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A FRAME INVARIANT AND PATH-INDEPENDENT GEOMETRICALLY EXACT THIN WALLED COMPOSITE BEAM ELEMENT

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Abstract. A geometrically exact, frame invariant and path independent beam finite element for composite thin-walled beams is presented. In the proposed formulation the virtual work equations are written as a function of generalized strain components, which are parametrized in terms of the director field and its derivatives. The generalized strains and forces are obtained by introducing a transformation that maps generalized components into physical components. Finite rotations are parametrized with the total rotation vector and a multiplicative update of the directors is performed via the total rotation tensor. A curvilinear transformation is applied to the strain vector and the transformed deformations are used to write the constitutive relations. As a result, the proposed formulation is valid for both isotropic and anisotropic beams. The frame invariance and path independence of the presented formulation is ensured by the use of interpolation to obtain the derivatives of the director field.

1 INTRODUCTION

A deep study of the mechanics of thin-walled composite beams demands a good knowledge of mathematical methods for treating geometrical nonlinearities. The study of the geometrically exact nonlinear behavior of beams requires the development of effective methods and procedures to deal with finite rotations.

Several authors have developed geometrically exact beam finite formulations. As a starting point, Reissner provided a 2D exact beam theory capable of describing arbitrary large displacements and rotations and a 3D theory for second order rotations (Reissner, 1981).

Simo (1985) and Simo and Vu-Quoc (1986, 1988) developed the first 3D geometrically exact formulation for isotropic hyperelastic beams. They used the Reissner relationships between the variation of the rotation tensor and the infinitesimal rotations to derive the strain-configuration relations, maintaining the geometric exactness of the theory. Another important contribution to the subject was done by Cardona and Geradin (1988), who presented a different alternative of parametrization, using the total rotational pseudo-vector to update the 3D rotations on the basis of the initial configuration.

An extension of the formulation of Simo to curved beams was presented by Ibrahimbegovic (1995). He also proposed an incremental rotation vector formulation (Ibrahimbegovic, 1997) to solve the nonlinear dynamics of space beams.

The use of the Green-Lagrange strain measures in a geometrically exact finite element formulation for 3D beams was introduced by Gruttmann (1998, 2000).

During the last years, great efforts were made to shed light to the problem of loss of objectivity introduced by the interpolation of rotations variables, a problem first noted by Crisfield and Jelenic (1999). Jelenic and Crisfield (1999) implemented the ideas proposed in (Crisfield et al., 1999) to complete de development of a strain-invariant and path independent geometrically exact 3D beam element.

Also, Ibrahimbegovic and Taylor (2002) re-examine the geometrically exact models to clarify the frame invariance issues concerning multiplicative and additive updates of rotations. Betsch and Steinmann (2002), Armero and Romero (2001) and Romero and Armero (2002) further contributed to the subject presenting frame-invariant formulations for geometrically exact beams using the director field to parametrize the equations of motion. Additional treatment of frame invariance can be found in references (Ghosh and Roy, 2009, Sansour and Wagner, 2003).

All the aforementioned formulations deal with isotropic beams with solid cross section beams. As a consequence, the extension of these formulations to thin-walled beams is not natural. The advantage of thin-walled beam formulations is that the inclusion of generally anisotropic material is greatly facilitated.

This work presents a new implementation of the thin-walled beam theory presented in (Saravia et al., 2010). In this case, the parametrization of finite rotations is done with the total rotation vector, thus, using the terminology of Cardona (1988), this formulation belongs the class of Total Lagrangian formulations. In this new implementation we also turn to some ideas proposed in previous works (Armero and Romero, 2001, Betsch and Steinmann, 2002, Gruttmann et al., 2000, Romero and Armero, 2002), which are based on the interpolation of the director field and its variation to obtain their derivatives, to avoid the need of curvatures and greatly simplifying the linearization of the virtual strains.

Regarding the frame invariance and path independence of geometrically exact formulations, it has been shown that in the presence of finite three dimensional rotations the concept of objectivity of strain measures does not extend naturally from the theory to the finite element formulation (Crisfield et al., 1999). Hence, despite being some formulations frame indifferent, they suffer from interpolation induced non-objectivity. In the present formulation these difficulties are overcome avoiding the use of interpolated rotations in the stiffness matrix, this is achieved by the use of interpolation of the nodal triads to obtain the derivatives of the director field instead of the classical approach based on curvatures. As a result, the formulation is path-independent and frame-invariant.

2 KINEMATICS

The kinematic description of the beam is extracted from the relations between two states of a beam, an undeformed reference state, denoted by $\boldsymbol{\mathcal{B}_0}$, and a deformed state, denoted by $\boldsymbol{\mathcal{B}}$, as shown in Fig. 1. Being $\boldsymbol{a}_i$ a spatial frame of reference, we define a reference frame $\boldsymbol{E}_i$ and a current frame $\boldsymbol{e}_i$ (both frames being orthonormal).

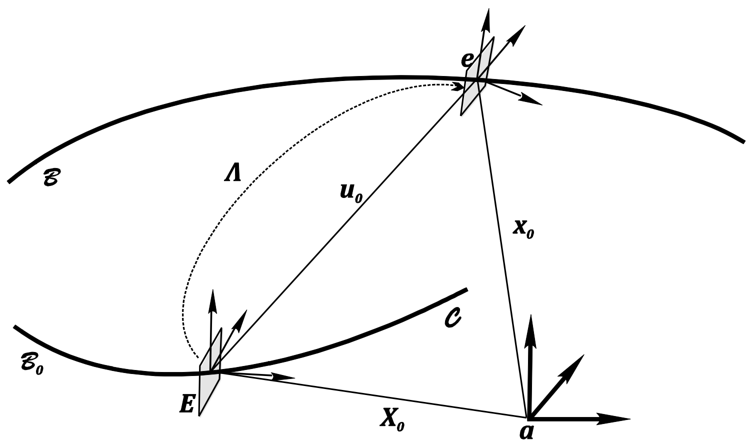

Figure 1. 3D beam kinematics.

The displacement of any point in the deformed beam measured with respect to the undeformed reference state can be expressed in the global coordinate system $\boldsymbol{a}_i$ in terms of a vector $\boldsymbol{u} = (u_1, u_2, u_3)$.

The current frame $\boldsymbol{e_i}$ is a function of a running length coordinate along the reference line of the beam, denoted as *x*, and is fixed to the beam cross-section. For convenience, we choose the reference curve $\mathcal{C}$ to be the locus of cross-sectional inertia centroids. The origin of $\boldsymbol{e}_i$ is located on the reference line of the beam and is called: *pole*. The cross-section of the beam is arbitrary and initially normal to the reference line of the beam.

The relations between the orthonormal frames are given by the linear transformations:

$$\boldsymbol{E}_i = \boldsymbol{\Lambda_0}(x)\boldsymbol{a}_i, \qquad \boldsymbol{e}_i = \boldsymbol{\Lambda}(x)\boldsymbol{E}_i, \tag{1}$$

where $\boldsymbol{\Lambda}_0(x)$ and $\boldsymbol{\Lambda}(x)$ are two-point tensor fields $\in \mathrm{SO}(3)$; the special orthogonal (Lie) group. Thus, it's satisfied that $\boldsymbol{\Lambda_0}^{\boldsymbol{T}}\boldsymbol{\Lambda_0} = \mathbf{1}$, $\boldsymbol{\Lambda}^{\boldsymbol{T}}\boldsymbol{\Lambda} = \mathbf{1}$. We will consider that the beam element is straight, so we set $\boldsymbol{\Lambda_0} = \mathbf{1}$.

Recalling the relations (1), we express the position vectors of a point in the beam in the undeformed and deformed configuration respectively as:

$$\boldsymbol{X}(x, \xi_2, \xi_3) = \boldsymbol{X}_0(x) + \sum_{i=2}^{3} \xi_i \, \boldsymbol{E}_i, \qquad \boldsymbol{x}(x, \xi_2, \xi_3, t) = \boldsymbol{x}_0(x, t) + \sum_{i=2}^{3} \xi_i \, \boldsymbol{e}_i. \tag{2}$$

Where in both equations the first term stands for the position of the pole and the second term stands for the position a point in the cross section relative to the pole. Note that, x is the running length coordinate and ξ_2 and ξ_3 are cross section coordinates.

Also, it is possible to express the displacement field as:

$$\boldsymbol{u}(x,\xi_2,\xi_3,t) = \boldsymbol{x} - \boldsymbol{X} = \boldsymbol{u}_0(x,t) + (\boldsymbol{\Lambda} - \mathbf{I})\sum_{2}^{3} \xi_i \boldsymbol{E}_i, \tag{3}$$

where $\boldsymbol{u}_0$ represents the displacement of the kinematic center of reduction, i.e. the pole. The nonlinear manifold of 3D rotation transformations $\boldsymbol{\Lambda}(\boldsymbol{\theta})$ (a special orthogonal Lie Group SO(3)) is described mathematically via the exponential map as:

$$\boldsymbol{\Lambda}(\boldsymbol{\theta}) = \cos\theta\, \boldsymbol{I} + \frac{\sin\theta}{\theta}\boldsymbol{\Theta} + \frac{1-\cos\theta}{\theta^2}\boldsymbol{\theta}\otimes\boldsymbol{\theta}, \tag{4}$$

where $\boldsymbol{\theta} = [\theta_1 \;\; \theta_2 \; \theta_3]^T$ is the rotation vector, θ its modulus and $\boldsymbol{\Theta}$ is its skew symmetric matrix.

The set of kinematic variables is defined by three displacements and three rotations as:

$$\mathcal{V} \coloneqq \{\boldsymbol{\phi} = [\boldsymbol{u},\boldsymbol{\theta}]^T : [0,\ell] \to R^3\}, \qquad [\boldsymbol{u},\boldsymbol{\theta}]^{\boldsymbol{T}} = [u_1,u_2,u_3,\theta_1,\theta_2,\theta_3]^{\boldsymbol{T}}. \tag{5}$$

3 BEAM MECHANICS

3.1 The Strain Tensor

Following the procedures described in Saravia et.al. (2010), we use five independent curvilinear strain measures to describe the strain state of the thin-walled beam, o more exactly: a laminate (see (Barbero, 2008)). Thus, the strain state of the beam will be described by the curvilinear strain vector:

$$\boldsymbol{\epsilon}_s = [\varepsilon_{xx} \quad \gamma_{xs} \quad \gamma_{xn} \quad \varkappa_{xx} \quad \varkappa_{xs}]^T. \tag{6}$$

We now propose de following generalized strain vector to represent the strain state of the beam:

$$\boldsymbol{\varepsilon} = \begin{bmatrix} \epsilon \\ \kappa_2 \\ \kappa_3 \\ \gamma_2 \\ \gamma_3 \\ \kappa_1 \\ \chi_2 \\ \chi_3 \\ \chi_{23} \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(\boldsymbol{x}_0' \cdot \boldsymbol{x}_0' - \boldsymbol{X}_0' \cdot \boldsymbol{X}_0') \\ \boldsymbol{x}_0' \cdot \boldsymbol{e}_3' - \boldsymbol{X}_0' \cdot \boldsymbol{E}_3' \\ \boldsymbol{x}_0' \cdot \boldsymbol{e}_2' - \boldsymbol{X}_0' \cdot \boldsymbol{E}_2' \\ \boldsymbol{x}_0' \cdot \boldsymbol{e}_2 - \boldsymbol{X}_0' \cdot \boldsymbol{E}_2 \\ \boldsymbol{x}_0' \cdot \boldsymbol{e}_3 - \boldsymbol{X}_0' \cdot \boldsymbol{E}_3 \\ \boldsymbol{e}_2' \cdot \boldsymbol{e}_3 - \boldsymbol{E}_2' \cdot \boldsymbol{E}_3 \\ \boldsymbol{e}_2' \cdot \boldsymbol{e}_2' - \boldsymbol{E}_2' \cdot \boldsymbol{E}_2' \\ \boldsymbol{e}_3' \cdot \boldsymbol{e}_3' - \boldsymbol{E}_3' \cdot \boldsymbol{E}_3' \\ \boldsymbol{e}_2' \cdot \boldsymbol{e}_3' - \boldsymbol{E}_2' \cdot \boldsymbol{E}_3' \end{bmatrix}. \tag{7}$$

And finally we find the matrix $\boldsymbol{\mathcal{T}}$ that satisfies the following relationship between the GL curvilinear strains and the generalized strains:

$$\boldsymbol{\epsilon}_s = \boldsymbol{\mathcal{T}}\, \boldsymbol{\varepsilon}. \tag{8}$$

We obtain:

$$\boldsymbol{\mathcal{T}}(s)=\begin{bmatrix}1 & \bar{\xi}_3 & \bar{\xi}_2 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2}\bar{\xi}_2^{\,2} & \frac{1}{2}\bar{\xi}_3^{\,2} & \bar{\xi}_2\bar{\xi}_3 \\ 0 & 0 & 0 & \bar{\xi}_2' & \bar{\xi}_3' & \bar{\xi}_2\bar{\xi}_3'-\bar{\xi}_3\bar{\xi}_2' & \bar{\xi}_2\bar{\xi}_2' & \bar{\xi}_3\bar{\xi}_3' & 0 & 0 & 0 \\ 0 & 0 & 0 & -\bar{\xi}_3' & \bar{\xi}_2' & \bar{\xi}_2\bar{\xi}_2'+\bar{\xi}_3\bar{\xi}_3' & -\bar{\xi}_2\bar{\xi}_3' & \bar{\xi}_3\bar{\xi}_2' & 0 & 0 & 0 \\ 0 & \bar{\xi}_2' & -\bar{\xi}_3' & 0 & 0 & 0 & 0 & 0 & -\bar{\xi}_2\bar{\xi}_3' & \bar{\xi}_3\bar{\xi}_2' & \left(\bar{\xi}_2\bar{\xi}_2'-\bar{\xi}_3\bar{\xi}_3'\right) \\ 0 & 0 & 0 & 0 & 0 & -\left(\bar{\xi}_2'^{\,2}+\bar{\xi}_3'^{\,2}\right) & -\bar{\xi}_2'\bar{\xi}_3' & \bar{\xi}_2'\bar{\xi}_3' & 0 & 0 & 0\end{bmatrix}, \tag{9}$$

Where we have neglected the terms in n^2. The matrix $\boldsymbol{\mathcal{T}}$ can be understood as a double transformation matrix, transforming the generalized strains into the curvilinear GL strains.

3.2 Constitutive Relations

According to the theory of thin-walled composite laminates (Barbero, 2008), we can express:

$$\begin{bmatrix}N_{xx}\\N_{xs}\\N_{xn}\\M_{xx}\\M_{xs}\end{bmatrix}=\begin{bmatrix}\bar{A}_{11} & \bar{A}_{16} & 0 & \bar{B}_{11} & \bar{B}_{16}\\ \bar{A}_{16} & \bar{A}_{66} & 0 & \bar{B}_{16} & \bar{B}_{66}\\ 0 & 0 & \bar{A}_{55}^{H} & 0 & 0\\ \bar{B}_{11} & \bar{B}_{16} & 0 & \bar{D}_{11} & \bar{D}_{16}\\ \bar{B}_{16} & \bar{B}_{66} & 0 & \bar{D}_{16} & \bar{D}_{66}\end{bmatrix}\begin{bmatrix}\varepsilon_{xx}\\ \gamma_{xs}\\ \gamma_{xn}\\ \varkappa_{xx}\\ \varkappa_{xs}\end{bmatrix}, \tag{10}$$

where $\bar{A}_{\mathrm{ij}}$ are components of the laminate reduced in-plane stiffness matrix, $\bar{B}_{\mathrm{ij}}$ are components of the reduced bending-extension coupling matrix, $\bar{D}_{\mathrm{ij}}$ are components of the reduced bending stiffness matrix and $\bar{A}_{55}^{H}$ is the component of the reduced transverse shear stiffness matrix.

We can express the above relation in matrix form as:

$$\boldsymbol{N}_s=\boldsymbol{C}\,\boldsymbol{\epsilon}_s, \tag{11}$$

where $\boldsymbol{C}$ is the composite shell constitutive matrix and $\boldsymbol{\epsilon}_s$ is the curvilinear shell strain vector defined in (8).

Now, it is possible to express the shell forces as a function of the generalized strains. Replacing (8) into (11) we obtain;

$$\boldsymbol{N}_s=\boldsymbol{C}\,\boldsymbol{\mathcal{T}}\,\boldsymbol{\varepsilon}. \tag{12}$$

Now, we transform the shell forces in (12) back to the "generalized space" by using the double transformation matrix $\boldsymbol{\mathcal{T}}$**.** Thus, we obtain a new entity, a sort of transformed back shell strain:

$$\boldsymbol{N}_s^G=\boldsymbol{\mathcal{T}}^T\boldsymbol{N}_s=\boldsymbol{\mathcal{T}}^T\boldsymbol{C}\,\boldsymbol{\mathcal{T}}\,\boldsymbol{\varepsilon}. \tag{13}$$

Although at first glance this stress could seem contrived, it can be observed that it is a vector of generalized shell stresses defined in the global coordinate system. Since $\boldsymbol{N}_s^G$ is a function of the cross section contour, integration over the contour gives the vector of generalized beam forces, work conjugate with the generalized strains, as:

$$\boldsymbol{S}(x)=\int_{\boldsymbol{S}}\boldsymbol{N}_s^G ds=\left(\int_S\boldsymbol{\mathcal{T}}^T\boldsymbol{C}\,\boldsymbol{\mathcal{T}}\,ds\right)\boldsymbol{\varepsilon}(x), \tag{14}$$

$$\boldsymbol{S}(x)=\boldsymbol{\mathcal{D}}\,\boldsymbol{\varepsilon}(x). \tag{15}$$

Note that since the generalized strain vector $\boldsymbol{\varepsilon}$ is not a function of the curvilinear coordinate s, see (7), it was taken out of the integral over the contour. Also, the matrix $\boldsymbol{\mathcal{D}}$ was defined as:

$$\mathbb{D} = \int_S \boldsymbol{\mathcal{T}}^T \boldsymbol{C}\, \boldsymbol{\mathcal{T}}\, ds. \tag{16}$$

The matrix $\mathbb{D}$ contains functions $\bar{\xi}_i$ that define the cross section mid-contour and also all the necessary material constants. It's of crucial importance the correct evaluation and formulation of $\mathbb{D}$ since it contains not only all geometrical couplings but also all material couplings.

The last derivations complete the formulation of the constitutive relations of the thin-walled beam theory. In contrast with most existing thin-walled beam formulations, the beam forces were not defined but deduced from the shell stresses (or forces) expression.

4 VARIATIONAL FORMULATION

The weak form of equilibrium of a three dimensional body $\boldsymbol{\mathcal{B}}$ is given by (Washizu, 1968, Zienkiewicz, 2000):

$$G(\boldsymbol{\phi}, \delta\boldsymbol{\phi}) = \int_{\mathcal{B}} \boldsymbol{\sigma} \cdot \delta\boldsymbol{\epsilon}\, dV - \int_{\mathcal{B}} \boldsymbol{\rho_0 b} \cdot \delta\boldsymbol{\phi}\, dV - \int_{\partial\mathcal{B}} (\boldsymbol{p} \cdot \delta\boldsymbol{u} + \boldsymbol{m} \cdot \delta\boldsymbol{\theta})\, d\Omega, \tag{17}$$

where $\boldsymbol{b}$, $\boldsymbol{p}$ and $\boldsymbol{m}$ are: body forces, prescribed external forces and prescribed external moments per unit length respectively. $\boldsymbol{\epsilon}$ is the GL strain tensor, work conjugate to the Second Piola-Kirchhoff stress tensor $\boldsymbol{\sigma}$. Where $\boldsymbol{\sigma}$ could be defined in either a rectangular or a curvilinear coordinate system, such a distinction is, at least at this point, unnecessary.

To maintain the variational formulation parametrized in terms of the director field, its admissible variation must be found. Once a set of kinematically admissible variations is obtained, the generalized virtual strains can be obtained, so the virtual work of the internal and external forces can be derived. Therefore, we aim to express the virtual work principle as a function of the generalized virtual strain vector and its work conjugate beam forces vector.

4.1 Finite Rotations and Director Variations

To obtain the generalized strains variations, the admissible variation of the director field is required. In order to find the variations (first and second) of the director field we must find first the perturbed rotation tensor and its variation.

Remembering that we set $\boldsymbol{\Lambda_0} = \boldsymbol{I}$ and recalling (1), we can write:

$$\delta\boldsymbol{e}_i = \delta(\boldsymbol{\Lambda}(\mathrm{s})\boldsymbol{E}_i) = \delta\boldsymbol{\Lambda}(x)\boldsymbol{E}_i. \tag{18}$$

The admissible variation of the rotation tensor (Lie variation) can be obtained introducing an infinitesimal virtual rotation superposed onto the existing finite rotation, see e.g. (Betsch, 1998, Corrêa and Camotim, 2002). This virtual rotation lies in the tangent space at $\boldsymbol{\Lambda}$ (spatial virtual rotation), or in the tangent space at $\mathbf{I}$ (material virtual rotation), and is represented by a skew symmetric matrix $\delta\boldsymbol{W}$, or $\delta\boldsymbol{\Psi}$ (see Fig. 2). These variables will be called: "*spins*" (Crisfield, 1997). The perturbed rotation tensor is constructed by exponentiating the spatial spin as:

$$\boldsymbol{\Lambda}_\epsilon = exp(\epsilon\, \delta\boldsymbol{W})\boldsymbol{\Lambda}. \tag{19}$$

Another way to construct the perturbed finite rotation tensor can be devised by making use of the rotation vector as:

$$\boldsymbol{\Lambda}_\epsilon = exp(\boldsymbol{\Theta} + \epsilon\, \delta\boldsymbol{\Theta}). \tag{20}$$

Recalling (19) and remembering that $\boldsymbol{\Lambda} = exp(\boldsymbol{\Theta})$ we have:

$$exp(\boldsymbol{\Theta} + \epsilon\, \delta\boldsymbol{\Theta}) = exp(\epsilon\, \delta\boldsymbol{W})exp(\boldsymbol{\Theta}), \tag{21}$$

Where we are trying to find an incremental rotation tensor, i.e. the virtual rotation tensor $\delta\boldsymbol{\Theta}$, that belongs to the tangent space as the infinitesimal rotation tensor $\boldsymbol{\Theta}$, this is $T_I SO(3)$. The vector $\boldsymbol{\theta}$ whose skew matrix is $\boldsymbol{\Theta}$ is called *total rotation vector*.

Taking derivatives with respect to the parameter ϵ we obtain (see e.g. (Ibrahimbegović et al., 1995, Mäkinen, 2007)):

$$\delta\boldsymbol{w} = \boldsymbol{T}(\boldsymbol{\theta})\, \delta\boldsymbol{\theta}, \tag{22}$$

Where $\boldsymbol{T}$ is a spatial tangential transformation, it reads:

$$\boldsymbol{T}(\boldsymbol{\theta}) = \frac{sin\,\theta}{\theta}\boldsymbol{I} + \frac{1 - cos\,\theta}{\theta^2}\boldsymbol{\Theta} + \frac{\theta - sin\,\theta}{\theta^3}\boldsymbol{\theta}\otimes\boldsymbol{\theta}. \tag{23}$$

These different choices in the construction of a kinematically admissible representation of the perturbed rotation tensor together with the type of algorithm chosen to perform the configuration update lead to different finite element formulations, i.e. Total Lagrangian, Updated Lagrangian and Eulerian formulations.

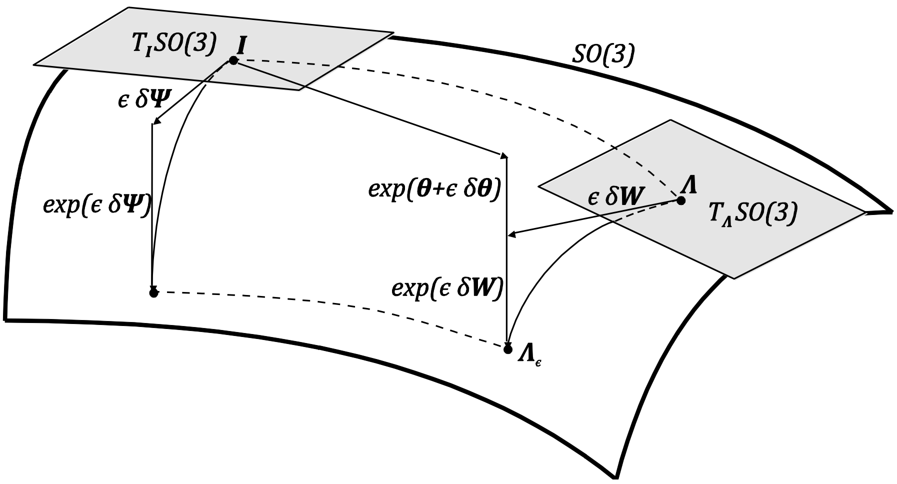

Figure 2 – Geometric interpretation of the exponential map.

Since the weak form of the equations of motion was parametrized in terms of the current directors and its derivatives, to ease the derivation of the virtual work it is necessary to use rotation variables that belong to the same tangent space as the directors, i.e. the tangent space at $\boldsymbol{\Lambda}$. Considering the latter, we will use the spatial version of virtual rotations (i.e. $\delta\boldsymbol{W}$) to obtain the kinematically admissible variation of the rotation tensor. Recalling (19) we can express the variation of the rotation tensor in terms of the spatial spin as:

$$\delta\boldsymbol{\Lambda} = \frac{d}{d\epsilon}[exp(\epsilon\, \delta\boldsymbol{W})\boldsymbol{\Lambda}]|_{\epsilon=0} = \delta\boldsymbol{W}\boldsymbol{\Lambda}. \tag{24}$$

Again, $\delta\boldsymbol{W}$ is a skew symmetric matrix such that $\delta\boldsymbol{W}\boldsymbol{a} = \delta\boldsymbol{w} \times \boldsymbol{a}$. Therefore, we can rewrite (18) as:

$$\delta\boldsymbol{e}_i = \delta\boldsymbol{w} \times \boldsymbol{e}_i. \tag{25}$$

Recalling (22) we can write the last equation as a function of the total rotation vector:

$$\delta \boldsymbol{e}_i = (\boldsymbol{T}\,\delta\boldsymbol{\theta}) \times \boldsymbol{e}_i. \tag{26}$$

The set of kinematically admissible variations can now be defined as:

$$\delta\mathcal{V} \coloneqq \{\delta\boldsymbol{\phi} = [\delta\boldsymbol{u}, \delta\boldsymbol{\theta}]^T : [0,\ell] \to R^3 | \delta\boldsymbol{\phi} = 0 \;\text{ on }\; \mathcal{S}\}, \tag{27}$$

where $\mathcal{S}$ describes de boundaries with prescribed displacements and rotations.

Noting that $\boldsymbol{e}' = \boldsymbol{T}\,\boldsymbol{\theta}'$ we can find the variation of the director's derivative as:

$$\delta\boldsymbol{e}_i' = (\delta\boldsymbol{T}\,\boldsymbol{\theta}' + \boldsymbol{T}\,\delta\boldsymbol{\theta}') \times \boldsymbol{e}_i + (\boldsymbol{T}\,\boldsymbol{\theta}') \times [(\boldsymbol{T}\,\delta\boldsymbol{\theta}) \times \boldsymbol{e}_i]. \tag{28}$$

The second variation of directors can be obtained using (26) as:

$$\Delta\delta\boldsymbol{e}_i = (\Delta\boldsymbol{T}\,\delta\boldsymbol{\theta}) \times \boldsymbol{e}_i + (\boldsymbol{T}\,\delta\boldsymbol{\theta}) \times [(\boldsymbol{T}\,\Delta\boldsymbol{\theta}) \times \boldsymbol{e}_i]. \tag{29}$$

The second variation of the director derivatives is in this case equal to the linearization of the virtual strains, it has a complicated expression that has not been presented yet, it reads:

$$\begin{aligned}\Delta\delta\boldsymbol{e}_i' = {} & \Delta(\delta\boldsymbol{T}\,\boldsymbol{\theta}' + \boldsymbol{T}\,\delta\boldsymbol{\theta}') \times \boldsymbol{e}_i + (\delta\boldsymbol{T}\,\boldsymbol{\theta}' + \boldsymbol{T}\,\delta\boldsymbol{\theta}') \times \Delta\boldsymbol{e}_i + \Delta(\boldsymbol{T}\,\boldsymbol{\theta}') \times [(\boldsymbol{T}\,\delta\boldsymbol{\theta}) \times \boldsymbol{e}_i] \\ & + (\boldsymbol{T}\,\boldsymbol{\theta}') \times \Delta[(\boldsymbol{T}\,\delta\boldsymbol{\theta}) \times \boldsymbol{e}_i]\end{aligned} \tag{30}$$

4.1 Virtual Generalized Strains

The variations of the directors and its derivatives are now used to obtain the virtual generalized strains. Considering that $\delta\boldsymbol{E_i} = 0$ and that $\delta\boldsymbol{X}_0' = 0$, and performing the variation to (7) we obtain:

$$\delta\boldsymbol{\varepsilon} = \begin{bmatrix} \boldsymbol{x}_0' \cdot \delta\boldsymbol{u}' \\ \boldsymbol{e}_3' \cdot \delta\boldsymbol{u}' + \boldsymbol{x}_0' \cdot \delta\boldsymbol{e}_3' \\ \boldsymbol{e}_2' \cdot \delta\boldsymbol{u}' + \boldsymbol{x}_0' \cdot \delta\boldsymbol{e}_2' \\ \boldsymbol{e}_2 \cdot \delta\boldsymbol{u}' + \boldsymbol{x}_0' \cdot \delta\boldsymbol{e}_2 \\ \boldsymbol{e}_3 \cdot \delta\boldsymbol{u}' + \boldsymbol{x}_0' \cdot \delta\boldsymbol{e}_3 \\ \delta\boldsymbol{e}_2' \cdot \boldsymbol{e}_3 + \boldsymbol{e}_2' \cdot \delta\boldsymbol{e}_3 \\ 2(\delta\boldsymbol{e}_2' \cdot \boldsymbol{e}_2') \\ 2(\delta\boldsymbol{e}_3' \cdot \boldsymbol{e}_3') \\ \delta\boldsymbol{e}_2' \cdot \boldsymbol{e}_3' + \boldsymbol{e}_2' \cdot \delta\boldsymbol{e}_3' \end{bmatrix}. \tag{31}$$

We can write the virtual strains as a function of a new set of kinematic variables $\delta\boldsymbol{\varphi}$ as:

$$\delta\boldsymbol{\varepsilon} = \mathbb{H}\,\delta\boldsymbol{\varphi},$$

$$\mathbb{H} = \begin{bmatrix} \boldsymbol{x}_0' & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{e}_3' & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{x}_0' \\ \boldsymbol{e}_2' & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{x}_0' & \boldsymbol{0} \\ \boldsymbol{e}_2 & \boldsymbol{0} & \boldsymbol{x}_0' & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{e}_3 & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{x}_0' & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{e}_2' & \boldsymbol{e}_3 & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{e}_2' & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{e}_3' \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{e}_3' & \boldsymbol{e}_2' \end{bmatrix}, \qquad \delta\boldsymbol{\varphi} = \begin{bmatrix} \delta\boldsymbol{u}' \\ \delta\boldsymbol{w} \\ \delta\boldsymbol{e}_2 \\ \delta\boldsymbol{e}_3 \\ \delta\boldsymbol{e}_2' \\ \delta\boldsymbol{e}_3' \end{bmatrix}. \tag{32}$$

The second variation of the generalized strains gives:

$$\Delta\delta\boldsymbol{\varepsilon} = \begin{bmatrix} \delta\boldsymbol{u}' \cdot \Delta\boldsymbol{u}' \\ \delta\boldsymbol{u}' \cdot \Delta\boldsymbol{e}_3' + \delta\boldsymbol{e}_3' \cdot \Delta\boldsymbol{u}' + \boldsymbol{x}_0' \cdot \Delta\delta\boldsymbol{e}_3' \\ \delta\boldsymbol{u}' \cdot \Delta\boldsymbol{e}_2' + \delta\boldsymbol{e}_2' \cdot \Delta\boldsymbol{u}' + \boldsymbol{x}_0' \cdot \Delta\delta\boldsymbol{e}_2' \\ \delta\boldsymbol{u}' \cdot \Delta\boldsymbol{e}_2 + \delta\boldsymbol{e}_2 \cdot \Delta\boldsymbol{u}' + \boldsymbol{x}_0' \cdot \Delta\delta\boldsymbol{e}_2 \\ \delta\boldsymbol{u}' \cdot \Delta\boldsymbol{e}_3 + \delta\boldsymbol{e}_3 \cdot \Delta\boldsymbol{u}' + \boldsymbol{x}_0' \cdot \Delta\delta\boldsymbol{e}_3 \\ \delta\boldsymbol{e}_2' \cdot \Delta\boldsymbol{e}_3 + \delta\boldsymbol{e}_3 \cdot \Delta\boldsymbol{e}_2' + \boldsymbol{e}_3 \cdot \Delta\delta\boldsymbol{e}_2' + \boldsymbol{e}_2' \cdot \Delta\delta\boldsymbol{e}_3 \\ 2(\boldsymbol{e}_2' \cdot \Delta\delta\boldsymbol{e}_2' + \delta\boldsymbol{e}_2' \cdot \Delta\boldsymbol{e}_2') \\ 2(\boldsymbol{e}_3' \cdot \Delta\delta\boldsymbol{e}_3' + \delta\boldsymbol{e}_3' \cdot \Delta\boldsymbol{e}_3') \\ \delta\boldsymbol{e}_2' \cdot \Delta\boldsymbol{e}_3' + \delta\boldsymbol{e}_3' \cdot \Delta\boldsymbol{e}_2' + \boldsymbol{e}_3' \cdot \Delta\delta\boldsymbol{e}_2' + \boldsymbol{e}_2' \cdot \Delta\delta\boldsymbol{e}_3' \end{bmatrix}. \tag{33}$$

The presented derivation of the virtual generalized strains in terms of the variations of the directors and its derivatives is independent of the parametrization of finite rotations.

4.2 Internal Virtual Work

Having derived the expressions for the admissible variations of the basis vectors and strains we develop in this section the expressions for the internal virtual work of the beam. Recalling (17), the internal virtual work of a three dimensional body can be written in vector form as:

$$G_{int}(\boldsymbol{\phi}, \delta\boldsymbol{\phi}) = \int_{\mathcal{B}_0} \delta\boldsymbol{\epsilon}^T \boldsymbol{\sigma}\, dV, \tag{34}$$

which in the curvilinear coordinate system is:

$$G_{int}(\boldsymbol{\phi}, \delta\boldsymbol{\phi}) = \int_{\ell} \int_{S} \int_{e} (\delta\boldsymbol{\epsilon}^T \boldsymbol{\sigma})\; dn\, ds\, dx. \tag{35}$$

We can now use the definition of the shell resultant forces to reduce the 3D formulation to a 2D formulation. Therefore, performing integration of (35) in the n direction we can write the internal virtual work in terms of shell quantities as:

$$G_{int}(\boldsymbol{\phi}, \delta\boldsymbol{\phi}) = \int_{\ell} \int_{S} \delta\boldsymbol{\epsilon}_s^T\, \boldsymbol{N}_s\, ds\;\, dx. \tag{36}$$

The reduction to a 1D formulation is aided by the deduction of 1D beam forces presented in (14). Transforming the virtual curvilinear shell strains into virtual generalized strains we can rewrite the last expression as:

$$G_{int}(\boldsymbol{\phi}, \delta\boldsymbol{\phi}) = \int_{\ell} \delta\boldsymbol{\varepsilon}^T \left(\int_{S} \boldsymbol{\mathcal{T}}^T\, \boldsymbol{N}_s\, ds \right) dx \tag{37}$$

In which the term in parentheses is the generalized beam forces vector (see (14)). Using (13) the beam forces vector can be expressed a slightly different form as:

$$\boldsymbol{S}(x) = \int_{S} \boldsymbol{\mathcal{T}}^T\, \boldsymbol{N}_s\, ds. \tag{38}$$

The explicit expression of the beam forces can be found in Saravia et.al. (2010).

Finally, it's possible write the 1D virtual work in terms of the generalized strains and the generalized beam forces as:

$$G_{int}(\boldsymbol{\phi}, \delta\boldsymbol{\phi}) = \int_{\ell} \delta\boldsymbol{\varepsilon}^T \boldsymbol{S}\, dx. \tag{39}$$

4.3 External Virtual Work

In this section we derive the expression of the external virtual work. In order to simplify this derivation we neglect the body forces. For this particular case, the external virtual work can be written as:

$$G_{ext}(\boldsymbol{\phi}, \delta\boldsymbol{\phi}) = \int_{l} (\boldsymbol{p} \cdot \delta\boldsymbol{u}_0 + \boldsymbol{m} \cdot \delta\boldsymbol{\theta})\, dx, \tag{40}$$

where $\boldsymbol{p}$ is the external forces vector and $\boldsymbol{m}$ the external moments vector.

4.4 Weak Form of Equilibrium

The variational equilibrium statement can now be presented in terms of generalized components of 1D forces and strains. Recalling (39) and (40) the virtual work of a composite beam is written in its one dimensional form as:

$$G(\boldsymbol{\phi}, \delta\boldsymbol{\phi}) = \int_{\ell} \delta\boldsymbol{\varepsilon}^T \boldsymbol{S}\, dx - \int_{l} (\boldsymbol{p} \cdot \delta\boldsymbol{u}_0 + \boldsymbol{m} \cdot \delta\boldsymbol{\theta})\, dx. \tag{41}$$

Recalling (32) it's possible to re-write the last expression as:

$$G(\boldsymbol{\phi}, \delta\boldsymbol{\phi}) = \int_{\ell} [\mathbb{H}\, \delta\boldsymbol{\varphi}]^T\, \boldsymbol{S}\, dx - \int_{l} (\boldsymbol{p} \cdot \delta\boldsymbol{u}_0 + \boldsymbol{m} \cdot \delta\boldsymbol{\theta})\, dx. \tag{42}$$

Thus, the equilibrium of the geometrically nonlinear beam is available in its variational form.

5 LINEARIZATION OF THE WEAK FORM

Being $L[G(\boldsymbol{\phi}, \delta\boldsymbol{\phi})]$ the linear part of the functional $G(\phi, \delta\phi)$, we have:

$$L\big[G\big(\widehat{\boldsymbol{\phi}}, \delta\boldsymbol{\phi}\big)\big] = G\big(\widehat{\boldsymbol{\phi}}, \delta\boldsymbol{\phi}\big) + D\, G\big(\widehat{\boldsymbol{\phi}}, \delta\boldsymbol{\phi}\big) \cdot \Delta\boldsymbol{\phi}, \tag{43}$$

where the first term $G\big(\widehat{\boldsymbol{\phi}}, \delta\boldsymbol{\phi}\big)$ is the unbalanced force at the configuration $\widehat{\boldsymbol{\phi}}$. The Frechet differential in the second term is obtained in a standard way as:

$$D\, G(\boldsymbol{\phi}, \delta\boldsymbol{\phi}) \cdot \Delta\boldsymbol{\phi} = \frac{d}{d\epsilon}\Big|_{\epsilon=0} G(\boldsymbol{\phi} + \epsilon\, \Delta\boldsymbol{\phi}), \tag{44}$$

where $\Delta\boldsymbol{\phi}$ fulfills the boundary conditions. Applying the definition (44) and recalling (39) and (31), we obtain the tangent stiffness as:

$$D\, G_{int}(\boldsymbol{\phi}, \delta\boldsymbol{\phi}) \cdot \Delta\boldsymbol{\phi} = \int_{\ell} (\delta\boldsymbol{\varepsilon}^T\, \mathbb{D}\, \Delta\boldsymbol{\varepsilon} + \Delta\delta\boldsymbol{\varepsilon}^T\, \boldsymbol{S}) dx, \tag{45}$$

where ℓ is the length of the undeformed beam. The integral of the first term gives raise to the material stiffness matrix and from the integral of the second term evolves the geometric stiffness matrix.

Recalling (32) the first term takes the form:

$$D_1 G_{int}(\boldsymbol{\phi}, \delta\boldsymbol{\phi}) \cdot \Delta\boldsymbol{\phi} = \int_{\ell} \delta\boldsymbol{\varphi}^T \mathbb{H}^T \mathbb{D} \mathbb{H} \Delta\boldsymbol{\varphi} \, dx. \tag{46}$$

Then, the general expression of the geometric stiffness operator gives:

$$D_2 \, G_{int}(\boldsymbol{\phi}, \delta\boldsymbol{\phi}) \cdot \Delta\boldsymbol{\phi} = \int_{\ell} \Delta\delta\boldsymbol{\varepsilon}^T \boldsymbol{S} \, dx. \tag{47}$$

6 FINITE ELEMENT FORMULATION

The implementation of the proposed finite element is based on linear interpolation and one point reduced integration. The most important procedure of this finite element is the use of interpolation to obtain the derivatives of the director field.

6.1 Interpolations and Directors Update

We interpolate the position vectors in the undeformed and deformed configuration as:

$$\boldsymbol{X} = \sum_{j=1}^{nn} \boldsymbol{N}_j \, \widehat{\boldsymbol{X}}_j, \qquad \boldsymbol{x} = \sum_{j=1}^{nn} \boldsymbol{N}_j \left(\widehat{\boldsymbol{X}}_j + \widehat{\boldsymbol{u}}_j\right), \tag{48}$$

The same finite element interpolation is also applied to the configuration and the variations of the configuration variables.

Recalling (1) the director at the iteration $n+1$ is found as:

$${}^{n+1}\boldsymbol{e}_i = \boldsymbol{\Lambda}({}^{n}\boldsymbol{\theta}) \, \boldsymbol{E}_i, \tag{49}$$

where $\boldsymbol{\Lambda}$ is the *total rotation tensor*.

According to (49), we could find the derivative of the directors as:

$$\boldsymbol{e}_i' = \boldsymbol{\Lambda}' \, \boldsymbol{E}_i = \boldsymbol{\kappa} \, \boldsymbol{e}_i = \boldsymbol{T}\boldsymbol{\theta}' \boldsymbol{e}_i, \tag{50}$$

as done in most Total Lagrangian formulations, but this requires the derivative of the total rotation tensor and in pursue of a simpler way to obtain the triad derivative we use (51) to obtain this derivatives.

Being $\boldsymbol{N}_j$ linear Lagrangian shape functions, it will be assumed that:

$$\boldsymbol{e}_i' \cong \sum_{j=1}^{nn} \boldsymbol{N}_j' \, \hat{\boldsymbol{e}}_i^j \tag{51}$$

Where $\boldsymbol{\kappa}$ is the curvature tensor, $\hat{\boldsymbol{e}}_i^j$ is the node j director in the direction i and nn is the number of nodes per element. This approximation is expected to be accurate enough to be used in almost every practical situation. However, we will analyze in the numerical investigations sections the impact of this approximation in the accuracy of the solution.

As a distinct consequence of the use of interpolation to obtain the derivative of the director field is that the finite element results to be path independent.

6.2 Discrete Virtual Strains

Although the expression (28) can be obtained relatively simply, the second variation of the directors derivative is more difficult to obtain. A simpler way to obtain it would help to simplify the expression of the tangent stiffness very much. In this direction, some works have proposed to use interpolation of the directors to obtain their derivatives (Armero and Romero, 2001, Betsch and Steinmann, 2002, Gruttmann et al., 2000, Romero and Armero, 2002).

Assuming holonomic constraints we may interchange the variations and derivatives, i.e. $\delta(\boldsymbol{e}') = (\delta\boldsymbol{e})'$. Using this property, we can use (51) to obtain the variation of the directors and its derivatives as:

$$\delta\boldsymbol{e}_i = \sum_{j=1}^{nn} \boldsymbol{N}_j\, \delta\hat{\boldsymbol{e}}_i^j, \qquad \delta\boldsymbol{e}_i' = \sum_{j=1}^{nn} \boldsymbol{N}_j'\, \delta\hat{\boldsymbol{e}}_i^j, \tag{52}$$

The obtention of the second variation of the directors and its derivatives is more involved and requires the linearization of the tangential transformation (23). Observing that the second variation (or linearization of the variation) of the directors is always pre multiplied by some vector $\boldsymbol{a}$, for simplicity in the arranging of terms, it's preferable to obtain the expression for the product and not only for the second variation. Thus it can be shown that:

$$\boldsymbol{a}\cdot\Delta\delta\boldsymbol{e}_i = \delta\hat{\boldsymbol{\theta}}^T \left[\sum_{j=1}^{nn} \boldsymbol{N}_j\, \Xi\left(\boldsymbol{a}, \hat{\boldsymbol{e}}_i^j\right)\right] \Delta\hat{\boldsymbol{\theta}} \tag{53}$$

The expression for $\Xi(\boldsymbol{a}, \boldsymbol{e}_i)$ results from the linearization of the tangential transformation and is given in Appendix A. As it was mentioned before, we use interpolation to obtain the derivatives of the director field, this helps to obtain an expression for the second variation of the director's derivatives as:

$$\boldsymbol{a}\cdot\Delta\delta\boldsymbol{e}_i' = \delta\hat{\boldsymbol{\theta}}^T \left[\sum_{j=1}^{nn} \boldsymbol{N}_j'\, \Xi\left(\boldsymbol{a}, \hat{\boldsymbol{e}}_i^j\right)\right] \Delta\hat{\boldsymbol{\theta}} \tag{54}$$

Additionally, it's possible to relate the two kinematic vectors $\delta\boldsymbol{\varphi}$ and $\delta\boldsymbol{\phi}$ via the matrix $\mathbb{B}$ as:

$$\delta\boldsymbol{\varphi} \cong \sum_{j=1}^{nn} \mathbb{B}^j\, \delta\hat{\boldsymbol{\phi}}^j, \tag{55}$$

where:

$$\mathbb{B}^j = \begin{bmatrix} \overline{\boldsymbol{N}}_j' & \boldsymbol{0} \\ \boldsymbol{0} & \overline{\boldsymbol{N}}_j \boldsymbol{T}^j \\ \boldsymbol{0} & -\tilde{\boldsymbol{e}}_2^j\, \overline{\boldsymbol{N}}_j\, \boldsymbol{T}^j \\ \boldsymbol{0} & -\tilde{\boldsymbol{e}}_3^j\, \overline{\boldsymbol{N}}_j\, \boldsymbol{T}^j \\ \boldsymbol{0} & -\tilde{\boldsymbol{e}}_2^j\, \overline{\boldsymbol{N}}_j'\, \boldsymbol{T}^j \\ \boldsymbol{0} & -\tilde{\boldsymbol{e}}_3^j\, \overline{\boldsymbol{N}}_j'\, \boldsymbol{T}^j \end{bmatrix}, \qquad \delta\hat{\boldsymbol{\phi}}^j = \begin{bmatrix} \delta\hat{\boldsymbol{u}}^j \\ \delta\hat{\boldsymbol{\theta}}^j \end{bmatrix}. \tag{56}$$

Where $\sim$ indicates the skew symmetric matrix of a vector and $-$ indicates the diagonal matrix of a vector. Thus $\tilde{\boldsymbol{e}}_j^i$ is a skew director in the direction j of the node i, $\overline{\boldsymbol{N}}_j$ is a diagonal shape function matrix, and $\boldsymbol{T}^j$ is a tangential transformation at the node i. Henceforth summation over index i will be implicitly defined, so we will omit the summation symbol and the node index i.

Finally, we can write the virtual generalized strains as:

$$\delta\boldsymbol{\varepsilon} \cong \mathbb{H}\,\mathbb{B}\,\delta\widehat{\boldsymbol{\phi}}, \tag{57}$$

The second variation of the generalized strains, i.e. $\Delta\delta\boldsymbol{\varepsilon}$, is more difficult to obtain. Having in mind the structure of the geometric stiffness operator (47) we aim to obtain a matrix $\mathbb{G}$ as to satisfy the equality $\Delta\delta\boldsymbol{\varepsilon}^T\boldsymbol{S} = \delta\boldsymbol{\varphi}^T\,\mathbb{G}\,\Delta\boldsymbol{\varphi}$, a lengthy manipulation gives:

$$\mathbb{G} = \begin{bmatrix} \bar{S}_1 & \mathbf{0} & \bar{Q}_2 & \bar{Q}_3 & \bar{M}_3 & \bar{M}_2 \\ & \boldsymbol{c} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ & & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ & & & \mathbf{0} & \bar{M}_1 & \mathbf{0} \\ & Sym & & & 2\bar{P}_2 & \bar{P}_{23} \\ & & & & & 2\bar{P}_3 \end{bmatrix}. \tag{58}$$

The matrix $\boldsymbol{c}$ results from the linearization of the virtual directors, see (53) and (54). Its expression gives:

$$\boldsymbol{c} = \bar{M}_2\tilde{\boldsymbol{x}}_0'\tilde{\boldsymbol{e}}_3' + \bar{M}_3\tilde{\boldsymbol{x}}_0'\tilde{\boldsymbol{e}}_2' + \bar{Q}_2\tilde{\boldsymbol{x}}_0'\tilde{\boldsymbol{e}}_2 + \bar{Q}_3\tilde{\boldsymbol{x}}_0'\tilde{\boldsymbol{e}}_3 + \boldsymbol{c}_{\Delta\boldsymbol{T}} \tag{59}$$

Where $\boldsymbol{c}_{\Delta\boldsymbol{T}}$ is given in Appendix A.

6.3 Tangent Stiffness Matrix

Introducing (55) into (46) we can obtain the discrete form of the virtual work as:

$$D_1G_{int}\left(\widehat{\boldsymbol{\phi}},\delta\widehat{\boldsymbol{\phi}}\right)\cdot\Delta\widehat{\boldsymbol{\phi}} = \int_\ell \left(\mathbb{B}\,\delta\widehat{\boldsymbol{\phi}}\right)^T\mathbb{H}^T\,\mathbb{D}\,\mathbb{H}\left(\mathbb{B}\,\Delta\widehat{\boldsymbol{\phi}}\right)dx. \tag{60}$$

Then, the element material stiffness matrix is:

$$\boldsymbol{k}_M = \int_\ell \mathbb{B}^T\mathbb{H}^T\mathbb{D}\,\mathbb{H}\,\mathbb{B}\,dx. \tag{61}$$

Proceeding in a similar way, we use (58) and (47) to obtain the discrete geometric stiffness terms as:

$$D_2G_{int}\left(\widehat{\boldsymbol{\phi}},\delta\widehat{\boldsymbol{\phi}}\right)\cdot\Delta\widehat{\boldsymbol{\phi}} = \int_\ell \left(\mathbb{B}\,\delta\widehat{\boldsymbol{\phi}}\right)^T\mathbb{G}\left(\mathbb{B}\,\Delta\widehat{\boldsymbol{\phi}}\right)dx. \tag{62}$$

Therefore, the element geometric stiffness matrix becomes:

$$\boldsymbol{k}_G = \int_\ell \mathbb{B}^T\mathbb{G}\,\mathbb{B}\,dx. \tag{63}$$

Following the common steps of the finite element method, the element and global tangent stiffness matrices are:

$$\begin{gathered} \boldsymbol{k}_T = \int_\ell \mathbb{B}^T(\mathbb{H}^T\mathbb{D}\,\mathbb{H} + \mathbb{G})\,\mathbb{B}\,dx, \\ \boldsymbol{K}_T = \sum_{e=1}^{els}\boldsymbol{k}_T, \end{gathered} \tag{64}$$

where the summation operator is used to represent the finite element assembly process.

7 NUMERICAL INVESTIGATIONS

We present in this section several examples that show the performance of the proposed finite element. We investigate both the isotropic and the anisotropic cases, choosing some benchmark tests proposed in the literature.

We compare the present finite element against existing finite elements, including an Eulerian formulation that uses the same beam theory (Saravia et al., 2010), because of that this is the best theory that we could choose to actually show the benefits and drawbacks of the present formulation. In the analysis of anisotropic beams we only compare the present formulation against Abaqus 3D shell models and the finite element in (Saravia et al., 2010) since other similar finite elements for thin-walled composite beams have not been reported.

7.1 Accuracy Assessment – Pure bending of a cantilever beam

We test in this example the behavior of the accuracy of the present formulation in a full three dimensional problem, a curved cantilever beam with out of plane loading (see. Fig. 3). The curved beam's reference configuration given is a 45º circular segment with radius $R=100$ and laying in the x-y plane, the beam is loaded with a vertical load (z direction). The properties of the isotropic material are: $E=1.0\times10^7$ and $\nu=0.3$. The cross section is a box with $b=1$, $h=1$ and $t=0.1$.

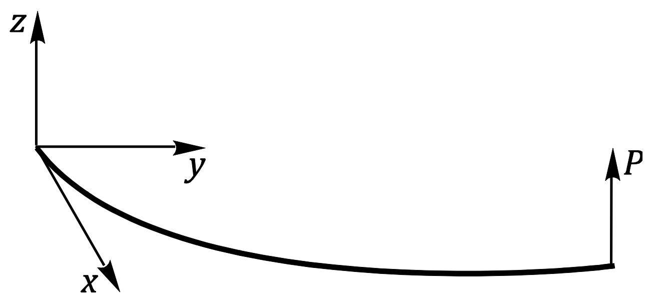

Figure 3 – 45º arc bending

Table 1 shows the results of the bending test for $P=100$. We have used an Abaqus 3D shell model as the reference model. As it can be seen, the present finite element formulation behaves better than to the Simo & Vu-Quoc element (Simo and Vu-Quoc, 1986) available in FEAP and the Abaqus B31 beam element. The results obtained with the present implementation and the path dependent implementation presented in (Saravia et al., 2010) are essentially the same.

	Tip y Disp.	Tip x Disp.	Max z Disp.	Elements
Abaqus Shell	-2.090	-3.641	22.611	-
Abaqus B31	-2.574	-3.570	22.734	50
Simo & Vu-Quoc (FEAP)	-1.986	-3.325	22.001	50
Saravia et. al. (Saravia et al., 2010)	-2.068	-3.495	22.366	50
Present	-2.069	-3.449	22.367	50

Table 1 – Maximum displacements in a 45º arc bending test ($P=100$).

The solution was reached in 5 load steps using an average of 8 iterations per step.

Increasing the load to *P=400* we obtain also excellent results (see Table 2). Note that we added to the comparison the Abaqus parabolic beam element B32. The present finite element represents the kinematic behavior of the beam better than all the presented finite elements.

	Tip *y* Disp.	Tip *x* Disp.	Max *z* Disp.	Elements
Abaqus Shell	-12.201	-21.546	50.997	-
Abaqus B31	-12.401	-21.311	-51.110	50
Abaqus B32	-12.416	-21.310	-51.111	50
Simo & Vu-Quoc (FEAP)	-12.008	-20.692	50.067	50
Saravia et. al. (Saravia et al., 2010)	-12.205	-21.015	50.880	50
Present	-12.206	-21.019	50.884	50

Table 2 – Maximum displacements in a 45º arc bending test (*P=400*).

We present next a comparison of the displacement path of the beam using an anisotropic laminate, we analyze the 45º arc of Fig. 2 laminated with a {45,-45,-45,45} configuration. The laminas are made of E-Glass fibers and an Epoxy matrix (Barbero, 2008), the material properties are given in Table 3.

E_{11}	E_{22}	G_{12}	G_{23}	ν_{12}
45.0×10^9	12.0×10^9	5.5×10^9	5.5×10^9	0.3

Table 3 – Material properties of E-Glass Fiber-Epoxy lamina.

To increase the complexity of the stress state in the beam we modify the applied load to have components $P_x=4.0\times10^5$, $P_y=-4.0\times10^5$, $P_z=8.0\times10^5$. Figure 3 presents the curves that describe the evolution of the displacements along the load path (LPF: Load Proportional Factor).

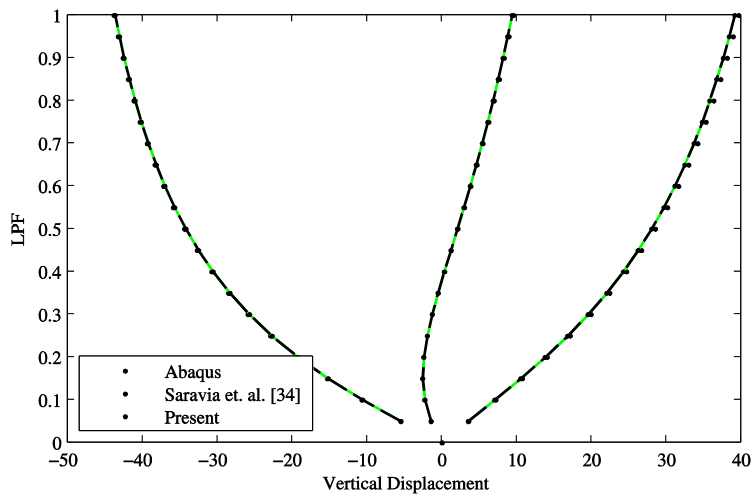

Figure 3. Displacements vs. Load Proportional Factor.

It can be seen from Fig. 3 that the correlation of the present formulation against the Abaqus shell model and also the Saravia et.al. (2010) formulation is excellent.

the y axis with a length of 5. The cross section is boxed with dimensions h=1, b=1 and thickness=0.1 made from 4 layers of E-Glass Fiber-Epoxy laminated as {45,-45,-45,45}. The material properties are given in Table 3.

The first load case consist on a tip force of 2 10^7 fixed in the *z* direction, the second load is applied in three different ways: i) rotation around the *z* axis, ii) rotation around the y axis and iii) rotation around the x axis. For both i, ii, and iii the rotation is imposed in 4000 increments of $\pi/20$ radians each, which is equivalent to 100 revolutions.

Figure 4 shows the evolution of displacements after completing a revolution; as expected from a frame-indifferent formulation, the displacements remain constant along the revolutions. Since the constant displacements are the result of the first load case and we have maintained this load case unaltered, the picture coincides exactly for both; i, ii, and iii.

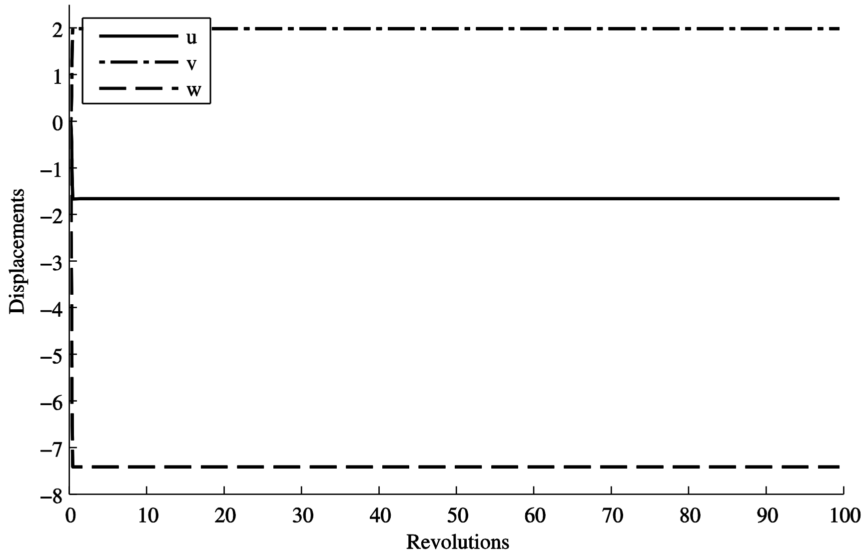

Figure 4. Displacements vs. Revolutions

The following figures show the deformed shapes of the frame in the full revolution path. It can be observed that for the three loading schemes the deformed shapes are identical for every revolution. It may be noted that the displacements in the beam are really large; this was induced on purpose to emphasize the fact that there isn't any nontrivial work generated by the fixed force.

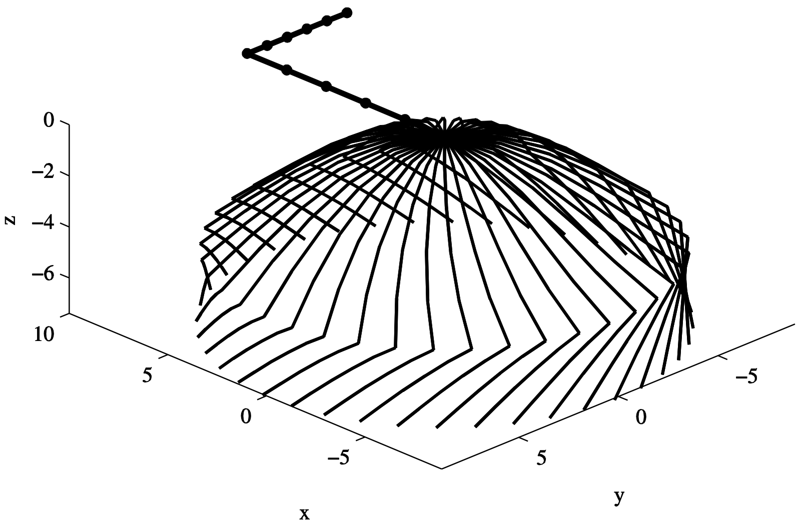

Figure 5. Displacements vs. Revolutions

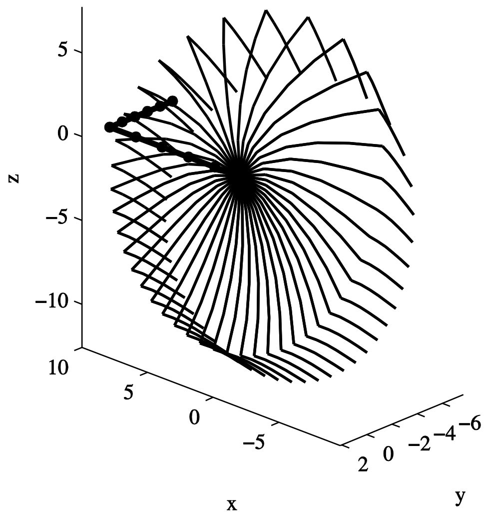

Figure 6. Displacements vs. Revolutions

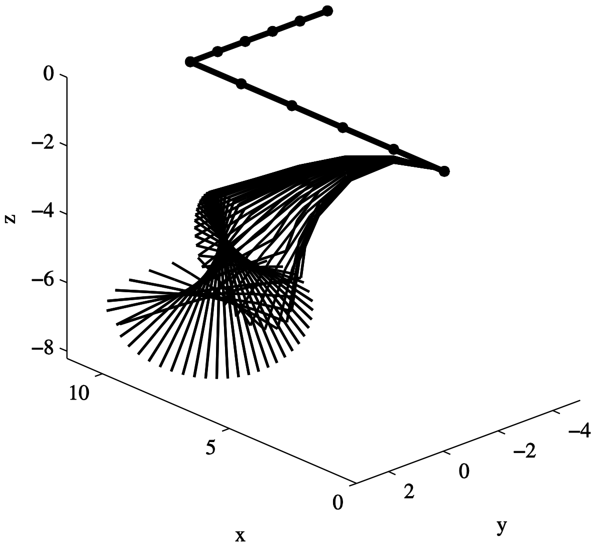

Figure 7. Displacements vs. Revolutions

Considering the case where the tip load is a follower force (initially oriented in the z direction) that rotates with the frame around the y axis, we observe that the present formulation is also frame-invariant. Figure 8 shows the deformed shapes for the full rotation path of 100 revolutions, it can be observed that these deformed shapes coincide for each revolution.

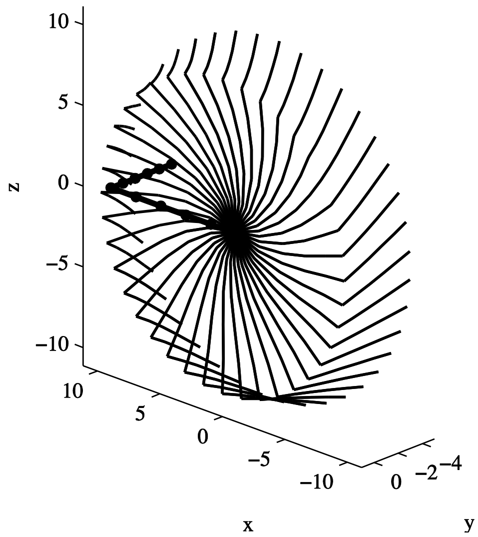

Figure 8. Deformed shapes for follower force.

8 CONCLUSIONS

A Total Lagrangian geometrical exact nonlinear beam finite element for composite thin-walled beams has been presented. The proposed formulation relies on the parametrization of the weak form of equilibrium in terms of the director field and its derivatives. Finite rotations were parametrized with the rotation vector.

The weak form of equilibrium was written in terms of generalized strains, which result from a dual transformation of the rectangular GL strains. The generalized strains work conjugate variables, i.e. the generalized beam forces, were deduced from the curvilinear shell stresses before the obtention of the weak form.

The presented numerical investigations show that the present finite element has an excellent accuracy when the deformation is not extreme. Still, the accuracy is very good when extreme deformation scenarios are accompanied by sufficient mesh refinement. Also, the presented finite element formulation guaranteed the path independence and frame invariance properties.

The possibility of using any type of composite material represents an important advantage of the present formulation over existing displacement based geometrically exact finite element beam formulations.

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