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NUMERICAL CALCULATION OF CONSERVATION LAWS IN SOLITONIC INTERACTIONS

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Abstract. The aim of this paper was the numerical computing of the first conservation law present in the interactions between solitons generated from numerical solutions of the Korteweg de Vries (KdV) equation.

As follows from the theory of partial differential equations in which appears solitons, like some solutions of the KdV, many interesting behaviors come associated with them, in particular the conservation laws derived from it. Only the first three ones have direct interpretation or are associated with physical quantities, such as conservation of energy, momentum or mass density. The remaining are conservation laws in the strict sense of physics but do not have a clearly associated with magnitudes that usually appear in physical processes.

The KdV equation constitutes one of the called "integrable systems" and in this way the conservations laws derived from it, have special interest for applications on the study of solitons's propagation.

These non-linear traveling waves that can be found from tsunamis in the oceans, transmission of information by transoceanic fiber optic communications, transmitted information through neural microtubules in living beings have a relevant technological and scientific interest.

So the knowledge of conservation laws in the interaction between solitons becomes very interesting from both point of view technological and for the study of certain process in living beings.

To obtain the explicit form of the conservation laws derived from the KdV equation, mainly due to the large number of terms that arise in the mathematical expression of them, there were developed a computational code implemented in a symbolic calculus platform that facilitated their algebraic handling.

The conservation laws thus obtained were numerically evaluated during the computing integration of the KdV equation, based on an spectral type scheme. Their solutions represent the interaction of two solitons.

Among the obtained results it can be pointed out that if the generation of solitons by solving the KdV equation has a high degree of approximation, the integrals of motion does not require highly sophisticated numerical algorithms to be satisfied and also make a numerical validation of the integration schemes of the nonlinear equation under study.

1 INTRODUCTION

The study and characterization of soliton wave propagation has several applications. These can be found in oceanography (Osborne, 2010) and medicine (Demiray, 2009) (Georgiev, 2004) and spanning a wide variety of cutting-edge technological applications.

In the present work arises from a type of iterative algebraic formalism to find the first eleven conservation laws coming from the soliton wave solutions of the KDV equation. The solutions of the KdV equations were obtained from computer integrations with the aim to characterize the conservation laws in two distinct times, one prior to the interaction between two solitons and other after that interaction.

The results contribute to better understand the above conservation laws, as well as to constitute a novel way to monitor the soliton propagation and interactions among them.

2 KORTEWEG DEVRIES EQUATION

The KdV equation is a non linear partial differential equation (NLPDE), and its most usual expression for a single dimension and time is (frequently noted (1+1)D) (Dickey, 1991)

$$\frac{\partial u(x,t)}{\partial t} + \delta u(x,t)\frac{\partial u(x,t)}{\partial x} + \lambda\frac{\partial^3 u(x,t)}{\partial x^3} = 0 \quad x \in (0,\infty), \tag{1}$$

where $u(x,t)$ is amplitude of the wave, x the spatial variable, t is the time, and λ and δ are two constants commonly fixed at λ=1, and δ=6, this reason will be given later.

Equation (1) is a NLPDE, where the second term introduces the non linearity and the third one has a dispersive behavior. It is very important to remark that the dispersive behavior might be given by a derivate of odd order or through a combination of functions having odd symmetry (Drazin and Johnson, 1989).

The mathematical structures of the expression (1) are very important to understand how some special traveling waves propagate without distortion like solitons.

Figure 1 shows the propagation of a wave whose analytical expression is

$$u(x,t) = \frac{v}{2}\operatorname{sech}^2\left(\frac{\sqrt{v}}{2}(x - vt) + c\right) \quad , c \in \mathfrak{R}, v \in \mathfrak{R} > 0, x \in \mathfrak{R}, t \geq 0. \tag{2}$$

The expression (2) is the soliton solution of KdV equation when λ=1, and δ=6 with periodic boundary condition that would be replacing the usual boundary condition for solving asymptotic type which is given by

$$\lim\big(u(x,t)\big) \to 0, when\ x \to \infty, \tag{3}$$

in order to avoid any boundary effect in the propagation of waves under study.

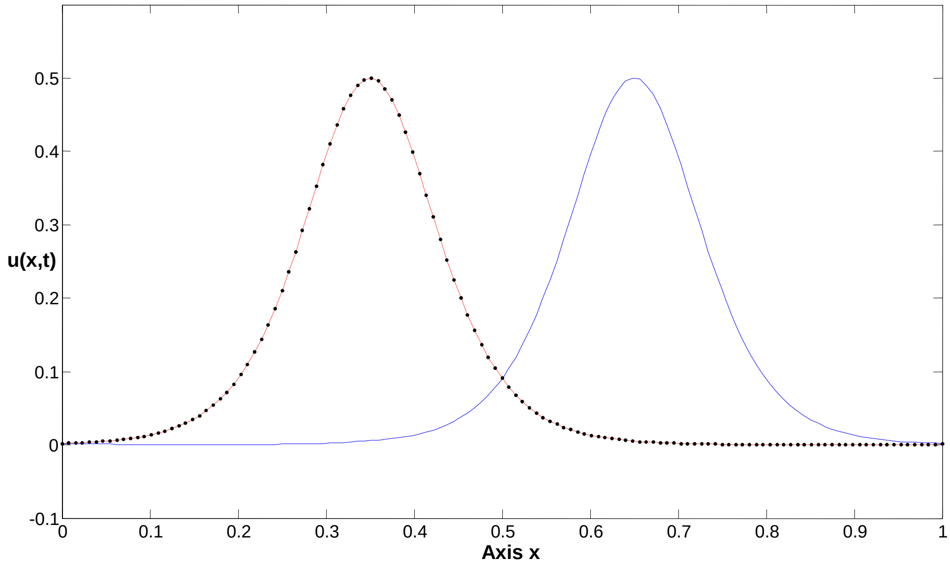

Figure 1: Propagation of solitonic waves for λ=1, and δ=6. The dashed line and dots (red) represents the initial condition and the analytical solution respectively, and the continuous (blue) the propagated wave in the numerical integration of the KdV equation.

The expression (2) can be obtained by analytical solution using many methods (see for example (Evans, 1997), (Strauss, 1992)).

In the case of Figures 1, 2 and 3 the expression (1) was solved numerically using a pseudospectral method in the spatial domain and a Runge Kutta scheme of fourth order in time integration that will be described in detail in the section of *Results*.

The mechanism by which the wave propagates without dispersion, despite containing a dispersive term, is that this effect is offset by the amplifying effect of the nonlinear component. These results can be seen in Figures 2 and 3. In the first, the nonlinear amplification is greater than the dispersive effect; but in the second one, the dissipative effect is greater than the nonlinear contribution.

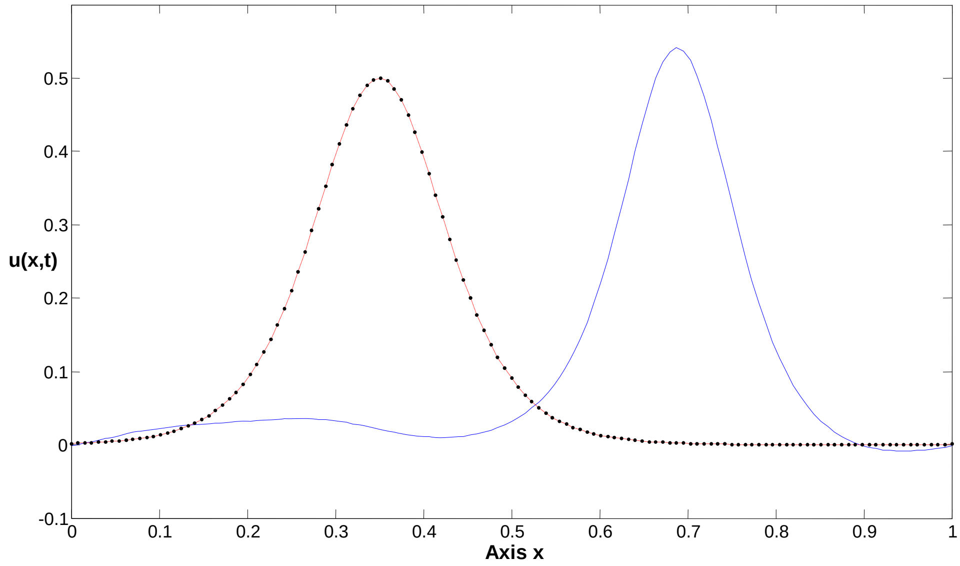

Figure 2: Propagation of non linear waves for λ=0.75, and δ=6. The dashed line and dots (red) represents the initial condition and the analytical solution respectively, and the continuous (blue) the propagated wave in the numerical integration of the KdV equation.

To achieve the balance between dispersion and nonlinear amplification values, the coefficients λ and δ were selected to λ=1 and δ=6 in order to maintain balanced both effects and thus lead to the spread of wave profiles as exposed in the expression (2) giving solutions such as solitary wave which propagates without changes, resulting in one of the distinguished features of solitons.

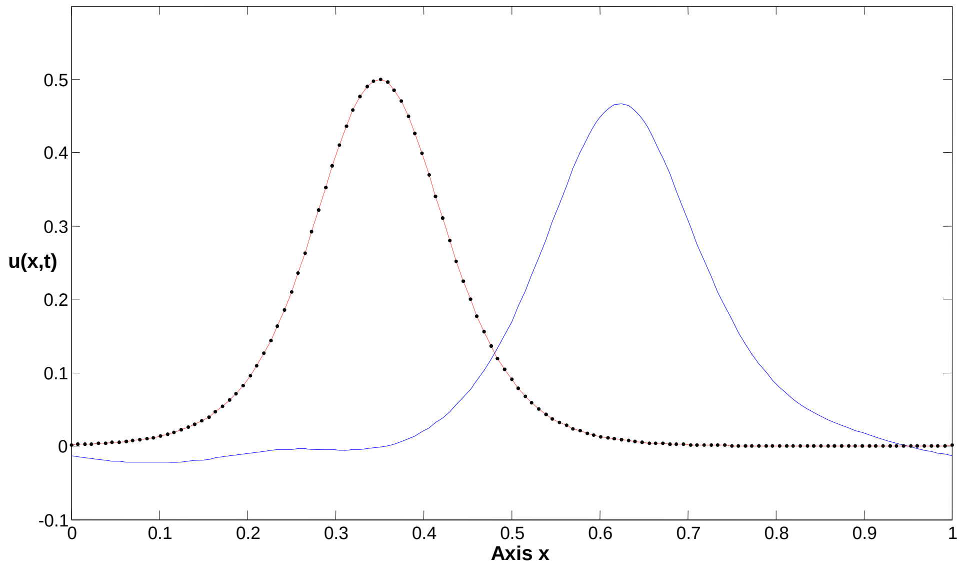

Figure 3: Propagation of non linear waves for λ=1.2, and δ=6. The dashed line and dots (red) represents the initial condition and the analytical solution respectively, and the continuous (blue) the propagated wave in the numerical integration of the KdV equation

On the other hand the expression (1) has and distinguished issue, it is invariant under some transformations of variables that become a continuous group of Lie transformations, in this way it is suggested the existence of invariant properties of the solution (Drazin and Johnson, 1989).

3 CONSERVATIONS LAWS OF THE KdV EQUATION

The first notice about the conservations laws derived from the KdV equation was prompted by Kruskal (Gardner et al., 1967) in his co-workers in Princeton when they were analytically investigating the NLPDE under consideration. In particular Miura (Gardner et al., 1967), (Miura, 1976) investigated local conservation laws like

$$\partial_t \varrho(x,t) + \partial_x(\varrho v) = 0, \tag{4}$$

where $\varrho(x,t)$ is the macroscopic particle density, and $v(x,t)$ is the streaming velocity. Equation (4) expresses the conservation law in the differential form of a continuity equation, and $\delta_h(.)$ means partial derivative respect to the independent variable h.

When the flux of the conserved quantity (ϱv) is integrated over the spatial domain $x \epsilon (-\infty,\infty)$, results

$$\frac{d}{dt}\left(\int_{-\infty}^{\infty} \varrho dx\right) = -\varrho v|_{-\infty}^{\infty}] = 0. \tag{5}$$

Equation (5) implies that

$$\int_{-\infty}^{\infty} \varrho\, dx = const. \tag{6}$$

The equation (6) is an integral form for the conserved quantity or a global conservation against the local one expressed by the equation (4).

In a general way of thinking it allows to introduce a general density T, and its current J, so the related conservation law can be written as

$$\partial_t T + \partial_x J = 0. \tag{7}$$

Following this reasoning it is easy to write the general continuity equation to nonlinear partial differential equation, for which it is necessary to assume that T and J are dependent on t, u, u_x, u_{xx}, and so on (from here suscripts denotes partial derivatives).

Considering the premise of equation (5), when $J \to 0$, a new expression is found when equation (7) is integrated over the spatial domain $x \epsilon (-\infty,\infty)$ with the corresponding boundary conditions

$$\frac{d}{dt}\int_{-\infty}^{\infty} T dx = 0. \tag{8}$$

These can be able to present an integral of motion:

$$\int_{-\infty}^{\infty} T\, dx = const, \tag{9}$$

which looks identical to equation (6).

For the case of the KdV equation, identifying $T_1=u$, and $J=u_{xx}-3u^2$, the equation (1) can be rewritten as

$$u_t - 6uu_{x+}u_{xxx} = 0. \tag{10}$$

And following the previous reasoning integrating over the spatial domain results

$$\int_{-\infty}^{\infty} u(x,t)\,dx = const. \tag{11}$$

When equation (10) is multiplied by u, appears a second conserved quantity as can be seen in equation (12), and in a similar way as pointed out before, after integrating in the spatial domain, appears a new conserved quantity showed in expression (13)

$$\partial_t\left(\frac{1}{2}\,u^2\right) + \partial_x\left\{uu_{xx} - \frac{1}{2}\,u^2{}_x \quad - 2u^3\right\} = 0, \tag{12}$$

$$\int_{-\infty}^{\infty} u^2\,dx = const. \tag{13}$$

In this way there have been appeared two conserved quantities from KdV equation, it means, equation (11) shows the conservation of mass, denoted as T_1, and equation (13) the conservation of momentum, denoted as T_2. Other physical conserved quantities can be identified as the energy density which is denoted as T_3

$$\mathrm{T}_3 = u^3 + \frac{1}{2}\,u^2{}_x \quad . \tag{14}$$

4 DERIVATION OF CONSERVATIONS LAWS

The procedure described trough equations (10) to (14) is done in an algebraic way and is hard to extend to higher orders than tree.

To solve this problem Miura and co-workers (Gardner et al, 1967) developed a special method to found the infinite hierarchy of conservation laws, precisely named the Miura transformation. It starts introducing the modified KdV equation called mKdV

$$v_t + 6v^2v_x + v_{xxx} = 0. \tag{15}$$

The theory applied to carry out with this transformation can be found in (Ashok, 1989) and (Baumann, 1996).

The transformation of the field v into u is shown in the equation (16).

$$u(x,t) = v^2(x,t) + v_x(x,t). \tag{16}$$

Expression (16) is called sometimes a Riccati transformation (Ashok, 1989). Now the field given by v can be written in terms of a new variable w and an arbitrary parameter ε.

$$v = \frac{1}{2\varepsilon} + \varepsilon\omega, \quad \varepsilon \in \Re. \tag{17}$$

Combining expression (16) with (17) results

$$u = \frac{1}{4}\,\frac{1}{\varepsilon^2} + \omega + \varepsilon^2\,\omega^2 + \varepsilon\omega_x. \tag{18}$$

Assuming an additional Galilean invariance for u of the form

$$\tilde{u} = u + \lambda, \tag{19}$$

the expression (18) can be rewritten as

$$u = \omega + \varepsilon\omega_x + \varepsilon^2\,\omega^2. \tag{20}$$

The steps showed from equation (16) to (20) constitute the called Gardner transformation (Baumann, 1996). Finally replacing the transformation (20) into the KdV equation (1) gives

$$\begin{aligned} &u_t - 6uu_x + u_{xxx} \\ &= \omega_t + \varepsilon\omega_{xt} + 2\varepsilon^2\omega\omega_t - 6(\omega + \varepsilon\omega_x + \varepsilon^2\,\omega^2) \\ &\cdot (\omega_x + \varepsilon\omega_{xx} + 2\varepsilon^2\omega\omega_x) + \omega_{xxx} + \varepsilon\omega_{xxxx} + 2\varepsilon^2(\omega\omega_x)_{xx}\,. \end{aligned} \tag{21}$$

The Miura transformation assures that if *u* is a solution of the equation (1) *w* is a solution of the same equation too (Ashok, 1996) and (Gardner, 1989)

$$u_t - 6uu_x + u_{xxx} = \left(1 + \varepsilon\frac{\partial}{\partial x} + 2\varepsilon^2\omega\right)\cdot\{\omega_t + 6(\omega + \varepsilon^2\,\omega^2)\omega_x + \omega_{xxx}\}\,. \tag{22}$$

Due to *u* is a solution of the KdV equation, *w* is a solution too (equation (23)).

$$\omega_t - 6(\omega + \varepsilon^2\,\omega^2)\omega_x + \omega_{xxx} = 0. \tag{23}$$

Recognizing the conservation form in (23) which gives a conserved quantity

$$\partial_t\omega + \partial_x(\omega_{xx} - 3\omega^2 - 2\varepsilon^2\,\omega^3) = 0. \tag{24}$$

In the same way used before in equation (11)

$$\int_{-\infty}^{\infty}\omega\,dx = const. \tag{25}$$

Up to now, the scheme to obtain the infinite string of conservation laws starts expanding *w* as a power series in ε. It is important to highlight that when ε → 0, *w* converges to *u* (Baumann, 1996).

$$\omega(x,t;\varepsilon) = \sum_{n=0}^{\infty}\varepsilon^n\,\omega_n(x,t). \tag{26}$$

When the expansion given by (26) is replaced into the Miura transformation (20).

$$\int_{-\infty}^{\infty}\omega\,dx = \sum_{n=0}^{\infty}\varepsilon^n\int_{-\infty}^{\infty}\omega_n\,dx = const. \tag{27}$$

$$\int_{-\infty}^{\infty}\omega_n\,dx = const \quad for\; n = 0,1,2,\ldots \tag{28}$$

Replacing the power series expansion of *w* in (20) gives

$$\sum_{n=0}^{\infty}\varepsilon^n\,\omega_n = u - \varepsilon\sum_{n=0}^{\infty}\varepsilon^n\,\omega_{nx} - \varepsilon^2\left(\sum_{n=0}^{\infty}\varepsilon^n\,\omega_n\right)^2, \tag{29}$$

Finally after comparing the same order in the series expansion in both sides of the identity (29) appears the conserved quantities, whose first terms are

$$\omega_0 = u. \tag{30}$$

$$\omega_1 = -\omega_{0x} = -\,u_x, \tag{31}$$

$$\omega_2 = -\omega_{1x} - \,\omega_0^2 = u_{xx} \,-\, u^2, \tag{32}$$

$$\omega_3 = -\omega_{2x} - \,2\omega_0\omega_1 = -\,(u_{xx} - u^2)_x + 2uu_x\,. \tag{33}$$

At this point is clear that is easy to construct the conserved densities, in the sense of the equation (7), recursively and is important to keep clear that these conserved densities can be arbitrary up to multiplicative constants and addition of total derivatives.

5 RESULTS

The numerical integration of the KdV was done by using a pseudospectral method in space.

The last one was composed by 1024 Fourier modes, of Fourier harmonics, lower numbers of harmonic produces a numerical artifact in the soliton interaction and propagation (small numerical oscillations). The length of the physical domain (space) was constituted by a regular mesh of size 0.013 (in arbitrary units) with the corresponding 1024 wave numbers, for this reason the scheme used is called pseudospectral method reserving the word spectral when no mesh is used (Canuto et al, 2007). The total time of integration was 2 seconds, and to simplify the lecture of the results the length of the spatial domain was normalized to unity. The heights of solitons were 2 and 4 to ensure the difference in the respective velocities. For the time integration a Runge Kutta of order fourth was applied using stability criteria given by (Trefethen, 2000).

The expressions for the first eleven conserved quantities are shown in the Table 1.

Results evident that the conservations above the third have a large algebraic structure (this stand for many terms and high degree derivatives) which requires careful handling and is one of the aims of this paper to put in evidence the computational aid necessity.

As was pointed out before the recursive scheme to obtain the conservation densities encourage the implementation in an algorithmic procedure with a symbolic plataform. To carry out with this a small computational code was implemented upon the base of the proposal of (Baumann, 1996).

Order of expansion	Conserved Density
0	u
1	$-u^{(1,0)}$
2	$-u^{2}+u^{(2,0)}$
3	$u\ u^{(1,0)}-u^{(3,0)}$
4	$2\ u^{3}-5\ u^{(1,0)2}-6\ u\ u^{(2,0)}+u^{(4,0)}$
5	$-16\ u^{2}\ u^{(1,0)}+18\ u^{(1,0)}\ u^{(2,0)}+8\ u\ u^{(3,0)}-u^{(5,0)}$
6	$-5\ u^{4}+30\ u^{2}\ u^{(2,0)}-19\ u^{(2,0)2}-28\ u^{(1,0)}\ u^{(3,0)}+10\ u\ (5\ u^{(1,0)2}-u^{(4,0)})+u^{(6,0)}$
7	$64\ u^{3}\ u^{(1,0)}-60\ u^{(1,0)3}-48\ u^{2}\ u^{(3,0)}+68\ u^{(2,0)}\ u^{(3,0)}+40\ u^{(1,0)}\ u^{(4,0)}+12\ u\ (-18\ u^{(1,0)}\ u^{(2,0)}+u^{(5,0)})-u^{(7,0)}$
8	$14\ u^{5}-140\ u^{3}\ u^{(2,0)}+442\ u^{(1,0)2}\ u^{(2,0)}-69\ u^{(3,0)2}-70\ u^{2}\ (5\ u^{(1,0)2}-u^{(4,0)})-110\ u^{(2,0)}\ u^{(4,0)}-54\ u^{(1,0)}\ u^{(5,0)}+14\ u\ (19\ u^{(2,0)}+28\ u^{(1,0)}\ u^{(3,0)}-u^{(6,0)})+u^{(8,0)}$
9	$-256\ u^{4}\ u^{(1,0)}+256\ u^{3}\ u^{(3,0)}-900\ u^{(1,0)2}\ u^{(3,0)}+250\ u^{(3,0)}\ u^{(4,0)}+96\ u^{2}\ (18\ u^{(1,0)}\ u^{(2,0)}-u^{(5,0)})+166\ u^{(2,0)}\ u^{(5,0)}+u^{(1,0)}\ (-1224\ u^{(2,0)2}+70\ u^{(6,0)})+16\ u\ (60\ u^{(1,0)3}-68\ u^{(2,0)}\ u^{(3,0)}-40\ u^{(1,0)}\ u^{(4,0)}+u^{(7,0)})-u^{(9,0)}$
10	$-42\ u^{6}-1105\ u^{(1,0)4}+630\ u^{4}\ u^{(2,0)}+1262\ u^{(2,0)3}+420\ u^{3}\ (5\ u\hat{}(1,0)^{2}-u^{(4,0)})+1630\ u^{(1,0)2}\ u^{(4,0)}-251\ u^{(4,0)2}-418\ u^{(3,0)}\ u^{(5,0)}-126\ u^{2}\ (19\ u^{(2,0)2}+28\ u^{(1,0)}\ u^{(3,0)}-u^{(6,0)})-238\ u^{(2,0)}\ u^{(6,0)}+u^{(1,0)}\ (5564\ u^{(2,0)}\ u^{(3,0)}-88\ u^{(7,0)})-18\ u\ (442\ u^{(1,0)2}\ u^{(2,0)}-69\ u^{(3,0)2}-110\ u^{(2,0)}\ u^{(4,0)}-54\ u^{(1,0)}\ u^{(5,0)}+u^{(8,0)})+u^{(10,0)}$

Table 1: First eleven conserved densities obtained from equation (29)

The notation used in Table 1 is $u^{(n,0)} = \frac{\partial^n u}{\partial x^n}$, and the superscript corresponds to the usual power of the basis.

The experiment to compute numerically the expressions of the Table 1 is composed by the propagation of two solitons of different higths (this implies different velocity of propagation) and located in two distinct places along the line of propagation to differentiate very well the shape of each one avoiding the initial overlapping of them.

Some of the densities showed in the Table 1 are plotted in Figure 4. In the upper row appears plotted the soliton before the interaction (black line) and after the interaction (red line). In the following rows appears the densities plotted using the same color convention used for the soliton propagation. In the rest of the work the order "n" of expansion in Table 1 is associated with the name of the corresponding "density n" to do a short reference to a certain conserved density.

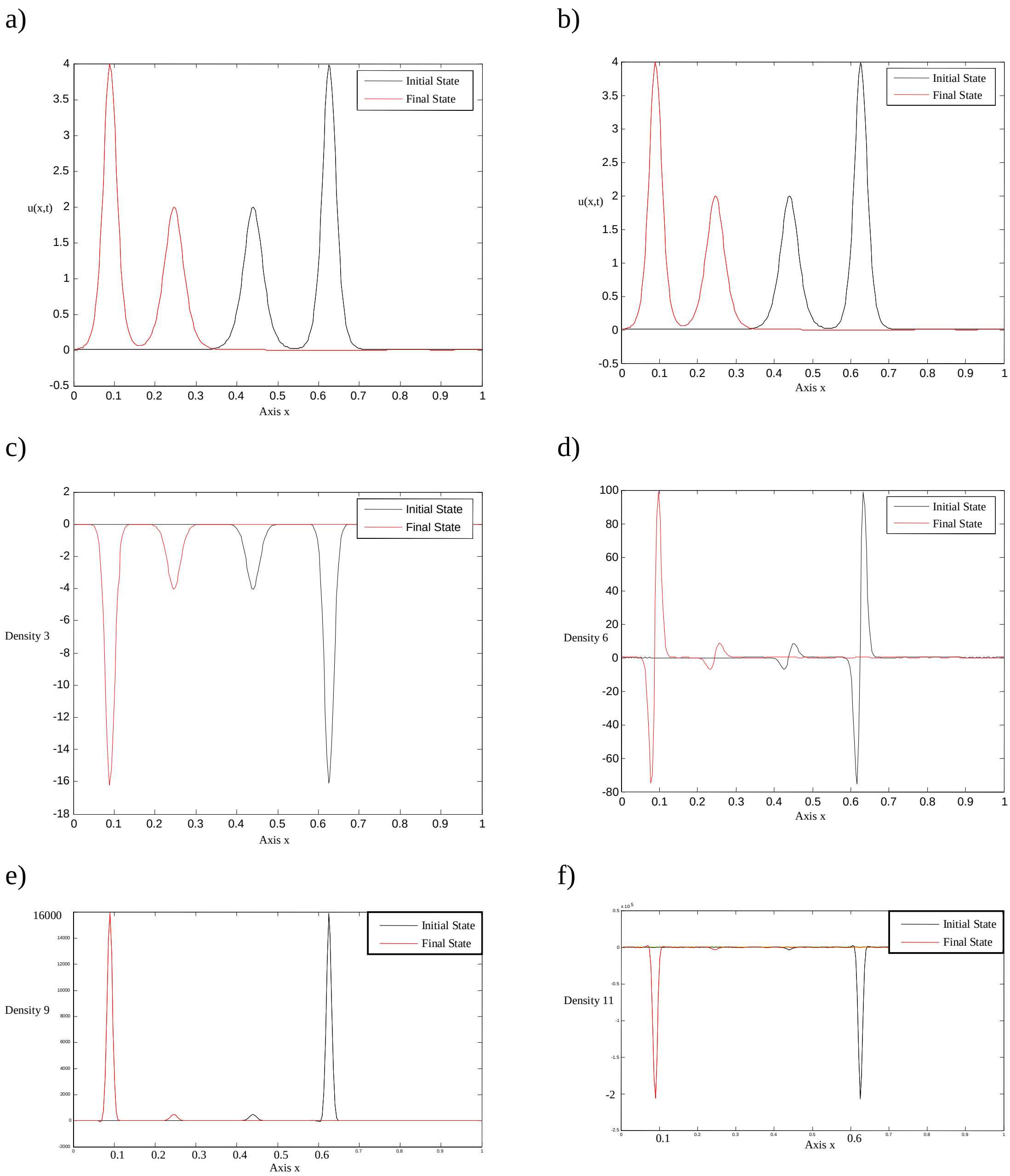

Figure 4: a) and b) soliton propagation, c) density 3, d) density 6, e) density 9, f) density 11

As can be seen in the Figure 4 the locations of the density looks to match with the position of the corresponding location of the solitons. This is a new concept that never has seen before in the literature. So the conservation law given by (9) is a global concept and is evaluated numerically in the next paragraph, but the densities make a trace of the soliton propagation.

Table 2 shows the results of the corresponding conservation laws in the initial state, previous to the soliton interaction and after the interaction when is numerically integrated over the spatial domain according to equation (9).

	Initial State	After Interaction	Percent of difference
0	7.7254834e+001	7.7254834e+001	0.00
1	5.9065900e-004	5.9065900e-004	0.00
2	-1.6335844e+002	-1.6335904e+002	0.00
3	-1.7625781e+000	-1.7576357e+000	0.28
4	9.1159517e+002	9.1100436e+002	0.06
5	1.1956318e+002	1.1931285e+002	0.21
6	-7.2540379e+003	-7.2455209e+003	0.12
7	-1.8443260e+003	-1.8398479e+003	0.24
8	6.8432711e+004	6.8318643e+004	0.17
9	2.8190447e+004	2.8109578e+004	0.29
10	-7.4539189e+005	-7.4377232e+005	0.22

Table 2: Results of the numerical integration of the densities of Table 1 and percents of difference.

Other integration techniques were applied to evaluate the conservation laws. In that sense the differences in the value of the conservation laws between the initial (previous of soliton interaction, black lines in Figure 4) and final state (after soliton interaction, red lines in Figure 4) using a Simpson 3/8 rule and a first generic order method of integration are compared in Table 3.

	First order generic method of integration	Simpson 3/8
0	0	0,00731216
1	0,0001	0,0001
2	-0,00082028	-0,00082028
3	-0,00036729	-0,00036729
4	0,064810567	0,06482483
5	0,209370477	0,20940412
6	0,11741047	0,11749039
7	0,242804146	0,24304975
8	0,166686367	0,16737992
9	0,286866682	0,28873779
10	0,217277652	0,22141691

Table 3: Comparison between the numerical evaluation of the conservation laws between the initial and final states using two different methods of integration, a generic first order (trapezoidal rule) and a Simpson 3/8.

6 CONCLUSIONS

The results of this paper contribute to see in a different way the densities conserved from the integration of the KdV equation.

As an aspect of the work the global amounts of the conserved quantities can be used, for the kind of equations composed by the KdV, as a validation of the numerical integration technique.

In an another way the densities itself constitutes a trace of the propagation of the solitons, are very well localized in space, as results from Figure 4 the centers of each soliton appears to match very well with the corresponding center of the shape given by the density associated with each density corresponding a conserved law derived from the KdV equation.

In a certain way the null contribution given increasing the precision of the integration technique showed in Table 3 is surprising because the theoretical improvement between a first order (trapezoidal rule) and the Simpson 3/8 is considerable but useless in this kind of application.

Finally the localization of the densities is one of the imminent lines of research to continue with the numerical studies of the soliton solutions of the KdV equation.

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