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EIGENFREQUENCIES OF GENERALLY RESTRAINED TIMOSHENKO BEAMS WITH AN INTERNAL HINGE

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Abstract. This paper deals with the free transverse vibration of a Timoshenko beam with ends elastically restrained against rotation and translation, and an arbitrarily located internal hinge including intermediate elastic constraints. A combination of the Ritz method and the Lagrange multiplier method is used to determine free vibrations characteristics of the mentioned beam. Trial functions denoting the transverse deflections and the normal rotations of the cross section of the beam are expressed in polynomial forms.
In order to obtain an indication of the accuracy of the developed mathematical model, some cases available in the literature have been considered. New results are presented for different end conditions and restraint conditions in the intermediate elastic constraints. Also a comparison with a crack model is included.

1 INTRODUCTION

Timoshenko proposed a beam theory which adds the effects of shear distortion and the rotatory inertia to the Euler-Bernoulli model (Timoshenko, 1921; Timoshenko, 1922). Afterwards there has been a considerable interest in developing techniques for the solutions of equations according to the Timoshenko theory. The problem of free vibration of Timoshenko beams with classical end conditions has been extensively treated and numerous papers have been devoted to it. The first papers are described in Quintana and Grossi (2009), but it is not possible to give a detailed account because of the great size of information, nevertheless some important references will be cited. The problem of elastic end restraints has also received considerable attention. Abbas (1984), treated the problem of free vibration of Timoshenko beams with elastically supported ends by using a finite element model (FEM) which satisfies all the geometric and natural boundary conditions. Farghaly (1994), investigated the natural frequencies and the critical buckling load coefficients for a multi-span Timoshenko beam elastically supported. Kocaturk and Simsek (2005a,b), analyzed the free vibrations of Timoshenko beams having classical and elastically supported ends by using the Lagrange equations with the trial functions expressed in the power series form. Zhou (2001), analyzed the free vibration of multi-span Timoshenko beams by the Rayleigh-Ritz method using static Timoshenko beam functions. Grossi and Aranda (1993), applied the Ritz method in the variational formulation of Timoshenko beams with elastically restrained ends. Han et al. (1999), presented a full development and analysis of four theories, including the Timoshenko model, for the transversely vibrating uniform beam.

A review of the literature further reveals that there is only a limited amount of information for the vibration of Bernoulli-Euler beams with internal hinges. Ewing and Mirsafian (1996), analyzed the forced vibrations of two beams joined with a non-linear rotational joint. Wang and Wang (2001), studied the fundamental frequency of a beam with an internal hinge and subjected to an axial force. Chang et al. (2006) investigated the dynamic response of a beam with an internal hinge, subjected to a random moving oscillator. Grossi and Quintana (2008) analyzed the free transverse vibration of a non-homogeneous tapered beam subjected to general axial forces, with arbitrarily located internal hinge and elastics supports and ends elastically restrained against rotation and translation.

The problem of vibration of Timoshenko beams with internal hinges, out of the context of cracks, has not been treated with exception of Lee et al. (2003) who considered a Timoshenko beam with an internal hinge by determining the exact vibration frequencies.

The aim of the present paper is to investigate the natural frequencies and mode shapes of a Timoshenko beam with several complicating effects such as intermediate elastic constraints, generally restrained ends and an intermediate internal hinge. Several cases are solved by a combination of the Ritz method and the Lagrange multiplier method in conjunction with sets of simple polynomials as trial functions. In order to obtain an indication of the accuracy of the developed mathematical model, some cases available in the literature have been considered and comparisons of numerical results are included. The algorithms developed can be applied to a wide range of the different elastic restraint conditions. A great number of problems were solved and, since this number of cases is prohibitively large, results are presented for only a few cases. Since the presence of intermediate elastic restraints and a hinge allow the simulation of a crack model, a comparison with results of Khaji et al. (2009) has been included.

2 THEORY AND FORMULATIONS

Let us consider a uniform Timoshenko of length l, which has elastically restrained ends, is constrained at an intermediate point and has an internal hinge elastically restrained against rotation, as shown in Figure 1.

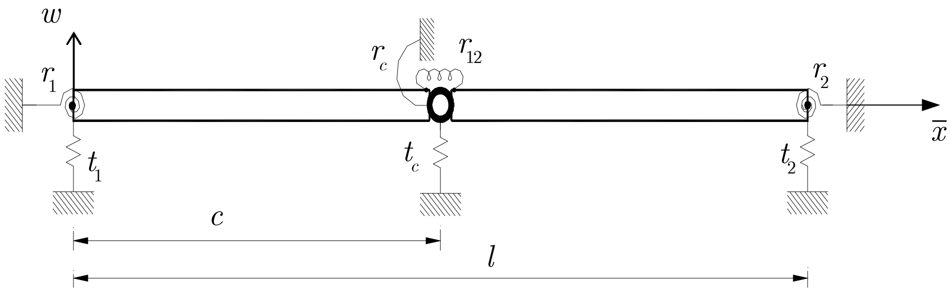


Figure 1: Beam model description

According to Timoshenko beam theory, two independent variables: transverse deflection w and normal rotational angle ϕ due to bending are used to describe the deformation of the beam. The elastic strain energy due to the beam and to the elastic restraints at any instant t is given by

$$U = \frac{1}{2}\int_0^l \left\{ EI\left(\frac{\partial \phi(\overline{x},t)}{\partial \overline{x}}\right)^2 + kGA\left(\frac{\partial w(\overline{x},t)}{\partial \overline{x}} - \phi(\overline{x},t)\right)^2 \right\} d\overline{x} + \frac{1}{2}\Big[t_1 w^2(0,t) + r_{12}\left(\phi\left(c^+,t\right) - \phi\left(c^-,t\right)\right)^2 + r_1\phi^2(0,t) + t_c w^2(c,t) + r_c\phi^2(c^-,t) + t_2 w^2(l,t) + r_2\phi^2(l,t)\Big], \tag{1}$$

where E is the Young's modulus, G is the transverse shear modulus, I is the moment of inertia, A is the area of the cross-section and k is the shear correction factor. The rotational restraints are characterized by the spring constants r_1, r_2, r_c and r_{12} and the translational restraints by the spring constants t_1, t_2 and t_c.

The kinetic energy of the beam at any instant t is given by

$$T = \frac{1}{2}\int_0^l \left\{ \rho A\left(\frac{\partial w(\overline{x},t)}{\partial t}\right)^2 + \rho I\left(\frac{\partial \phi(\overline{x},t)}{\partial t}\right)^2 \right\} d\overline{x}, \tag{2}$$

where ρ is the mass per unit volume.

When the beam executes free vibrations, transverse deflection and normal rotation can be written as

$$w(\overline{x},t) = \overline{W}(\overline{x})\sin(\omega t), \quad \phi(\overline{x},t) = \overline{\Phi}(\overline{x})\sin(\omega t), \tag{3}$$

where ω is the radian frequency.

By introducing the following non-dimensional parameters

$$x = \frac{\overline{x}}{l}, W = \frac{\overline{W}}{l}, \Phi = \overline{\Phi}, \tag{4}$$

the Lagrangian functional L_0 of the problem can be written as

$$\begin{aligned} L_0 &= U - T = \\ &= \frac{1}{2}\int_0^1 \left\{ \left(\frac{d\Phi}{dx}\right)^2 + \gamma\left(\frac{l}{r}\right)^2 \left(\frac{dW}{dx} - \Phi\right)^2 \right\} dx + \frac{1}{2}\Big[T_1 W^2(0) + R_1 \Phi^2(0) + \\ &+ R_{12}\left(\Phi^2(c_l^+) - \Phi^2(c_l^-)\right) + T_c W^2(c_l) + R_c \Phi^2(c_l) + \\ &+ T_2 W^2(1) + R_2 \Phi^2(1)\Big] - \frac{1}{2}\Omega^2 \int_0^1 \left[\left(\frac{r}{l}\right)^2 \Phi^2 + W^2 \right] dx, \end{aligned} \tag{5}$$

where:

$$\gamma = \frac{kG}{E};\ r = \sqrt{\frac{I}{A}},\ \Omega = \omega l^2 \sqrt{\frac{\rho A}{EI}},\ c_l = \frac{c}{l},$$

$$T_i = \frac{t_i l^3}{EI}, R_i = \frac{r_i l}{EI}, i = 1,2,\ R_c = \frac{r_c l}{EI}, R_{12} = \frac{r_{12} l}{EI},\ T_c = \frac{t_c l^3}{EI}.$$

2.1 Combination of the Ritz method and the Lagrange multiplier method.

Since it is difficult to construct a simple and adequate deflection function which can be applied to the entire beam and to show the continuity of displacement and the discontinuities of the slope crossing the hinge, the minimization of the functional given by Eq. (5) will be achieved using subsidiary conditions. In consequence we can assume that $W(x)$ and $\Phi(x)$ are given by

$$\begin{aligned} W(x) &= \begin{cases} W_1(x) \forall x \in [0, c_l] \\ W_2(x) \forall x \in [c_l, l] \end{cases}, \\ \Phi(x) &= \begin{cases} \Phi_1(x) \forall x \in [0, c_l] \\ \Phi_2(x) \forall x \in [c_l, l]. \end{cases} \end{aligned} \tag{6}$$

Considering the compatibility requirement on the intermediate elastically restrained point, the relationships between two adjacent spans can be expressed as

$$W_1(c_l) - W_2(c_l) = 0. \tag{7}$$

Now the problem can be posed as one of extremizing the given functional in Eq. (5) subjected to the following constraint:

$$H = W_1(c_l) - W_2(c_l). \tag{8}$$

This constraint may be incorporated into the energy functional given by Ec. (5) by using the Lagrange multiplier method (Reddy, 1986) as:

$$L_L = L_0 + \lambda H, \tag{9}$$

where L_L is called the Lagrangian functional, and $\lambda \in \mathbb{R}$ is a time independent Lagrangian multiplier.

The transverse deflection and the normal rotation can be represented by a set of characteristic polynomials $p_{ki}(x)$ and $q_{kj}(x)$, as:

$$\Phi_k = \sum_{i=1}^{N} a_{ki} p_{ki}\left(x\right), \quad k = 1,2 \tag{10}$$

$$W_k = \sum_{j=1}^{M} b_{kj} q_{kj}\left(x\right), \quad k = 1,2, \tag{11}$$

where both a_{ki} and b_{kj} are unknown coefficients to be determined and $p_{ki}(x)$, $q_{kj}(x)$ are the trial functions. It is sufficient that they satisfy the geometric boundary conditions of the beam since, as the number of trial functions approaches infinity, the natural boundary conditions will be exactly satisfied (Mikhiln, 1964). The first member of the set $p_{11}(x)$ is obtained as the simplest polynomial that satisfies at least the geometric boundary condition of the first span.

Assume that

$$p_{11}(x) = \sum_{i=1}^{5} \overline{a}_{1i} x^{i-1} \tag{12}$$

where the arbitrary constants $\overline{a}_{1i}$ are determined by substituting Eq. (12) into the above-mentioned boundary conditions. In the case of beam involving free edges or ends elastically restrained against rotation and translation simpler starting member of zero order are used.

The higher members of the set $\left\{p_1\right\}$ are obtained as:

$$p_{1i} = p_{11} x^{i-1}, \quad i = 2,3,..,N. \tag{13}$$

The polynomials set $\left\{p_2\right\}$ and $\left\{q_k\right\}$ are also generated using the same procedure. Thus

$$p_{ki} = p_{k1} x^{i-1}, \quad i = 2,3,..,N, \tag{14}$$

$$q_{kj} = q_{k1} x^{j-1}, \quad j = 2,3,..,N, \quad k = 1,2. \tag{15}$$

In the present paper, beams having a variety of boundary conditions are considered, and the starting functions used are given in the Appendix.

Substituting Eq. (10) and (11) into Eq. (9), and minimizing with respect to the unknown coefficients a_{ki}, b_{kj} and the Lagrangian multiplier λ, one obtains

$$\frac{\partial L_L}{\partial a_{ki}} = 0, \quad i = 1,2,...,N, \quad k = 1,2 \tag{16}$$

$$\frac{\partial L_L}{\partial b_{kj}} = 0, \quad j = 1,2,...,M, \quad k = 1,2 \tag{17}$$

$$\frac{\partial L_L}{\partial \lambda} = 0, \tag{18}$$

By using Eq. (13)-(15) the following simultaneous set of linear algebraic equations are obtained which can be expressed in the following matrix forms

$$\left(\left[K\right]-\Omega^2\left[M\right]\right)\left\{\overline{c}\right\}=\left\{0\right\} \tag{19}$$

where

$$\left[K\right]=\begin{bmatrix} \left[K_{aa}^{(1)}\right] & \left[K_{ab}^{(1)}\right] & \left[K_{aa}^{(1,2)}\right] & \left[0\right] & \left[L_{a\lambda}^{(1)}\right] \\ & \left[K_{bb}^{(1)}\right] & \left[0\right] & \left[0\right] & \left[0\right] \\ & & \left[K_{aa}^{(2)}\right] & \left[K_{ab}^{(2)}\right] & \left[L_{a\lambda}^{(2)}\right] \\ & \text{symm} & & \left[K_{bb}^{(2)}\right] & \left[0\right] \\ & & & & \left[0\right] \end{bmatrix}, \tag{20}$$

$$\left[M\right]=\begin{bmatrix} \left[M_{aa}^{(1)}\right] & \left[0\right] & \left[0\right] & \left[0\right] & \left[0\right] \\ & \left[M_{bb}^{(1)}\right] & \left[0\right] & \left[0\right] & \left[0\right] \\ & & \left[M_{aa}^{(2)}\right] & \left[0\right] & \left[0\right] \\ & & & \left[M_{bb}^{(2)}\right] & \left[0\right] \\ & \text{symm} & & & \left[0\right] \end{bmatrix}, \tag{21}$$

$$\left\{\overline{c}\right\}=\left\{\left\{a_1\right\},\left\{b_1\right\},\left\{a_2\right\},\left\{b_2\right\},\left\{\lambda\right\}\right\}^T, \tag{22}$$

with

$$\begin{aligned} \left\{a_k\right\} &= \left\{a_{k1},a_{k2},\ldots,a_{kN}\right\}^T \\ \left\{b_k\right\} &= \left\{b_{k1},b_{k2},\ldots,b_{kM}\right\}^T, \quad k=1,2. \end{aligned} \tag{23}$$

The expressions for the various elements of the stiffness matrix $\left[K\right]$ and the mass matrix $\left[M\right]$ are the following

$$\begin{aligned} K_{aaij}^{(1)} &= \int_0^{c_l}\left[\frac{dp_{1i}(x)}{dx}\frac{dp_{1j}(x)}{dx}+\gamma\left(\frac{l}{r}\right)^2 p_{1i}(x)p_{1j}(x)\right]dx+R_1p_{1i}(0)p_{1j}(0)+ \\ &+R_cp_{1i}(c_l)p_{1j}(c_l)+R_{12}p_{1i}(c_l)p_{1j}(c_l), \end{aligned} \tag{24}$$

$$K_{abij}^{(1)}=-\int_0^{c_l}\gamma\left(\frac{l}{r}\right)^2 p_{1i}\frac{dq_{1j}}{dx}dx, \tag{25}$$

$$K_{aaij}^{(1,2)}=-R_{12}p_{1i}(c_l)p_{2j}(c_l), \tag{26}$$

$$K_{bbjn}^{(1)} = \int_0^{c_l} \gamma \left(\frac{l}{r}\right)^2 \frac{dq_{1j}(x)}{dx}\frac{dq_{1n}(x)}{dx}dx + T_1 q_{1j}(0)q_{1n}(0) + T_c q_{1j}(c_l)q_{1n}(c_l), \tag{27}$$

$$L_{a\lambda i1}^{(1)} = q_{1j}\left(c_l\right), \tag{28}$$

$$K_{aaim}^{(2)} = \int_{c_l}^{1}\left[\frac{dp_{2i}(x)}{dx}\frac{dp_{2m}(x)}{dx} + \gamma\left(\frac{l}{r}\right)^2 p_{2i}(x)p_{2m}(x)\right]dx + R_2 p_{2i}(1)p_{2m}(1) + R_{12}p_{2i}(c_l)p_{2j}(c_l), \tag{29}$$

$$K_{abij}^{(2)} = -\int_{c_l}^{1} \gamma\left(\frac{l}{r}\right)^2 p_{2i}\frac{dq_{2j}}{dx}dx, \tag{30}$$

$$K_{bbjn}^{(2)} = \int_{c_l}^{1} \gamma\left(\frac{l}{r}\right)^2 \frac{dq_{2j}(x)}{dx}\frac{dq_{2n}(x)}{dx}dx + T_2 q_{2j}(1)q_{2n}(1), \tag{31}$$

$$L_{a\lambda i1}^{(2)} = -q_{2j}\left(c_l\right), \tag{32}$$

$$M_{aaim}^{(1)} = \int_0^{c_l}\left(\frac{r}{l}\right)^2 p_{1i}(x)p_{1m}(x)dx, \tag{33}$$

$$M_{bbjn}^{(1)} = \int_0^{c_l} q_{1j}(x)q_{1n}(x)dx, \tag{34}$$

$$M_{aaim}^{(2)} = \int_{c_l}^{1}\left(\frac{r}{l}\right)^2 p_{2i}(x)p_{2m}(x)dx, \tag{35}$$

$$M_{bbjn}^{(2)} = \int_{c_l}^{1} q_{2j}(x)q_{2n}(x)dx, \tag{36}$$

with

$$i,m = 1,2,\ldots,M,$$
$$j,n = 1,2,\ldots,N.$$

and

$$M_{bbij}^{(1)} = \int_0^{c_l} q_{1i}(x)q_{1j}(x)dx, \tag{37}$$

$$M_{aaij}^{(2)} = \int_{c_l}^{1}\left(\frac{r}{l}\right)^2 p_{2i}(x)p_{2j}(x)dx, \tag{38}$$

$$M_{bbij}^{(2)} = \int_{c_l}^{1} q_{2_i}(x) q_{2_j}(x) dx, \tag{39}$$

with

$$k, n = 1, 2, \ldots, N,$$
$$j, m = 1, 2, \ldots, M.$$

The eigenvalues Ω^2 are found from the condition that the determinant of the system of equations given by Eq. (19) must vanish.

3 CONVERGENCE AND COMPARISON STUDY

The entire beam was considered with the same material properties and beam section, therefore $\Omega_{1,i} = \Omega_{2,i} = \Omega_i$ for the i natural frequency, where $\Omega_{1,i}$ is the dimensionless natural frequency parameter of the first span and $\Omega_{2,i}$ the one of the second span. The values of the frequency parameter Ω were obtained for different end conditions and intermediate elastic restraints. Through all the present analysis, beams were modeled with shear correction factor $k = 5/6$ and Poisson's ratio $\mu = 0.3$.

The computations in this paper were performed by using Maple (TM). The routine computes in exact way the definite integral over the straight line from x_0 to x_1. The eigenvalues are computed by the QR method. The matrix is first balanced and transformed into upper Hessenberg form. Then the eigenvalues are computed.

A convergence study of the first six values of the dimensionless frequency parameter Ω of a simply-simply supported (S-S) and a clamped-clamped beam (C-C) with an intermediate support located at $c_l = 0.4$ for $\sqrt{12}\ r / l = 0.1$ are presented in Table 1. The convergence of the mentioned eigenvalues is studied by gradually increasing the number of the trial functions. A comparison of values with those of Zhou (2001) is also included. The table shows that $N = M = 11$ in the Ritz with Lagrange multipliers method is enough to reach stable convergence in all cases and to give results with the same precision and that the agreement with the values of Zhou (2001) is excellent. To compare results with those used in a crack model, a comparison with the model used in Khaji et al. (2009) is presented. The cracked section of the Timoshenko beam was modeled as local flexibility that was assumed to be a rotational spring. This model was first proposed by Ostachowicz and Krawczuk (1991) from a theory based on the stress intensity factor developed previously by Haisty and Springer (1988). Later, Narkis (1994) compared the results of this model with three different authors and a FEA model, Narkis (1994) and Khaji et al. (2009) used this model to solve the inverse problem of identify crack locations and crack depths from frequency data first obtained from a FEA model. The comparison of this works shown that the crack model proposed had an excellent performance.

The discontinuity in the slope of the beam was modeled as:

$$\left. \left(\frac{\partial W_2}{\partial x} - \frac{\partial W_1}{\partial x} \right) \right|_{x=c} = \theta \cdot \left. \frac{\partial \Phi_2}{\partial x} \right|_{x=c} \tag{40}$$

where $\theta = 6\pi\eta^2 f(\eta)\left(h/l\right)$ is the non-dimensional crack sectional flexibility and depends on the extension of the crack, $\eta = a/h$ is the crack depth ratio where a is the crack depth and h is the beam depth.

Boundary conditions	N=M	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6
S-S	3	35.9280	78.4428	139.1719	250.1963	679.3662	690.4346
	4	31.3524	67.3305	129.6826	226.1855	268.0760	429.2906
	5	31.3505	67.0022	104.4706	194.6074	255.8360	396.4166
	6	31.3372	66.9616	104.3777	186.7849	205.4627	349.6778
	7	31.3371	66.9553	103.9240	186.5327	204.6538	300.6535
	8	31.3371	66.9552	103.9223	185.3377	203.2441	300.4937
	9	31.3371	66.9552	103.9196	185.3368	203.2227	293.0330
	10	31.3371	66.9552	103.9196	185.3184	203.1968	293.0235
	11	31.3371	66.9552	103.9196	185.3184	203.1966	292.7686
	12	31.3371	66.9552	103.9196	185.3183	203.1965	292.7684
	13	31.3371	66.9552	103.9196	185.3183	203.1965	292.7653
	14	31.3371	66.9552	103.9196	185.3183	203.1965	292.7653
	Zhou (2001)	31.3365	66.9549	103.9197	185.3192	203.2250	292.8411
C-C	3	53.0692	103.8763	174.1909	276.5284	690.3621	721.8255
	4	44.9647	90.0285	141.7985	236.7698	290.3811	448.5260
	5	44.9174	89.4652	121.2447	214.0275	251.5873	403.2562
	6	44.8972	89.3835	120.6392	204.6194	222.1686	361.0645
	7	44.8970	89.3755	120.3072	203.0319	220.9901	313.1963
	8	44.8970	89.3751	120.3001	202.1035	220.3835	308.1262
	9	44.8970	89.3751	120.2983	202.0641	220.3560	304.0140
	10	44.8970	89.3751	120.2982	202.0523	220.3466	303.7682
	11	44.8970	89.3751	120.2982	202.0520	220.3463	303.6562
	12	44.8970	89.3751	120.2982	202.0519	220.3463	303.6523
	13	44.8970	89.3751	120.2982	202.0519	220.3463	303.6512
	14	44.8970	89.3751	120.2982	202.0519	220.3463	303.6512
	Zhou (2001)	44.8967	89.3762	120.3014	202.0662	220.4037	303.7840

Table 1: Convergence study of the first six values of the frequency parameter Ω of a two-span Timoshenko beam ($T_c \to \infty$ and $R_{12} \to \infty$) located at $c/l = 0.4$ for $\sqrt{12}\ r/l = 0.1$.

Assuming a one side open crack:

$$f(\eta) = 0.6384 - 1.035\eta + 3.7201\eta^2 - 5.1773\eta^3 + 7.553\eta^4 - 7.332\eta^5 + 2.4909\eta^6 \tag{41}$$

To perform a comparison between modal frequency results from this work and the ones obtained by Khaji et al. (2009), the relationship between the non dimensional hinge rigidity and the non-dimensional crack sectional flexibility is:

$$R_{12} = \frac{1}{\theta} \tag{42}$$

Table 2 provides a comparison of the first four modal frequencies for a S-S, S-C and C-C beam with η equal to 0.20, 0.35, 0.50 and 0.70, $c/l = 0.5$ for $r/l = 0.25$ with $N = M = 7$.

Boundary conditions	η	R_{12}	Khaji et al. (2009)				This work, N=M=7			
			Ω_1	Ω_2	Ω_3	Ω_4	Ω_1	Ω_2	Ω_3	Ω_4
S-S	0.20	9.6689	8.2760	29.6610	52.1525	80.6253	8.2733	29.6509	52.1351	80.5985
	0.35	2.9396	7.1126	29.6610	48.9134	80.6253	7.1102	29.6509	48.8969	80.5985
	0.50	1.2380	5.7693	29.6610	46.1053	80.6253	5.7674	29.6509	46.0894	80.5985
	0.70	0.5185	4.2726	29.6610	43.9407	80.6253	4.2711	29.6509	43.9256	80.5985
S-C	0.20	9.6689	12.0286	33.0248	54.3153	81.7510	12.0246	33.0135	54.2970	81.7239
	0.35	2.9396	11.1045	32.9459	51.0196	81.7050	11.1007	32.9348	51.0024	81.6781
	0.50	1.2380	10.1282	32.8581	48.2051	81.6598	10.1248	32.8470	48.1887	81.6327
	0.70	0.5185	9.1955	32.7702	46.0750	81.6194	9.1919	32.7591	46.0593	81.5924
C-C	0.20	9.6689	15.8527	36.0531	56.2394	82.7987	15.8474	36.0409	56.2204	82.7713
	0.35	2.9396	14.9282	36.0531	52.7601	82.7987	14.9231	36.0409	52.7421	82.7713
	0.50	1.2380	13.9791	36.0531	49.8068	82.7987	13.9743	36.0409	49.7897	82.7713
	0.70	0.5185	13.1041	36.0531	47.5784	82.7987	13.0997	36.0409	47.5622	82.7713

Table 2: Comparison study of the first four values of the frequency parameter Ω which correspond to the crack model proposed by Khaji et al. (2009) and the present work results, varying R_{12} values as a function of the crack depth with $N = M = 7$.

4 NUMERICAL EXAMPLES

In order to investigate the influence of stiffness of the intermediate elastic restraints on the free vibration characteristics of Timoshenko beams, numerical results were computed by using the combination of the Ritz method with the Lagrange multiplier method. A great number of problems were solved and, since the number of cases is extremely large, results are presented for only a few cases. All calculations have been performed taking $N = M = 7$, $k = 5/6$ and $\mu = 0.3$ unless otherwise specified. Mode shapes shown in the following tables corresponds to the bolted frequencies values indicated in each table.

Table 3 depicts values of the fundamental frequency parameter Ω_1 of a Timoshenko beam for different values of R_{12}, $T_c = R_c = 0$, located at $c/l = 0.1$, 0.3 and 0.5 for $\sqrt{12}\, r/l = 0.001$, 0.01 and 0.1. The results correspond to S-S, C-C, F-F, C-F, C-S and S-F boundary conditions.

Table 4 depicts the value of the fundamental frequency parameter Ω_1 of a Timoshenko beam with $R_c = 0$ and different values of T_c and R_{12} located at $c/l = 0.5$ for $\sqrt{12}\, r/l = 0.1$, 0.3 and 0.6, for S-S, C-C and F-F boundary conditions. Modal shapes shown correspond to $\sqrt{12}\, r/l = 0.1$ and $R_{12} = 0$.

Boundary conditions	R_{12}	$\sqrt{12}\,r/l$ = 0.001	0.01	0.1	0.001	0.01	0.1	0.001	0.01	0.1
		$\frac{c}{l}=0.1$			$\frac{c}{l}=0.3$			$\frac{c}{l}=0.5$		
S-S	0	17.8621	17.8545	17.1396	26.3351	26.3169	25.2004	39.4761	39.4510	37.0962
	1	8.9962	8.9948	8.8644	6.3947	6.3941	6.3317	5.6796	5.6791	5.6308
	10	9.7760	9.7744	9.6176	9.2746	9.2732	9.1353	9.0078	9.0065	8.8779
	100	9.8602	9.8585	9.6984	9.8055	9.8039	9.6460	9.7723	9.7707	9.6141
	1000	9.8686	9.8670	9.7066	9.8631	9.8615	9.7013	9.8597	9.8581	9.6980
C-C	0	18.9073	18.8972	17.9663	20.0982	20.0865	19.0201	14.0640	14.0596	13.6391
	1	19.6422	19.6312	18.6246	21.0117	20.9987	19.8173	16.8748	16.8678	16.2113
	10	21.4330	21.4194	20.1814	22.0828	22.0680	20.7290	20.9977	20.9849	19.8188
	100	22.2488	22.2337	20.1814	22.3405	22.3252	20.9450	22.2111	22.1960	20.8377
	1000	22.3604	22.3450	20.9616	22.3698	22.3545	20.9695	22.3566	22.3413	20.9586
F-F	0	26.3124	26.2932	24.7237	39.7090	39.6812	37.2193	61.6725	61.6083	56.2079
	1	21.9475	21.9392	21.1615	14.4029	14.3985	13.9808	11.8182	11.8154	11.5448
	10	22.3331	22.3250	21.5673	21.0996	21.0922	20.4056	19.9794	19.9729	19.3621
	100	22.3694	22.3612	21.6045	22.2396	22.2316	21.4827	22.0961	22.0882	21.3498
	1000	22.3730	22.3649	21.6082	22.3598	22.3517	21.5959	22.3450	22.3370	21.5822
C-F	0	18.8924	18.8853	18.2209	19.1291	19.1186	18.1464	9.8696	9.8665	9.5771
	1	19.5378	19.5299	18.7885	20.2577	20.2466	19.2208	14.2254	14.2202	13.7321
	10	21.1604	21.1499	20.1790	21.6476	21.6357	20.5404	20.1497	20.1398	19.2178
	100	21.9180	21.9061	20.8108	21.9907	21.9786	20.8656	21.8141	21.8023	20.7111
	1000	22.0224	22.0103	20.8970	22.0300	22.0179	20.9027	22.0120	21.9999	20.8870
C-S	0	12.1297	12.1268	11.8480	15.0959	15.0897	14.5074	9.0711	9.0688	8.8515
	1	12.8700	12.8666	12.5364	15.2344	15.2283	14.6492	11.4895	11.4862	11.1740
	10	14.5730	14.5679	14.0854	15.3805	15.3744	14.7979	14.5168	14.5114	14.0093
	100	15.3080	15.3020	14.7389	15.4139	15.4078	14.8318	15.3145	15.3084	14.7414
	1000	15.4068	15.4007	14.8261	15.4177	15.4116	14.8356	15.4076	15.4015	14.8264
S-F	0	25.9582	25.9442	24.6665	38.5079	38.4833	36.2888	46.0557	46.0191	42.8134
	1	13.2815	13.2785	12.9936	8.9482	8.9468	8.8109	8.6977	8.6962	8.5557
	10	15.1810	15.1771	14.8074	14.1399	14.1366	13.8198	14.0154	14.0121	13.6969
	100	15.3943	15.3903	15.0088	15.2763	15.2724	14.8973	15.2596	15.2557	14.8810
	1000	15.4158	15.4118	15.0290	15.4038	15.3998	15.0177	15.4021	15.3981	15.0161

Table 3: Values of the fundamental frequency parameter Ω_1 of a Timoshenko beam for different values of R_{12}, $T_c = R_c = 0$ located at $c/l = 0.1$, 0.3 and 0.5.

Table 5 depicts the first three values of the frequency parameter Ω of a Timoshenko beam with $R_c = 0$ and different values of T_c and R_{12} located at $c/l = 0.5$ with $\sqrt{12}r/l = 0.5$ for S-S, C-C and F-F boundary conditions. The figures shown correspond to the first three mode shapes with $T_c = 1000$ and $R_{12} = 0$.

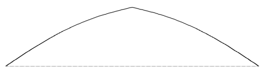
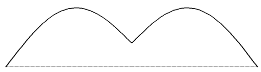
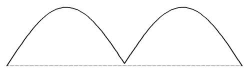
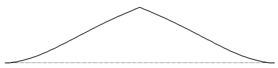
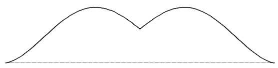
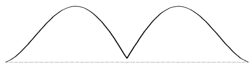
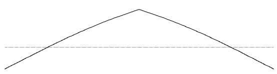
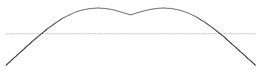
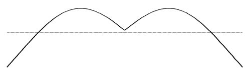
		$\sqrt{12}\,\frac{r}{l}$								
		0.1	0.3	0.6	0.1	0.3	0.6	0.1	0.3	0.6
Boundary conditions	R_{12}	$T_c = 100$			$T_c = 1000$			$T_c = 10000$		
S-S	0	**16.1447**	14.9548	11.9502	**32.2988**	25.3839	17.0112	**36.5931**	27.1249	17.5932
	100	16.8473	15.5080	12.7744	37.0962	27.3922	17.0147	37.0962	27.3147	17.6462
	1000	16.8602	15.5162	12.7802	37.0962	27.4332	17.0148	37.0962	27.3147	17.6462
										
C-C	0	**23.3060**	19.9094	14.8909	**44.9941**	30.1018	17.5648	**52.8442**	31.6270	17.7768
	100	25.9774	20.5242	14.8967	47.4799	31.7852	17.7990	53.7468	31.7852	17.7990
	1000	26.0304	20.5327	14.8967	47.5536	31.7852	17.7990	53.7468	31.7852	17.7990
										
F-F	0	**19.0466**	16.7598	12.0666	**44.8242**	33.0958	17.4364	**54.9988**	36.9666	18.2488
	100	8.4242	7.8422	6.5132	12.6584	10.7290	7.8240	13.3122	11.1183	7.9777
	1000	8.4757	7.8866	6.5425	12.8530	10.8555	7.8803	13.5358	11.2587	8.0378
										

Table 4: Values of the fundamental frequency parameter Ω_1 of a uniform Timoshenko beam with $R_c = 0$ and different values of T_c and R_{12} located at $c / l = 0.5$ for $\sqrt{12}\, r/l = 0.1$, 0.3 and 0.6, for S-S, C-C and F-F boundary conditions. The modal shapes correspond to $\sqrt{12}\, r/l = 0.1$ and $R_{12} = 0$.

Table 6 depicts the first three values of the fundamental frequency parameter Ω of an uniform Timoshenko beam with $T_c = R_c = R_{12} = 0$ at different locations for $\sqrt{12} r / l =$ 0.001 for F-F, S-S, C-C, S-F, and C-F boundary conditions and $N = M = 12$. The mode shapes which correspond to a hinge located at $c/l = 0.5$ are also presented.

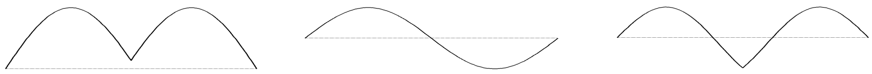
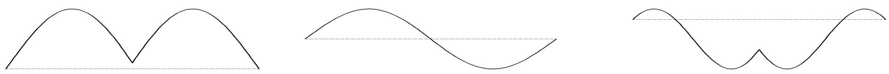
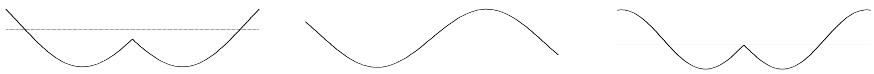
Boundary conditions	R_{12}	$\frac{c}{l}=0.5;\ \sqrt{12}\,\frac{r}{l}=0.5$								
		$T_c=100$			$T_c=1000$			$T_c=10000$		
		Ω_1	Ω_2	Ω_3	Ω_1	Ω_2	Ω_3	Ω_1	Ω_2	Ω_3
S-S	0	13.0863	20.1495	24.2519	**19.3340**	**20.1495**	**26.4141**	20.0728	20.1495	27.0917
	100	13.6916	20.1495	27.1746	19.6723	20.1495	27.1746	20.1495	20.8384	27.1746
	1000	13.6975	20.1495	27.1746	19.6753	20.1495	27.1746	20.1495	20.8451	27.1746
										
C-C	0	16.4047	20.9842	25.8990	**20.5666**	**20.9842**	**33.5814**	20.9448	20.9842	35.7162
	100	16.4239	20.9842	35.4292	20.9842	22.4179	35.9556	20.9842	23.5489	35.9556
	1000	16.4241	20.9842	35.5288	20.9842	22.4393	35.9556	20.9842	23.5816	35.9556
										
F-F	0	13.6452	23.5509	34.4165	**21.9081**	**23.5509**	**34.5278**	23.3820	23.5509	34.5427
	100	6.9676	17.3298	23.5509	8.6778	22.7043	23.5509	8.8852	23.5509	23.8724
	1000	7.0018	17.3904	23.5509	8.7513	22.7166	23.5509	8.9645	23.5509	23.8797
										

Table 5. – First three values frequencies parameter of a uniform Timoshenko beam with $R_c=0$ and different values of T_c and R_{12} located at $c/l=0.5$ with $\sqrt{12}r/l=0.5$ for S-S, C-C and F-F boundary conditions. Modal shapes shown correspond to $T_c=1000$ and $R_{12}=0$.

5 CONCLUSIONS

The free transverse vibration of a Timoshenko beam with ends elastically restrained against rotation and translation, and an arbitrarily located internal hinge including intermediate elastic constraints is studied. For this purpose, a simple and accurate approach has been developed based on a combination of the Ritz method and the Lagrange multiplier method for the determination of natural frequencies. The algorithm is very general and it is characterized by a low computational cost and high accuracy. Close agreement with results presented by previous investigators is demonstrated for some examples and for a crack model.

These results obtained may provide useful information for structural designers and engineers. The algorithms developed can be easily extended to a beam with an arbitrary number of hinges and intermediate points elastically restrained.

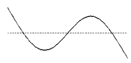
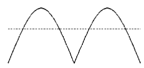
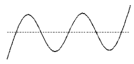
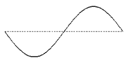
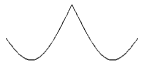
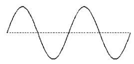
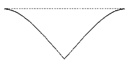
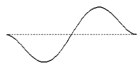
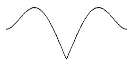
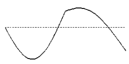
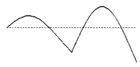
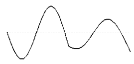
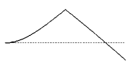
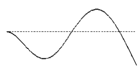
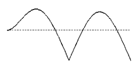
Boundary conditions	c/l	Ω_1	Mode shape	Ω_2	Mode shape	Ω_3	Mode shape
F-F	0.5	**61.6728**		**89.4931**		**199.8594**	
	0.4	51.1142		112.5862		162.1347	
	0.3	39.7090		110.8393		194.1497	
	0.2	31.8087		88.7651		175.1775	
	0.1	26.3124		72.7993		143.1623	
S-S	0.5	**39.4784**		**61.6728**		**157.9137**	
	0.4	33.4385		76.8753		127.9821	
	0.3	26.3352		86.0614		138.7773	
	0.2	21.3289		70.4180		147.9077	
	0.1	17.8622		58.2367		122.1138	
C-C	0.5	**14.0641**		**61.6728**		**88.1380**	
	0.4	15.6294		51.5618		111.5344	
	0.3	20.0983		43.3521		111.8010	
	0.2	22.1521		52.2125		94.8284	
	0.1	18.9074		60.1749		120.7269	
S-F	0.9	18.1166		58.9283		123.3481	
	0.8	21.8457		71.7316		150.8768	
	0.7	27.1872		89.8417		178.0350	
	0.6	35.0781		103.3874		139.2128	
	0.5	**46.0561**		**79.6851**		**171.3698**	
	0.4	47.8922		81.9546		155.4564	
	0.3	38.5079		104.0083		147.0098	
	0.2	31.1090		87.1791		171.3272	
	0.1	25.9581		71.9805		141.7791	
C-F	0.9	4.1174		25.9183		72.8272	
	0.8	4.9107		31.3479		88.7963	
	0.7	5.9945		39.1706		110.8688	
	0.6	7.5413		50.5287		112.5308	
	0.5	**9.8696**		**61.6728**		**88.8264**	
	0.4	13.5648		52.1706		111.5896	
	0.3	19.1292		43.8673		111.7699	
	0.2	21.9266		52.3588		94.7969	
	0.1	18.8923		60.1824		120.7251	

Table 6. First three values of the frequencies parameter Ω of a uniform Timoshenko beam with $T_c = R_c = R_{12} = 0$ at different locations, for $\sqrt{12}r\,/\,l =$ 0.001 with different boundary conditions. The mode shapes shown correspond to $c/l = 0.5$.

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NOTATION

A	cross-sectional area
$c_l = c / l$	geometrical parameter
E	Young's modulus
G	transverse shear modulus
I	moment of inertia
l	length of the beam.
h	beam depth.
a	crack depth.
η	crack depth ratio
θ	non-dimensional crack sectional flexibility
$r = \sqrt{I / A}$	radius of gyration of cross section
r_1, r_2	rotational stiffness at the left and right ends respectively
r_{12}	rotational stiffness at the internal hinge
r_c	rotational stiffness at the point $\bar{x} = c$.
$R_c, R_{12}, R_i, \quad i = 1,2$	dimensionless rotational parameters.
t	time.
t_1, t_2	translational stiffness at the left and right ends respectively.
t_c	translational stiffness at the point $\bar{x} = c$.
T	kinetic energy.
$T_c, T_i, \quad i = 1,2$	dimensionless translational parameters.
U	strain energy.
x	dimensionless abscissa.
$\bar{x}$	abscissa.
$\Omega = \omega l^2 \sqrt{\rho A / EI}$	dimensionless natural frequency parameter.
ω	radian frequency.
ρ	mass density.

APPENDIX

First members of the set of polynomials $\left\{p_i^{(k)}(x)\right\}$ and $\left\{q_j^{(k)}(x)\right\}$ for all possible combinations of classical boundary conditions and with intermediate elastic restraints.

Classical boundary conditions and intermediate elastic restraints hinge at $x = c_l$.	$p_1^{(1)}$	$q_1^{(1)}$	$p_1^{(2)}$	$q_1^{(2)}$
S-S	1	x	1	$x-1$
S-F	1	x	1	1
F-F	1	1	1	1
C-C	x	x	$x-1$	$x-1$
C-S	x	x	1	$x-1$
C-F	x	x	1	1
Classical boundary conditions with intermediate point support at $x = c_l$.	$p_1^{(1)}$	$q_1^{(1)}$	$p_1^{(2)}$	$q_1^{(2)}$
S-S	1	$x-c_l$	1	$\left(x-1\right)\left(x-c_l\right)$
S-F	1	$x-c_l$	1	$x-c_l$
F-F	1	$x-c_l$	1	$x-c_l$
C-C	x	$x\left(x-c_l\right)$	$x-1$	$\left(x-1\right)\left(x-c_l\right)$
C-S	x	$x\left(x-c_l\right)$	1	$x-1$
C-F	x	$x\left(x-c_l\right)$	1	1