7. The correspondence is as follows, translations for i varying from 1 to 3, rate of thickness variation for $i=4$ and generalized vectors for i varying from 5 to 7.

The conservation of energy in a mechanical system is guaranteed if the input and output of energy are at balance. If there is some kind of dissipation the total energy of the system changes along time. It can be understood writing the total potential energy of a system as follows:

$$\Pi = \Pi_0 - Q(t, x) \quad (28)$$

where, following Lanczos (1970), $Q(t, x)$ can be stated as the quantity of energy withdrawn from the simple conservative idealized energy Π_0 . As a consequence, Π is the remaining (current) mechanical energy of the system and Eq. (28) can be rewritten as:

$$\Pi_0 = \Pi + Q(t, x) \quad (29)$$

This equation can be understood as recovering the possibility of using stationary properties for the mechanical system analysis, i.e., one can use the minimum potential energy theorem on the ideal energy function Π_0 for equilibrium analysis.

For a structural problem associated with a fixed reference system, Fig. 5, the ideal potential energy function can be written as the composition of the strain energy (U_e), the potential energy of applied conservative forces (P), the kinetic energy (K) and dissipation (Q), as follows.

$$\Pi_0 = U_e - P + K + Q \quad (30)$$

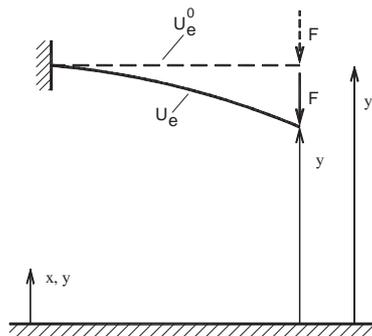


Fig. 5 Deformation of a beam under the action of a force

In this work the non-conservative forces will be considered as part of the dissipative potential. The strain energy function of the body, shell for instance, is considered stored in the initial volume of the body V_0 and is written as an integral of the specific strain energy value u_e , Eq. (3), as

$$U_e = \int_{V_0} u_e dV_0 \quad (31)$$

The strain energy is assumed to be zero at the initial position, called non-deformed. The potential energy of the applied conservative forces is written as:

$$P = F_i Y_i + \int_{s_0} t_i y_i ds_0 \quad (32)$$

where F_i represents forces applied in direction "i" and Y_i is the i th current position of the point where the load is applied, t_i is the distributed force applied in direction "i" and y_i is the current position of mid surface points ($i=1,2,3$ only). Gravitational force has not been mentioned, however as it is a conservative force, it can be introduced directly into the integral of Eq. (32). The letter ds_0 represents the initial differential area of elements. The kinetic energy is written as

$$K = \frac{1}{2} \int_{V_0} \rho_0 \dot{y}_i \dot{y}_i dV_0 \quad (33)$$

where \dot{y}_i are velocities and ρ_0 is the mass density, relative to the initial volume V_0 . The dissipative term, including normal distributed forces, is written in its differential form as

$$\frac{\partial}{\partial y_i} Q(t, y) = \int_{V_0} \frac{\partial}{\partial y_i} \bar{q}(y, t) dV_0 = \int_{V_0} \lambda_m \rho_0 \dot{y}_i dV_0 - \int_{s_0} q_i ds_0 \quad (34)$$

where \bar{q} is the specific dissipative functional, λ_m is a proportional damping constant, \dot{y}_i are velocities at any point and q_i are components of the normal distributed forces given by:

$$q_i = q \phi_i V_{\epsilon_i}^1 \quad (35)$$

In equation (35) q is the normal distributed force over the element and $V_{\epsilon_i}^1$ the generalized vector corresponding to the values Y_{ϵ_i} for i varying from 5 to 7. The integral of these forces respects the direction of the current position but its integral is performed over the initial surface as the load magnitude is written regarding initial position. The current load is easily known by multiplying these magnitudes by ds_0 / ds , for usual applications of thin structures the value of $ds_0 / ds \cong 1$.

Substituting Eqs. (31), (32) and (33) in Eq. (30) results:

$$\Pi_0 = \int_{V_0} u_e dV_0 - F_i Y_i - \int_{s_0} t_i y_i ds_0 + \frac{1}{2} \int_{V_0} \rho_0 \dot{y}_i \dot{y}_i dV_0 + Q \quad (36)$$

This energy function can be written substituting the exact position field by its approximation described in section 4, i.e.:

$$\Pi_0 = \int_{V_0} u_e(\xi, Y_i) dV_0 - F_i Y_i - \int_{s_0} t_i y_i(\xi, Y_i) ds_0 + \frac{1}{2} \int_{V_0} \rho_0 \dot{y}_i(\xi, Y_i) \dot{y}_i(\xi, Y_i) dV_0 + Q(\xi, Y_i) \quad (37)$$

The minimum potential energy theorem is used on Π_0 by differentiating Eq. (37) regarding a generic nodal position Y_{ϵ_j} , resulting:

$$\begin{aligned} \frac{\partial \Pi_0}{\partial Y_{\epsilon_j}} = & \int_{V_0} \frac{\partial u_e(\xi, Y_i)}{\partial Y_{\epsilon_j}} dV_0 + \int_{V_0} \rho_0 \phi_{\epsilon_j}(\xi) \phi_{\epsilon_i}(\xi) dV_0 \ddot{Y}_{\epsilon_i} + \int_{V_0} \lambda_m \rho_0 \phi_{\epsilon_j}(\xi) \phi_{\epsilon_i}(\xi) dV_0 \dot{Y}_{\epsilon_i} \\ & - F_{\epsilon_j} - \int_{s_0} \phi_{\epsilon_j}(\xi) \phi_{\epsilon_i}(\xi) ds_0 t_{\epsilon_i} - \int_{s_0} q_j \phi_{\epsilon_j}(\xi) ds_0 = 0 \end{aligned} \quad (38)$$

It is worth noting that in this equation the dissipative potential is differentiated regarding nodal positions, differently from Eq. (34), so Eq. (35) should be introduced in the last integral of Eq. (38) to perform the numerical integration. Moreover, as the vibration frequency of the thickness variation is too high, when compared to the other movements, the mass matrix is generated neglecting this term. One can rewrite Eq. (38) in a simple vector form as

$$g_{\ell j} = \frac{\partial U_e}{\partial Y_{\ell j}} + F_{\ell j}^{inert.} + F_{\ell j}^{damp.} - F_{\ell j}^c - F_{\ell j}^{nc} = 0 \quad (39)$$

or

$$g_{\ell j} = \frac{\partial U_e}{\partial Y_{\ell j}} + M\ddot{Y}_{\ell j} + C\dot{Y}_{\ell j} - F_{\ell j}^c - F_{\ell j}^{nc} = 0 \quad (40)$$

The involved forces are, inertial force $F_{\ell j}^{inert.}$ or $M\ddot{Y}_{\ell j}$, damping $F_{\ell j}^{damp.}$ or $C\dot{Y}_{\ell j}$ and the external force, divided into conservative $F_{\ell j}^c$ and following forces $F_{\ell j}^{nc}$. It is important to note that the second representation of the inertial and damping forces is possible because the simple vector mapping described in sections 3 and 4 generates constant mass matrix. Splitting the derivative of the specific strain energy, one writes:

$$\frac{1}{2} \frac{\partial}{\partial Y_{\ell j}} (E_{kl} C_{klim} E_{im}) = \frac{1}{2} \frac{\partial}{\partial E_{\alpha\beta}} (E_{kl} C_{klim} E_{im}) \frac{\partial E_{\alpha\beta}}{\partial Y_{\ell j}} = C_{\alpha\beta im} E_{im} \frac{\partial E_{\alpha\beta}}{\partial Y_{\ell j}} = S_{\alpha\beta} \frac{\partial E_{\alpha\beta}}{\partial Y_{\ell j}} \quad (41)$$

Consequently:

$$F_{\ell j}^{int.} = \int_{V_0} C_{\alpha\beta im} E_{im} \frac{\partial E_{\alpha\beta}}{\partial Y_{\ell j}} dV_0 \quad (42)$$

where $F_{\ell j}^{int.}$ is the first gradient vector of the strain energy potential regarding positions, understood as internal force. Eq. (39) represents the dynamic equilibrium of the shell in the D'Alembert sense. If not, vector $g_{\ell j}$ can be understood as the unbalanced force of the mechanical system.

The current position is the unknown of the problem, so it is necessary to solve the non-linear Eq. (39) regarding $Y_{\ell j}$ and time. The first solution step is to integrate Eq. (39) regarding time. For an implicit approach this step is of great importance regarding the momentum conserving properties of the adopted time integrator. In this work a proof (alternative to the one given by Kane et al. (2000)) that the Newmark β method conserves linear momentum and angular momentum for any adopted time step, is given. This proof is restrict to total Lagrangian formulation (not co-rotational) and is trivially extended to energy conserving property for rigid bodies.

6 LINEAR AND ANGULAR PROPERTIES OF NEWMARK β METHOD FOR CONSTANT MASS MATRIX:

In this section no indexes are used, so the variables, as they appear, are vectors. It is important to mention that the proofs given here are restricted to total Lagrangian formulations. It is not extended to non-lagrangian or co-rotational formulations.

The Newmark β approximations, following the notation given by Argyris and Mlejnek (1991), for position description is:

$$Y_{S+1} = Y_S + \Delta t \dot{Y}_S + \Delta t^2 \left[\left(\frac{1}{2} - \beta \right) \ddot{Y}_S + \beta \ddot{Y}_{S+1} \right] \quad (43)$$

$$\dot{Y}_{S+1} = \dot{Y}_S + \Delta t (1 - \gamma) \ddot{Y}_S + \gamma \Delta t \ddot{Y}_{S+1} \quad (44)$$

The linear momentum expression for a total Lagrangian description is given as:

$$Q = \int_{V_0} \rho_0 \dot{Y} dV_0 \quad (45)$$

If the body does not develop any deformation and the external forces are zero, then the linear momentum does not vary along time, that is:

$$\frac{\partial Q}{\partial t} = \int_{V_0} \rho_0 \ddot{Y} dV_0 = 0 \quad (46)$$

From the continuity of the body, Ogden (1984), one concludes that, for any point of the continuum, holds

$$\ddot{Y} = 0 \quad (47)$$

Using Eq. (47), written for two times t_s and t_{s+1} , into Eq. (44), it results:

$$\dot{Y}_{s+1} = \dot{Y}_s \quad (48)$$

or by continuity

$$\int_{V_0} \rho_0 \dot{Y}_{s+1} dV_0 = \int_{V_0} \rho_0 \dot{Y}_s dV_0 \quad (49)$$

i.e, the linear momentum is conserved for any adopted time step and Newmark constants.

To prove the conservation of angular momentum more steps are required. The Lagrangian expression of angular momentum is:

$$J = \int_{V_0} \rho_0 \dot{Y} \times Y dV_0 \quad (50)$$

where \times is the vector product.

The angular momentum is constant when there is a fixed axis around which a rigid body turns with constant angular velocity. As the body is considered rigid, no transfer of energy from kinetics to strain energy occurs; as a consequence the conservation of momentum means the conservation of energy for an isothermal situation. Assuming this hypothesis one writes:

$$\frac{\partial J}{\partial t} = \int_{V_0} \rho_0 (\ddot{Y} \times Y + \dot{Y} \times \dot{Y}) dV_0 = \int_{V_0} \rho_0 (\ddot{Y} \times Y) dV_0 = 0 \quad (51)$$

Form the continuity assumption the equality

$$\ddot{Y} \times Y = 0 \quad (52)$$

must hold for any point of the continuum. It occurs in two situations, the trivial undesired one, i.e., $\ddot{Y} = 0$ and the desired one,

$$\ddot{Y} = -\omega^2 Y \quad (53)$$

where ω is the angular velocity of the body around the rotation axis and Y is, without loss of generality, the position vector of the point related to its projection over the rotation axis. Using Eq. (53) for time t_{s+1} into the Newmark β Eqs. (43) and (44), one writes

$$Y_{s+1} = Y_s + \Delta t \dot{Y}_s - \Delta t^2 (1/2 - \beta) \omega^2 Y_s - \Delta t^2 \beta \omega^2 Y_{s+1} \quad (54)$$

$$\Delta t \dot{Y}_{s+1} = \Delta t \dot{Y}_s - \Delta t^2 (1 - \gamma) \omega^2 Y_s - \Delta t^2 \gamma \omega^2 Y_{s+1} \quad (55)$$

Rearranging terms of Eq. (54) one has,

$$\Delta t \dot{Y}_s = (\Delta t^2 \beta \omega^2 + 1) Y_{s+1} - (1 - \Delta t^2 (1/2 - \beta) \omega^2) Y_s \quad (56)$$

substituting Eq. (56) into Eq. (55) results:

$$\Delta t \dot{Y}_{s+1} = (\Delta t^2 \beta \omega^2 + 1 - \Delta t^2 \gamma \omega^2) Y_{s+1} - (1 - \Delta t^2 (1/2 - \beta) \omega^2 + \Delta t^2 (1 - \gamma) \omega^2) Y_s \quad (57)$$

Post-vector-multiplying Eqs. (56) and (57) by Y_s and Y_{s+1} , respectively, and subtracting results one achieves:

$$\Delta t (\dot{Y}_{s+1} \times Y_{s+1} - \dot{Y}_s \times Y_s) = (1 - \Delta t^2 (1/2 - \beta) \omega^2 + \Delta t^2 (1 - \gamma) \omega^2 - \Delta t^2 \beta \omega^2 - 1) (Y_{s+1} \times Y_s) \quad (58)$$

Finally Eq. (58) simplifies to:

$$(\dot{Y}_{s+1}xY_{s+1} - \dot{Y}_s x Y_s) = \Delta t \omega^2 \left(\frac{1}{2} - \gamma \right) (Y_{s+1}xY_s) \quad (59)$$

By continuity one writes

$$\int_{V_0} \rho_0 \dot{Y}_{s+1}xY_{s+1}dV_0 = \int_{V_0} \rho_0 \dot{Y}_s x Y_s dV_0 + \int_{V_0} \rho_0 \Delta t \omega^2 \left(\frac{1}{2} - \gamma \right) (Y_{s+1}xY_s) dV_0 \quad (60)$$

or in a shorter notation

$$J_{s+1} = J_s + \int_{V_0} \rho_0 \Delta t \omega^2 \left(\frac{1}{2} - \gamma \right) (Y_{s+1}xY_s) dV_0 \quad (61)$$

Therefore, the Newmark β method is angular momentum conserving for constant mass matrix and $\gamma = 1/2$, despite the adopted time step, angular velocity or β parameter.

7 TIME MARCHING PROCESS AND THE NEWTON RAPSON PROCEDURE

From the previous developments Eq. (39) can be written in a simpler form as

$$g = \frac{\partial U_e}{\partial Y} - F^c - F^{nc} + M\ddot{Y} + C\dot{Y} = 0 \quad (62)$$

Expression (62) represents the dynamic equilibrium equation at any time and has to be solved. In order to do so the first step is to write this equilibrium for a specific instant t_{s+1} , as follows:

$$\frac{\partial \Pi}{\partial Y} \Big|_{s+1} = \frac{\partial U_e}{\partial Y} \Big|_{s+1} - F_{s+1} + M\ddot{Y}_{s+1} + C\dot{Y}_{s+1} = 0 \quad (63)$$

Substituting the Newmark approximations (43) and (44) into Eq. (63) results

$$g(Y_{s+1}) = \frac{\partial \Pi}{\partial Y} \Big|_{s+1} = \frac{\partial U_e}{\partial Y} \Big|_{s+1} - F_{s+1} + \frac{M}{\beta \Delta t^2} Y_{s+1} - M Q_s + C R_s + \frac{\gamma C}{\beta \Delta t} Y_{s+1} - \gamma \Delta t C Q_s = 0 \quad (64)$$

where vectors Q_s and R_s represent the dynamic contribution of the past, and are given by

$$Q_s = \frac{Y_s}{\beta \Delta t^2} + \frac{\dot{Y}_s}{\beta \Delta t} + \left(\frac{1}{2\beta} - 1 \right) \ddot{Y}_s \quad (65)$$

$$R_s = \dot{Y}_s + \Delta t (1 - \gamma) \ddot{Y}_s \quad (66)$$

Eq. (64) can be understood simply by $g(Y_{s+1}) = 0$ and is clearly non-linear regarding (Y_{s+1}) . A Taylor expansion to solve this non-linear equation is necessary. The second derivative of the total energy potential is then given by:

$$\frac{\partial^2 \Pi}{\partial Y^2} \Big|_{s+1} = \nabla g(Y_{s+1}) = \frac{\partial^2 U_e}{\partial Y^2} \Big|_{s+1} + \frac{M}{\beta \Delta t^2} + \frac{\gamma C}{\beta \Delta t} \quad (67)$$

One builds the Taylor series of first order as:

$$0 = g(Y) \cong g(Y^0) + \nabla g(Y^0) \Delta Y \quad (68)$$

and derives the Newton-Raphson procedure to solve the non-linear Eq. (64), i.e.,

$$\nabla g(Y^0) \Delta Y = -g(Y^0) \quad (69)$$

where Y^0 is a trial position (usually Y_s) for Y_{s+1} used in Eq. (64) to calculate $g(Y^0)$. Solving ΔY one calculates a new trial for Y_{s+1} as

$$Y_{s+1} = Y^0 + \Delta Y \quad (70)$$

The acceleration must be corrected for each iteration by an expression obtained from Eq. (43), i.e.,

$$\ddot{Y}_{s+1} = \frac{Y_{s+1}}{\beta \Delta t^2} - Q_s \quad (71)$$

This equation is used in Eq. (44) to correct velocity. The stop criterion is given by Eq. (72), when a chosen tolerance (TOL) is satisfied, i.e.,

$$\|g(Y^0)\| \leq TOL \quad \text{or} \quad \|\Delta Y\| \leq TOL \quad (72)$$

It must be noted that, before the first time step, the initial acceleration must be calculated as follows

$$\ddot{Y}_0 = M^{-1} \left[F_0 - \frac{\partial U_e}{\partial Y} \Big|_0 - C \dot{Y}_0 \right] \quad (73)$$

8 THE DERIVATIVES OF THE SPECIFIC STRAIN ENERGY

In order to conclude the description of the formulation the second derivatives of the strain energy regarding nodal positions should be given as it has been done for the first derivative in Eq. (42). From Eqs. (41) and (42) one writes:

$$\frac{\partial^2 U_e}{\partial Y_k \partial Y_j} = \int_{V_0} \frac{\partial}{\partial Y_k} \left(C_{\alpha\beta im} E_{im} \frac{\partial E_{\alpha\beta}}{\partial Y_j} \right) dV_0 = \int_{V_0} \left(\frac{\partial E_{im}}{\partial Y_k} C_{\alpha\beta im} \frac{\partial E_{\alpha\beta}}{\partial Y_j} + E_{im} C_{\alpha\beta im} \frac{\partial^2 E_{\alpha\beta}}{\partial Y_j \partial Y_k} \right) dV_0 \quad (74)$$

Finally, the first and second derivatives of the Green strain regarding current nodal positions should be done. Firstly the necessary derivatives of the Cauchy-Green stretch tensor are presented. Next the derivatives of strains are straightforward achieved. Recalling that the Cauchy-Green stretch tensor is given by:

$$C = A^T A \quad (75)$$

and omitting, for simplicity, extra indices, one applies the proposed mapping, i.e., $A = A^1 (A^0)^{-1}$, and writes:

$$C = [(A^0)^T]^{-1} (A^1)^T (Y_i) A^1 (Y_i) (A^0)^{-1} \quad (76)$$

Remembering that A_0 is constant regarding the current position, the first derivative is performed as:

$$\frac{\partial C}{\partial Y_j} = [(A^0)^T]^{-1} \frac{\partial (A^1)^T (Y_i)}{\partial Y_j} A^1 (Y_i) (A_0)^{-1} + [(A^0)^T]^{-1} (A^1)^T (Y_i) \frac{\partial A^1 (Y_i)}{\partial Y_j} (A^0)^{-1} \quad (77)$$

and, from Eqs. (25) to (27), one has the following (not null) values of the current positional mapping gradient:

$$\frac{\partial A_{11}^1}{\partial Y_{\ell 1}} = \phi_{\ell,1}(\xi_1, \xi_2), \quad \frac{\partial A_{12}^1}{\partial Y_{\ell 1}} = \phi_{\ell,2}(\xi_1, \xi_2) \quad (78)$$

$$\frac{\partial A_{21}^1}{\partial Y_{\ell 2}} = \phi_{\ell,1}(\xi_1, \xi_2), \quad \frac{\partial A_{22}^1}{\partial Y_{\ell 2}} = \phi_{\ell,2}(\xi_1, \xi_2) \quad (79)$$

$$\frac{\partial A_{31}^1}{\partial Y_{\ell 3}} = \phi_{\ell,1}(\xi_1, \xi_2), \quad \frac{\partial A_{32}^1}{\partial Y_{\ell 3}} = \phi_{\ell,2}(\xi_1, \xi_2) \quad (80)$$

$$\frac{\partial A_{11}^1}{\partial Y_{\ell 4}} = \frac{h_0}{2} \xi_3^2 \left\{ \phi_{\ell,1}(\xi_1, \xi_2) V_k \phi_k(\xi_1, \xi_2) + \phi_{\ell,2}(\xi_1, \xi_2) V_k \phi_{k,1}(\xi_1, \xi_2) \right\} \quad (81)$$

$$\frac{\partial A_{12}^1}{\partial Y_{\ell 4}} = \frac{h_0}{2} \xi_3^2 \{ \phi_{\ell,2}(\xi_1, \xi_2) V_k \phi_k(\xi_1, \xi_2) + \phi_\ell(\xi_1, \xi_2) G V_k \phi_{k,2}(\xi_1, \xi_2) \} \quad (82)$$

$$\frac{\partial A_{13}^1}{\partial Y_{\ell 7}} = h_0 \xi_3 \phi_\ell(\xi_1, \xi_2) V_k \phi_k(\xi_1, \xi_2) \quad (83)$$

$$\frac{\partial A_{11}^1}{\partial Y_{\ell 5}} = \frac{h_0}{2} \{ \phi_{\ell,1}(\xi_1, \xi_2) \xi_3 + [\phi_{k,1}(\xi_1, \xi_2) A_k \phi_\ell(\xi_1, \xi_2) + \phi_k(\xi_1, \xi_2) A_k \phi_{\ell,1}(\xi_1, \xi_2)] \xi_3^2 \} \quad (84)$$

$$\frac{\partial A_{12}^1}{\partial Y_{\ell 6}} = \frac{h_0}{2} \{ \phi_{\ell,2}(\xi_1, \xi_2) \xi_3 + [\phi_{k,2}(\xi_1, \xi_2) A_k \phi_\ell(\xi_1, \xi_2) + \phi_k(\xi_1, \xi_2) A_k \phi_{\ell,2}(\xi_1, \xi_2)] \xi_3^2 \} \quad (85)$$

$$\frac{\partial A_{13}^1}{\partial Y_{\ell 7}} = \frac{h_0}{2} \{ \phi_\ell(\xi_1, \xi_2) + 2 \xi_3 \phi_k(\xi_1, \xi_2) A_k \phi_\ell(\xi_1, \xi_2) \} \quad (86)$$

$$\frac{\partial A_{21}^1}{\partial Y_{\ell 5}} = \frac{h_0}{2} \{ \phi_{\ell,1}(\xi_1, \xi_2) \xi_3 + [\phi_{k,1}(\xi_1, \xi_2) A_k \phi_\ell(\xi_1, \xi_2) + \phi_k(\xi_1, \xi_2) A_k \phi_{\ell,1}(\xi_1, \xi_2)] \xi_3^2 \} \quad (87)$$

$$\frac{\partial A_{22}^1}{\partial Y_{\ell 6}} = \frac{h_0}{2} \{ \phi_{\ell,2}(\xi_1, \xi_2) \xi_3 + [\phi_{k,2}(\xi_1, \xi_2) A_k \phi_\ell(\xi_1, \xi_2) + \phi_k(\xi_1, \xi_2) A_k \phi_{\ell,2}(\xi_1, \xi_2)] \xi_3^2 \} \quad (88)$$

$$\frac{\partial A_{23}^1}{\partial Y_{\ell 7}} = \frac{h_0}{2} \{ \phi_\ell(\xi_1, \xi_2) + 2 \xi_3 \phi_k(\xi_1, \xi_2) A_k \phi_\ell(\xi_1, \xi_2) \} \quad (89)$$

$$\frac{\partial A_{31}^1}{\partial Y_{\ell 5}} = \frac{h_0}{2} \{ \phi_{\ell,1}(\xi_1, \xi_2) \xi_3 + [\phi_{k,1}(\xi_1, \xi_2) A_k \phi_\ell(\xi_1, \xi_2) + \phi_k(\xi_1, \xi_2) A_k \phi_{\ell,1}(\xi_1, \xi_2)] \xi_3^2 \} \quad (90)$$

$$\frac{\partial A_{32}^1}{\partial Y_{\ell 6}} = \frac{h_0}{2} \{ \phi_{\ell,2}(\xi_1, \xi_2) \xi_3 + [\phi_{k,2}(\xi_1, \xi_2) A_k \phi_\ell(\xi_1, \xi_2) + \phi_k(\xi_1, \xi_2) A_k \phi_{\ell,2}(\xi_1, \xi_2)] \xi_3^2 \} \quad (91)$$

$$\frac{\partial A_{33}^1}{\partial Y_{\ell 7}} = \frac{h_0}{2} \{ \phi_\ell(\xi_1, \xi_2) + 2 \xi_3 \phi_k(\xi_1, \xi_2) A_k \phi_\ell(\xi_1, \xi_2) \} \quad (92)$$

The first index of $Y_{\ell m}$ is the element node and the second is the degree of freedom. The second derivative of the Cauchy-Green stretch is given by,

$$\begin{aligned} \frac{\partial^2 C}{\partial Y_j \partial Y_k} &= [(A^0)^j J^{-1} \frac{\partial (A^1)^j(Y_i)}{\partial Y_j} \frac{\partial A^1(Y_i)}{\partial Y_k} (A_0)^{-1} + [(A^0)^j J^{-1} \frac{\partial (A^1)^j(Y_i)}{\partial Y_k} \frac{\partial A^1(Y_i)}{\partial Y_j} (A^0)^{-1} + \\ & [(A^0)^j J^{-1} \frac{\partial^2 (A^1)^j(Y_i)}{\partial Y_j \partial Y_k} A^1(Y_i) (A_0)^{-1} + [(A^0)^j J^{-1} (A^1)^j(Y_i) \frac{\partial A^1(Y_i)}{\partial Y_j \partial Y_k} (A^0)^{-1} \end{aligned} \quad (93)$$

where $\partial A^1 / \partial Y_k$ are given by Eqs. (78-92) and the second derivatives, $\partial^2 A^1 / (\partial Y_k \partial Y_j)$ are straightforward. Recalling Eq. (2), one achieves directly

$$\frac{\partial E}{\partial Y_j} = \frac{1}{2} \frac{\partial C}{\partial Y_j}; \quad \frac{\partial^2 E}{\partial Y_j \partial Y_k} = \frac{1}{2} \frac{\partial^2 C}{\partial Y_j \partial Y_k} \quad (94)$$

It is important to mention that the present technique can be applied to any strain measure based on the Cauchy-Green stretch. Eq. (67) indicates that the proposed procedure can be operated by creating the Hessian matrix and internal forces for finite elements and building the global matrix and internal force vector by summation of coincident degrees of freedom, as it is done for usual FEM procedures. One should remember that all nodal parameters follow the global system of reference, avoiding the use of rotation schemes.

9 NUMERICAL EXAMPLES

This section provides eight examples covering selected tests to confirm the generality and accuracy of the proposed formulation for static and dynamic situations. No mention to units is made to keep a coherence with references. The important features are: objectivity, locking free behavior, Linear momentum conserving, angular momentum conserving, total energy conserving and generality in applications. More examples regarding static analysis can be seen in Coda and Paccola (2007).

9.1 Objectivity of the formulation regarding rotations

As mentioned in the introduction, this formulation is tested regarding mapping objectivity. The employed way to test this property follows well known methodologies; see for instance Criesfield and Jelenic (1999) and Ibrahimbegovic and Taylor (2002). A clamped vertical plate (shell in deformed configuration) is subject to a transverse load at its free end as depicted in Fig. 6. Static conditions are considered so inertial forces are neglected, gravitational forces are also not considered. The physical properties of the structure are $E = 100000$ and $\nu = 0$. The thickness of the shell is $h = 0.1$. The adopted discretization can also be seen in Fig. 6.

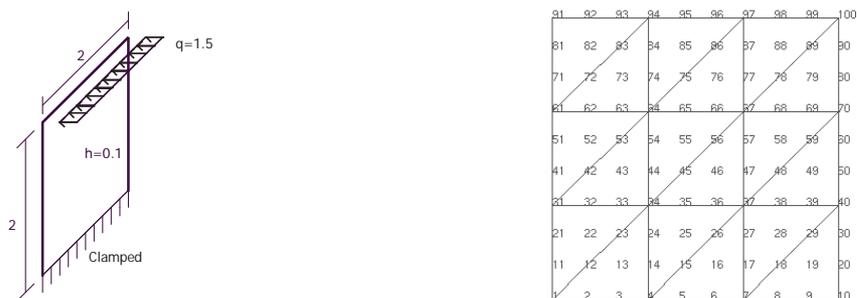


Fig. 6 - Geometrical characteristics of the problem and discretization.

Two situations are created. The first consists into applying a rotation over the shell regarding the clamping axis without applying any load. The objective is to show that no stress will be generated at any stage of rotation. One hundred turns are applied and no stress appears; moreover the positions are exactly the same after each turn. In Fig. 7 one can see an illustration of this situation for the first turn. The adopted rotation step is 0.1π .

In the second situation the process is divided into two phases. First, the load is increased to its final value in ten equal steps. The resulting stress, following the longitudinal direction of the shell, at the superior face of the shell, is depicted in Fig. 8.

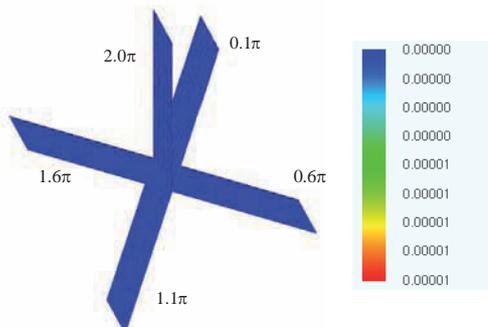


Fig. 7 - Stress values for the first turn -unloaded

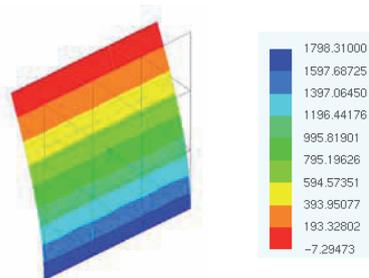


Fig. 8 Stress values for the first deformed configuration - no rotation

Then the load is kept constant and acting in the same sense and direction and the rotation, similar to the one used in the first situation, is applied. The adopted rotation step is 0.01π . At the beginning of the rotation process the action of rotation is against the action of the loading. At a quarter of the first turn the loading is compressing the shell and the stress values, following the longitudinal direction of the shell, are depicted in Fig. 9.

At the half of the first turn the shell is in opposite position to the beginning of the rotation process, and the initially superior face of the shell is now the inferior one. The stress values at this face are negative and their values are depicted in Fig. 10. The difference in stress magnitude from the first deformed configuration and this one is due to the normal traction force that increases the values in Fig. 8 and decreases the absolute values in Fig. 10.

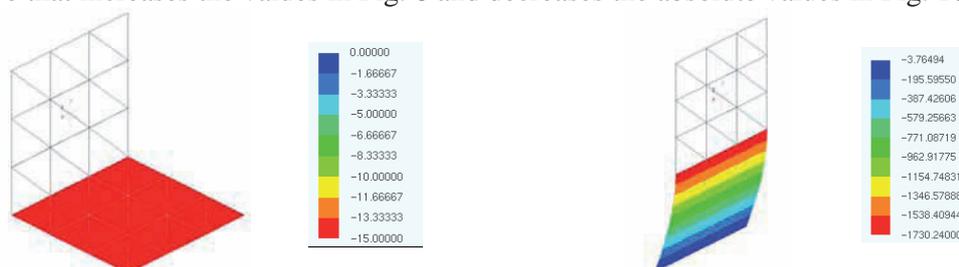


Fig. 9 Stress values for a quarter of the first turn ($\pi/2$) Fig. 10 Stress values for the half turn (π)

At three quarters of the first turn the stress values are the ones depicted in Fig. 11, exactly as expected. Finally, after a complete revolution the stress values depicted in Fig. 12 are exactly the same ones present in Fig. 8. Ninety nine more turns were performed and the results are repeated for each turn, revealing the total objectiveness of the generalized vector mapping.

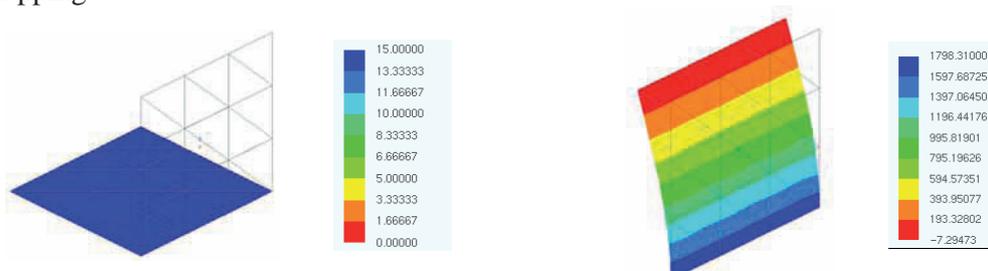


Fig. 11 Stress values for three quarters of a turn ($3\pi/2$) Fig. 12 Stress values for one complete turn (2π)

9.2 Shear and Volumetric locking analysis

This example is extracted from Bucalem and Nóbrega (2000) and is a benchmark to check FEM formulations regarding shear and volumetric locking. It is an important example as the solution for plates is very sensitive to the Poisson Ratio and some shell theories fail to reproduce the analytical solution by the presence of shear locking. It is the analysis of a simple supported square plate subjected to a static transverse concentrated load at its center. The numerical results are compared with the analytical solution obtained using Navier's series for Kirchhoff plate theory. The thin plate geometry is depicted in Fig. 13.

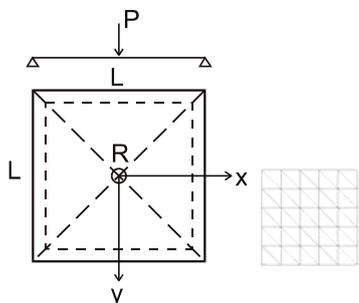


Fig. 13 Analyzed plate and adopted discretization

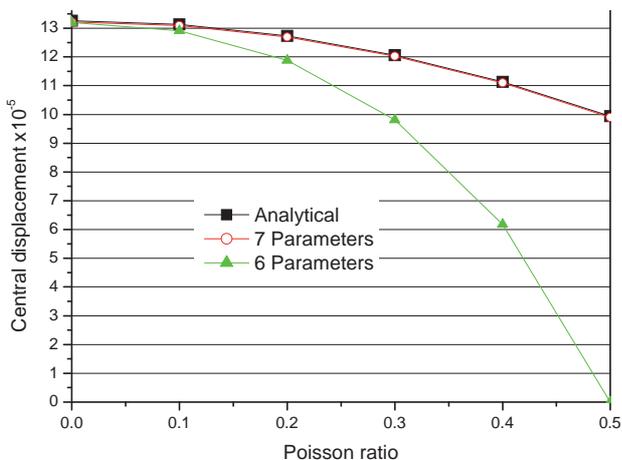


Fig. 14 Displacement at the centre of the plate versus Poisson Ratio.

The adopted physical data are $L = 2$, $E = 2.1 \times 10^9$, $h = 0.002$ and ν varying from 0 to 0.5. The applied load is $P = 0.4 \times 10^{-2}$. In Fig. 14 the results obtained using the presented improved formulation is compared to the analytical solution and the solution for constrained linear rate of thickness variation. This last situation is called simply six parameter shell formulation.

As one can observe the presented formulation is free from shear and volumetric locking and reproduces perfectly the analytical solution. For more examples regarding the shear locking free behavior of the proposed kinematics one is referred to Coda and Paccola (2007), where an extensive static analysis for a six parameter shell element is presented.

9.3 Pinched Cylinder with rigid diaphragms (static)

This benchmark consists in a cylinder with rigid diaphragms pinched by concentrated loads at two opposite points at its top and bottom, see Fig. 15. The adopted discretization ($2 \times 18 \times 6$ mesh) is also depicted in Fig. 15 comprising 1045 nodes. This example is also used to test the formulation regarding shear and Poisson locking. It should be noted that if a formulation suffers from any locking the correct results can not be achieved. The use of symmetry can hide some buckling modes of the problem, however to compare with the reference papers the symmetry is assumed. The number of modes one can get with accuracy (excluding non symmetric ones) is about 30 for a total of 220 node for the total circumference and cubic approximations. The results are compared to the ones achieved by Sansour and Kollman (2000) with a mesh of 1681 nodes. Taking advantage of symmetry only one octant of the cylinder is discretized. The adopted physical parameters are: $R = 100$, $h = 1$, $E = 3 \times 10^4$, $L = 200$. Two values are adopted for the Poisson ratio, $\nu = 0.3$ and $\nu = 0.49$, respectively, in order to check the locking free behavior of the presented formulation.

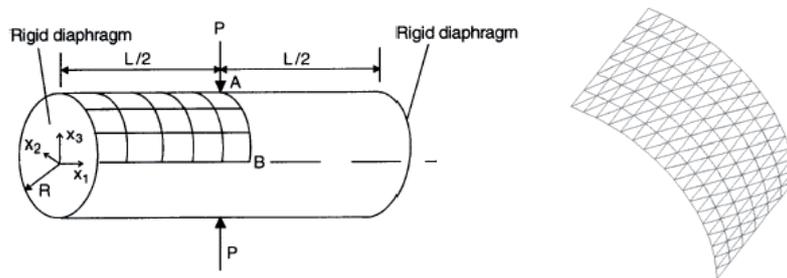


Fig. 15 Pinched Cylinder geometry, loading and the adopted discretization.

The problem is solved with the proposed formulation following two strategies. The first, called 7 parameters, does not constrain the linear rate of thickness. The second, called 6 parameters, totally constrains the linear rate of thickness variation. In Fig. 16 the results for $\nu = 0.3$ are compared with Sansour and Kollmman (2000) that employed a strain enhanced strain quadrilateral element. As one can see the formulation proposed here can capture the flexibility of the pinched shell for $\nu = 0.3$ even if the linear rate of thickness variation is totally constrained. A formulation suffering of shear locking is not able to run this example.

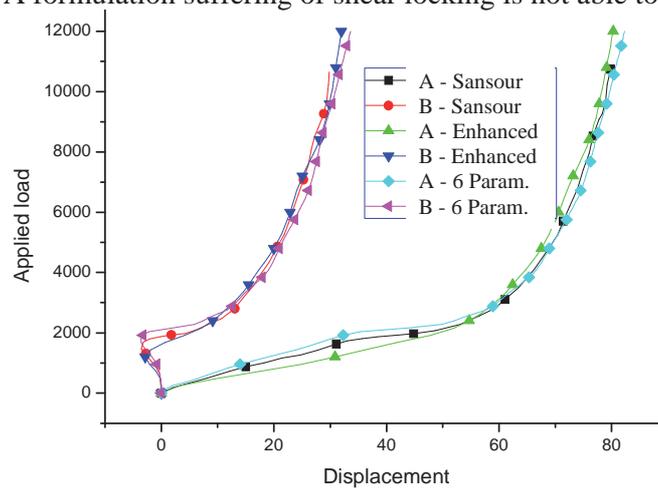


Fig. 16 Displacements for points A and B, $\nu = 0.3$.

In Fig. 17, the behavior of the proposed formulation (with the seventh parameter constrained or not) is depicted for $\nu = 0.49$. The reference value for this figure is the seven parameter result with $\nu = 0.3$. As expected, the proposed formulation does not lock for large Poisson ratio. However, the results for the totally constrained rate of thickness variation lock completely. Shell formulations based on six parameters and full constitutive relations can be free from shear locking, but not from Poisson locking.

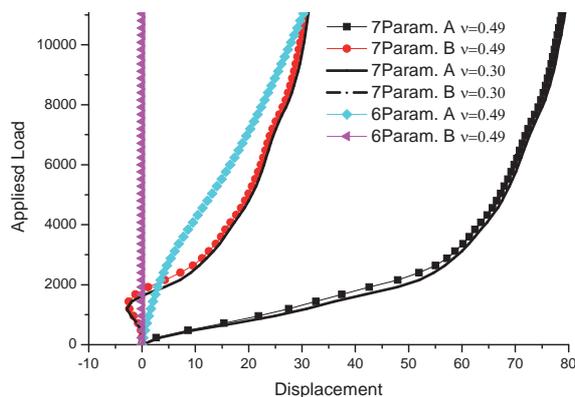


Fig. 17 Displacements for points A and B, $\nu = 0.49$.

From this result it is obvious that the positional improved formulation based on generalized vectors, together with high order curved elements, is able to solve geometrically non-linear shell problems with precision and reduced mesh. Additional and practical information is that the pinched cylinder benchmark problem is not very sensitive to the Poisson ratio intensity.

9.4 Linear momentum conservation

This example is used to confirm the proof given for the linear momentum conserving property of the Newmark β method when used with the generalized vector mapping to develop the positional formulation. A plate with the same dimensions of example 9.1 with no displacements restriction and mass density $\rho_0 = 1$ is subjected to an initial velocity in the vertical direction of value $V_2 = 1$. As the gravitational force is neglected and no displacement restrictions are imposed the plate does not deform and moves with constant velocity. The total applied kinetic energy is, of course, $k = 0.2$. Only two finite elements were used to run this problem. In Fig. 18 the numerical kinetic energy calculated for 1000 time steps using different time steps is depicted. Remembering that the Newmark parameter $\gamma = 0.5$ is mandatory the other one is adopted as $\beta = 1/4$.

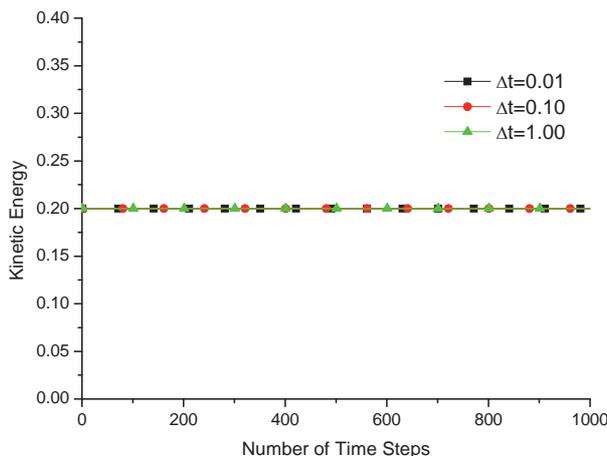


Fig. 18 Energy conserving for linear momentum

As one can see in Fig. 18 the Newmark β method preserves the total energy of the system when a linear momentum is applied. No strains were developed in this example.

9.5 Angular momentum conservation

In this example a vertical circular thick cylinder, see Fig. 19, is subjected to a field of initial velocity generated by an angular velocity of $w = 1$. The dimensions of the cylinder are: $h = 0.4$, $l = 1.0$ and $r = 1.0$. The physical parameters are: $E = 100$, $\nu = 0$ and $\rho_0 = 0.1$. The total energy introduced in the problem by this velocity field is (considering the thickness influence) $k = 1.3069$. Eighteen finite elements with forty eight nodes and $\beta = 0.25$ are used to run this problem, as shown in Fig. 19. One should note that the elements are curved despite the straight lines that appear in the Fig. 19. The achieved total energy is depicted along 1000 time steps in Fig. 20 for two different time steps. It is interesting noting that a small strain energy was computed for this analysis.

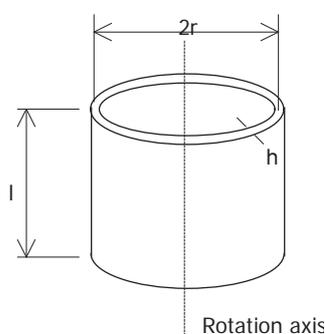


Fig. 19 Geometry and discretization

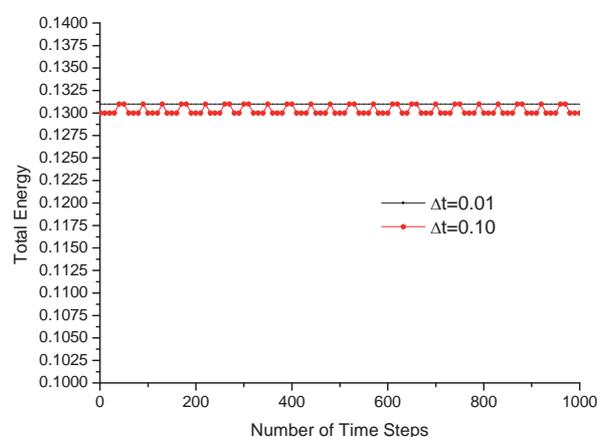


Fig. 20 Energy conserving for angular momentum

The exact solution is coincident with the numerical one for $\Delta t = 0.01$. The numerical result for the large time step is also energy conserving; the difference in results is floating and less than 1%. This difference is due to the better accuracy achieved when using a small time step. In Fig. 20 the number of time steps is the same, however for the larger one 15.9 turns are depicted and only 1.59 turns are depicted for the smaller one. The same total energy is found after 1,000,000 time steps for the smaller time step, i.e., performing 1590 turns. Even better results are achieved for thinner shells. As a consequence, the proof given for the momentum conserving property of the Newmark β method is confirmed also for angular momentum.

9.6 Transverse dynamic load over a clamped beam - Energy conservation check

This example adopts the proposed non-linear shell element to run the non-linear transversal dynamic vibration of a clamped beam subjected to an initial field of velocity see Fig. 21. It intends to show the transfer behavior of kinetic and strain energies for a deformable body. The beam have young Modulus $E = 0.2 \times 10^9$, mass density $\rho_0 = 500$, length $L = 1$, thickness $h = 0.01$, width $b = 0.20$ and Poisson's ratio $\nu = 0$. The applied velocity is proportional to the distance from the clamped end and has a maximum value of $v_0 = 1$, see Fig. 20. The adopted time step is $\Delta t = 0.001$. The exact total energy of the system is $k_0 = 0.16667$

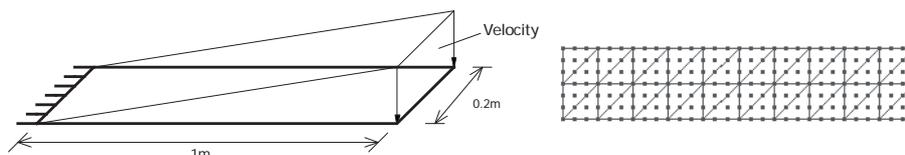


Fig. 21 Geometry, initial velocity and discretization.

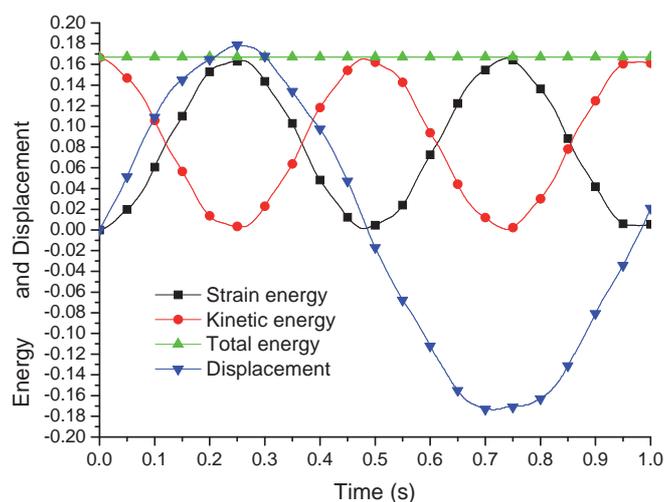


Fig. 22 Energy and free end displacement along time

Forty finite elements and 217 nodes are employed in the discretization. In Fig. 22, the kinetic, strain and total energies are depicted. The transversal displacement of the beam at the free end is also depicted in Fig. 22. As one can see the energy is completely conserved for deformable situations. It is important to mention that in non-linear applications the coupling of vertical and horizontal movements is present and this is the reason the peaks of the analysis does not repeat with the same shape, also there is not necessarily an instant for which the kinetic energy becomes zero, as it is expected in some simple linear analysis.

9.7 Simple airbag simulation

This simple example is inspired in the airbag simulation presented by Cirak and Radovitzky (2005) that includes fluid structure interaction. In the present work only the structure (airbag) is modeled subjected to a deterministic applied load in order to demonstrate the possibilities of the proposed formulation regarding the analysis of very thin membranes and general problems. A formulation suffering of shear locking is not able to run this example. The load is orthogonal to the airbag surface and, as no comparisons can be made, the initial structure discretization is simplified. The simulation corresponds to an initially-flat airbag made of an elastic fabric with a Young's modulus of $E = 6.0 \times 10^9$, Poisson's ratio of $\nu = 0.3$, and mass density of $\rho_0 = 1000$. The initial position of the real airbag can be seen in Fig. 23, extracted from the work of Cirak and Radovitzky (2005). The thickness of the airbag is $h = 7.3 \times 10^{-4}$ and the diameter in its flat initial configuration is $D = 0.74$. Our initial discretization (1/8th of the airbag due to the assumed double symmetry) consists of 800 cubic elements of ten nodes resulting in 3721 nodes, Fig. 24. The adopted boundary condition at the curved board is simple supported. The pressure of the fluid is simulated by the following function.

$$100te^{(0.999t-0.01)r^2} 5066.25 \quad \text{for } 0 < t < 0.01 \quad \text{and} \quad 5066.25 \quad \text{for } t > 0.01$$

where t is time and r is the distance from the centre of the airbag. The adopted time step is $\Delta t = 0.0002$ and the maximum tolerance for convergence is $tol = 0.0005$ in positions. An initial random vertical defect with maximum amplitude of $\delta = 0.001$ is assumed for the analysis.

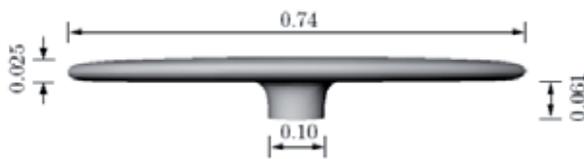


Fig. 23 Airbag geometry, Cirak and Radovitzky (2005)

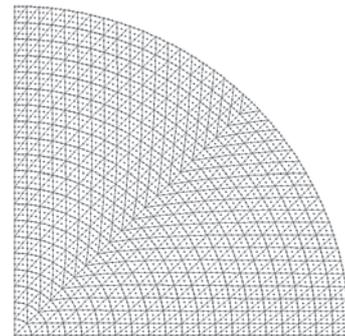


Fig. 24 Adopted discretization, initially flat

Some selected deformed positions are depicted in Figs. 25, 26 and 27. In Fig. 28 the time story of the displacement of the top of the airbag is depicted. As one can see in figure 28 the top of the airbag stabilizes, to the final value, by geometrical accommodation.

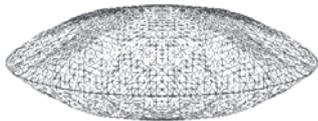


Fig. 25 Airbag at 0.0042s

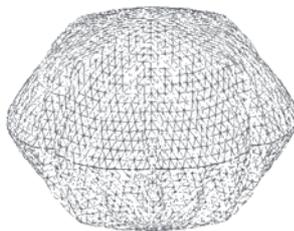


Fig. 26 Airbag at 0.0084s

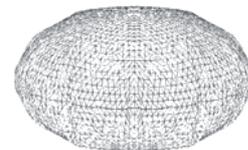


Fig. 27 Airbag at 0.0314s

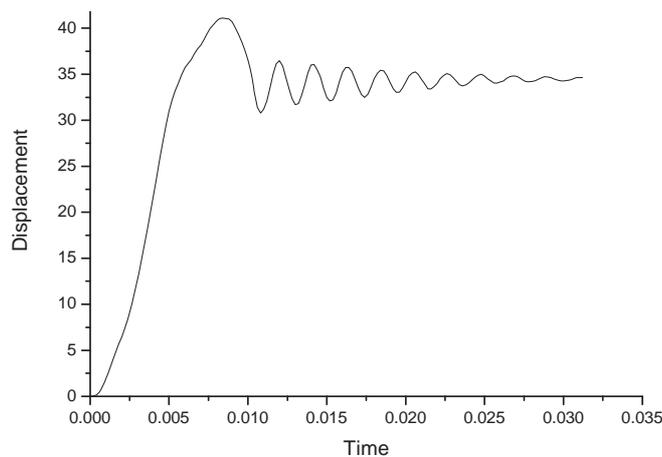


Fig. 28 Top displacement along time

9.8 Cylindrical shell with dynamic snap through

Following Argyris et al. (2003), the second example is a cylindrical shell exhibiting dynamic snap through, a severe nonlinearity. Important theoretical studies, related to severe non-linearities are presented by Breslavsky et al. (2008) and others cited therein. This is a typical benchmark example that has been used extensively as a test for all nonlinear shell dynamics formulations presented so far. Snap through problems in shells produce higher dynamical modes and this is the reason why it is believed that the standard integration schemes such as the Newmark method are not adequate to produce a stable and accurate solution and that only algorithms with numerical dissipation and energy decaying schemes can be applied with an acceptable time step. Despite this widespread belief, it will be shown that using the proposed shell element it is possible to obtain stable and accurate solutions with the Newmark integration for reasonable time step. Results will be compared to the TRIC element and an ABACUS solution, both presented by Argyris et al. (2003).

The geometry of the cylindrical shell is shown in Fig. 29. The two straight edges of the shell are simply supported, while the two curved edges are free. A concentrated load is applied at the central node of the shell. The value of this load increases linearly from 0 to 50×10^6 in a time of 0.2, after that it is held constant. To avoid any distortion in results the structure is totally discretized. The mesh used in the analysis is shown in Fig. 30 and consists of 32 curved finite elements with cubic approximation resulting in 169 nodes and 1183 degrees of freedom. Argyris et al. (2003) used (for a symmetric quarter of the structure) 128 TRIC elements with 407 degrees of freedom to run this example, see Fig. 30. The adopted Newmark Parameters are $\beta = 0.25$ and $\gamma = 0.5$. The TRIC element is designed to run this example using large time step within the context of co rotational formulation. by the coupling of rigid body movements, for the overall element, and strain modes. Argyris et al. (2003) found a maximum stable time step for their co rotational formulation of $\Delta t = 1 \times 10^{-3}$. Using the proposed formulation we vary the time step from $\Delta t = 0.0625 \times 10^{-3}$ to $\Delta t = 4 \times 10^{-3}$ and found not instability for the total Lagrangian formulation. The adopted physical properties are $E = 200 \times 10^9$, $\nu = 0.25$, $\rho = 10.000$ and thickness of $h = 0.1$

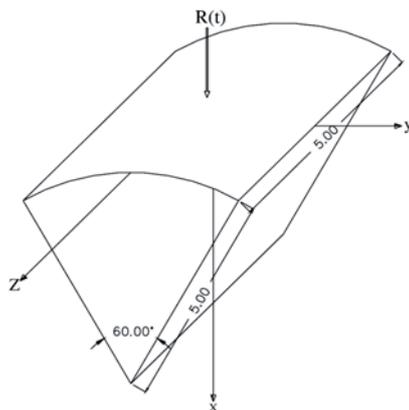


Fig. 29 Geometry

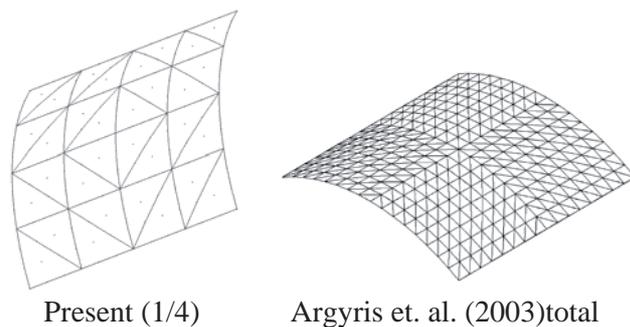


Fig. 30 Discretizations

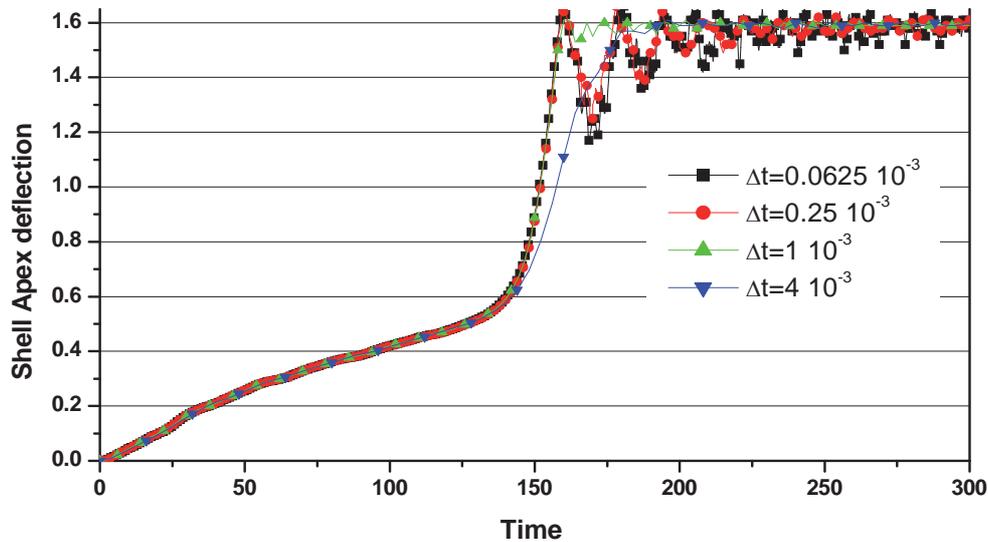


Fig. 31 Present solution for the apex displacement

As one can see, the results converge to the smaller time step, while for large time steps the accuracy is lost, but the stability is kept.

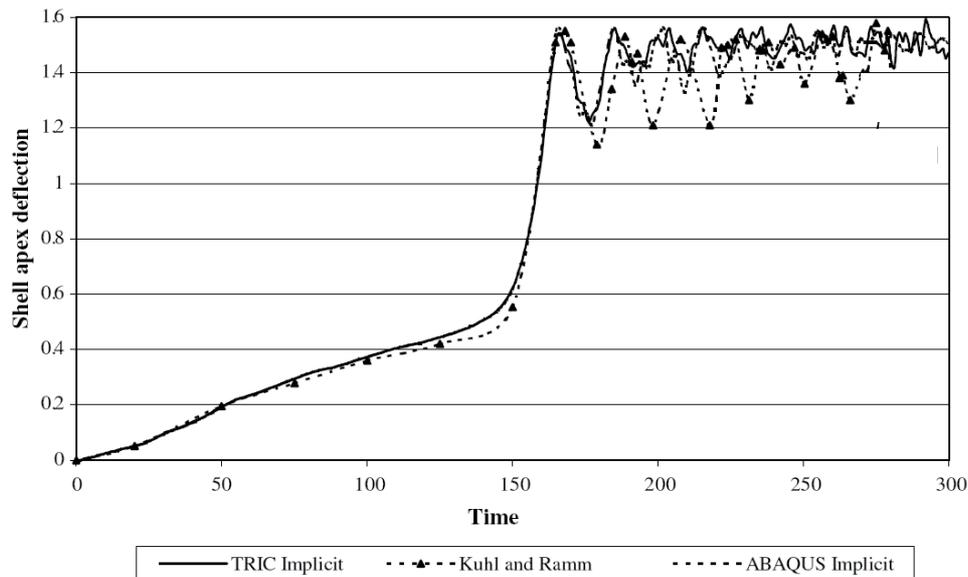


Fig. 32 Results presented by Argyris et. al. (2003)

10 CONCLUSIONS

A new, simple and robust formulation to solve dynamic geometrical non-linear problems with large deflections applied to shells is proposed and implemented. The formulation is based on unconstrained vector mapping of the continuum, called here position description, simplifying the understanding and the implementation of total Lagrangian geometrical non-linear analysis when compared to typical FEM shell formulations. The Newmark β method has been proved to be linear and angular momentum conserving, for the proposed total Lagrangian formulation. The high order curved triangular element with improved transverse position field is free from locking and does not need reduced integration or relaxed

constitutive relation to reproduce, with accuracy and small degrees of freedom, the static benchmarks of plate and shell analysis. Moreover, the formulation proved to be objective regarding rotation for unloaded and loaded structures. The general dynamic analysis of thin shells (airbag) and the snap through benchmark indicate that the formulation is promising and should be extended to include physical non-linearities (hyperelasticity and plasticity), fluid-structure interaction and impact.

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