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THE INDIRECT-BEM FOR 3D ELASTOSTATIC AND ELASTODYNAMIC PROBLEMS: CONSTRAINTS, CONVERGENCE AND COMPUTATIONAL COST

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Abstract. This work reports an investigation of some issues found in an implementation of the Indirect version of the Boundary Element Method (IBEM) for three-dimensional elastostatic and elastodynamic problems. Two different Green's functions, or auxiliary states, are used in the implementations. Numerical results show that the IBEM presents a much slower convergence rate with respect to the number of boundary elements than the Direct-BEM counterpart. The numerically synthesized non-singular dynamic auxiliary states is computationally much more demanding and expensive, but are much more versatile in terms of incorporating non-isotropic behavior and general linear viscoelastic models. These trade offs, which oppose computational efficiency and modeling abilities are addressed. Finally, it is investigated whether the discontinuity observed in the traction components in diagonal terms of the influence matrix S constrains the applicability of IBEM to open domains.

1 INTRODUCTION

The Boundary Element Method (BEM) is part of the group of numerical methods to approximate the solution of differential equations which involve some discretization. A remarkable feature of this particular method is that, under the application of a proper Green's function, it only requires the boundary of the problem to be discretized.

One of the characteristics of the Indirect version of BEM (IBEM) is that it allows the application of non-singular Green's functions, which considerably reduces the mathematical effort required by the method. In many other aspects, however, the IBEM resembles the formulation of its Direct-BEM (DBEM) counterpart. Both versions involve the integration of an auxiliary state or Green's function for the operator modeling the behavior of the medium that is being treated, and both are especially well-suitable for the study of infinite domains. In the DBEM, a final set of linear algebraic equations relates physically meaningful displacements and tractions at the prescribed boundary. Conversely, the IBEM relates actual tractions and displacements through a set of fictitious stresses.

This work reports an investigation of some issues found in a typical implementation of IBEM. Particularly, the computational cost and convergence rate of the method are compared with Direct-BEM implementations. Rajapakse and Shah (1986) point out that the discontinuity of some stress components that naturally arise in the formulation of IBEM constraints the application of the method to unbounded domains or static problems. This constraint is also investigated in the present work.

Two different Green's functions are used in the implementations, regarding the behavior of elastostatic and elastodynamic problems. These non-singular Green's functions are synthesized by the integration of classical fundamental solutions for concentrated loads applied in unbounded domains. Three-dimensional static and dynamic fundamental solutions are considered. For the elastodynamic case, a third non-singular Green's function is presented, which is numerically synthesized using a double Fourier integral.

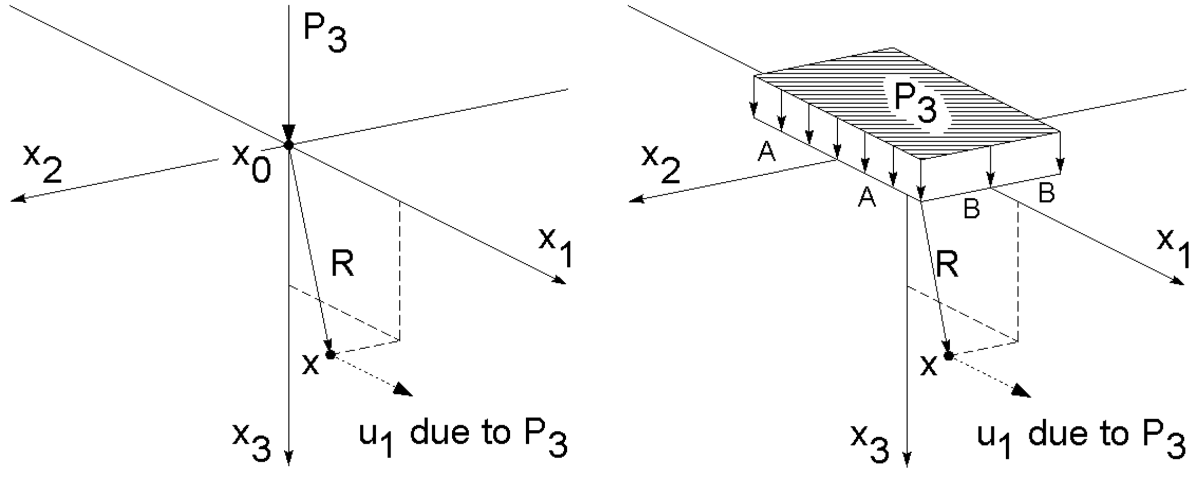
2 NON-SINGULAR GREEN'S FUNCTIONS

Green's functions play a fundamental role in both Direct and Indirect formulations of BEM. They represent the response of an auxiliary state, usually an unbounded full-space, to a loading applied to the interior of that full-space. The boundary element application is based on the superposition of these solutions throughout the discretized boundary of the real problem. The nature of the medium of the auxiliary state and the loading to which it is subjected should correspond to the ones in the boundary element application.

In the classical Direct version of BEM, singular Green's functions are used. Equation (1) shows an example of such solution, which corresponds to the displacements of a given point $\boldsymbol{x}$ in an isotropic, elastic three-dimensional full-space due to a static concentrated loading applied at a point $\boldsymbol{x_0}$ (Kane, 1994) (see Fig. 1a).

$$\bar{g}_{mn}(\boldsymbol{x},\boldsymbol{x}_0)=\frac{p_n(\boldsymbol{x}_0)}{16\pi\mu(1-\nu)}\frac{1}{R}\left\{\delta_{mn}(3-4\nu)+\frac{1}{R}x_m x_n\right\} \tag{1}$$

In Eq. (1), m and n stand respectively for the direction in which the displacement occurs and the direction in which the loading of intensity $p_n(\boldsymbol{x_0})$ is applied, (m,n=1,2,3). δ_{mn} represents the Kroenecker Delta, and R is the distance between $\boldsymbol{x}$ and $\boldsymbol{x_0}$.



(a) (b)

Figure 1: Examples of Green's functions of the displacement field of a 3D space, due to (a) concentrated and (b) distributed loads.

An analogous solution corresponding to the application of a dynamic loading with circular frequency ω is given by Kitahara (1985) as:

$$\tilde{g}_{mn}(\boldsymbol{x},\boldsymbol{x}_0) = \frac{p_n(\boldsymbol{x}_0)}{4\pi\mu}\left\{\frac{e^{ik_S R}}{R}\delta_{mn} + \frac{1}{k_S^2}\frac{\partial^2}{\partial x_m \partial x_n}\left(\frac{e^{ik_S R}}{R} - \frac{e^{ik_P R}}{R}\right)\right\} \tag{2}$$

In Eq. (2), $k^2_P=\omega^2\rho/(\lambda+2\mu)$ and $k^2_S=\omega^2\rho/\mu$ are the pressure and shear wave numbers, respectively; λ and μ are Lamé constants and ρ is the mass density of the medium.

The stress response in both static and dynamic cases can be obtained from Eq. (1) and (2) through the stress-strain relationship, $H_{mnp}(\boldsymbol{x}, \boldsymbol{x_0}) = \lambda\delta_{mn}G_{kp,k} + \mu(G_{mp,n} + G_{np,m})$, in which H_{mnp} is the stress component σ_{mn} due to a loading in the direction of p (m,n,p=1,2,3).

Both solutions can be extended to represent also *visco*elastic isotropic 3D full-spaces. This can be obtained by Christensen's elastic-viscoelastic correspondence principle (Christensen, 2003), in which the Lamé's elastic parameters μ and λ from Eqs. (1) and (2) are written as complex variables $\mu^*=\mu_{elastic}(1+i\eta)$, $\lambda^*=\lambda_{elastic}(1+i\eta)$. The parameter η encloses the medium's material damping.

The use of such solutions for concentrated loadings on DBEM brings along the inconvenient need to deal with singularities in the integrands. Conversely, IBEM allows the use of non-singular Green's functions in its formulation. A practical way of obtaining these functions is by integration of the singular ones over an area. A non-singular Green's function corresponding to Eqs. (1) and (2) can be given by:

$$\left[\tilde{G}_{mn}(\boldsymbol{x},\boldsymbol{x}_0), \bar{G}_{mn}(\boldsymbol{x},\boldsymbol{x}_0)\right] = \int_{\xi=-A}^{\xi=+A}\int_{\zeta=-B}^{\zeta=+B}\left[\tilde{g}_{mn}(\boldsymbol{x},\boldsymbol{x}_0), \bar{g}_{mn}(\boldsymbol{x},\boldsymbol{x}_0)\right] d\zeta\, d\xi \tag{3}$$

Equation (3) represents the responses of displacements of a medium subjected to a static or dynamic loading distributed over a rectangular area of sides 2A×2B (see Fig. 1b). Its integrands present integrable singularities and therefore its integrals can be performed, for example, by ordinary Gaussian Quadrature.

Another technique for obtaining non-singular Green's function is based on the solution of the general Navier's equilibrium equations, i.e., $\mu^* u_{i,jj}+(\lambda^*+\mu^*)u_{k,ki}=-\omega^2\rho u_i$ (i,j=x,y,z), in which ρ is the medium's density. This solution results in a series of generalized expressions for displacements and stress. The Green's functions are obtained when specific boundary conditions are applied to this series of expressions. These boundary conditions correspond to the loading applied in the interior of the full-space which is being treated. If distributed loads are taken as boundary conditions, then non-singular Green's functions are obtained.

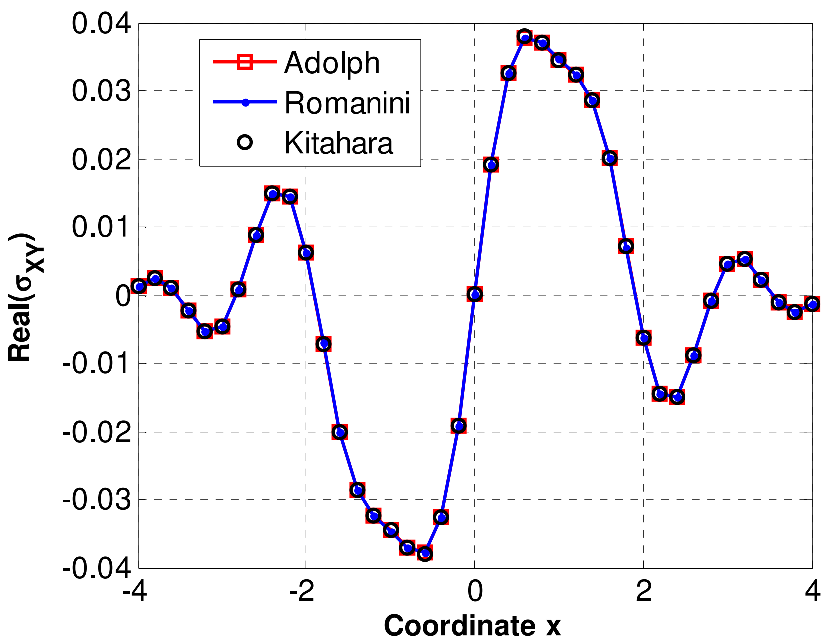


Figure 2. Comparison of Romanini's, Adolph's and Kitahara's solution of the component σ_{XY} for ω=5.0, ρ=μ=1.0, η=0.001, ν=0.25, Y=1.1, Z=0.5, A=2B=1.0, $x \in [-4, 4]$.

This second technique is more versatile than the first one because it allows the incorporation of general viscoelastic models, to deal with non-isotropic media and even to treat transient problems. One trade-off is that it usually results in computationally expensive final equations.

Romanini (1993) used this technique to synthesize a solution for a 3D isotropic medium subjected to a harmonic load. The load was uniformly distributed over a rectangular area of sides 2A×2B, immersed in the interior of the full-space. Fourier transforms were used to bring Navier's equation to a more convenient coordinate space whenever necessary. His final expressions involve double indefinite integrations to be performed numerically, which are computationally expensive.

Adolph (2006) developed a Green's function for the same problem. Radon transforms were used along with Fourier transforms. Adolph aimed to reduce the computational cost of the solution by eliminating one of the final indefinite integrals.

Figure 2 examples a validation of Romanini's and Adolph's solutions against a non-singular Green's function. This Green's function was obtained by the integration of Kitahara's solution (1985) (see Eq. 2) according to Eq. (3). In this particular validation, the numerical integration of Adolph's and Romanini's solutions took up to two orders of magnitude longer than the double integration of Kitahara's solution.

3 INDIRECT VERSION OF BEM

The displacement response of a physical problem can be written as a linear combination, or superposition, of the proper Green's function according to Eq. (4).

$$u_m(\boldsymbol{x},\boldsymbol{x}_0) = G_{mn}(\boldsymbol{x},\boldsymbol{x}_0)q_n(\boldsymbol{x}_0) \tag{4}$$

In Eq. (4), $G_{mn}(\boldsymbol{x}, \boldsymbol{x_0})$ represent the Green's function – given by Eqs. (1) or (2), for example. The influence of each of these terms is weighted by the so-called fictitious stress $q_n(\boldsymbol{x_0})$, applied to the point $\boldsymbol{x_0}$ in the direction of n, to compose the real displacement that occur at $\boldsymbol{x}$. The repeated indices in Eq. (4) should be interpreted according to Einstein summation convention, with m,n=x,y,z.

In a physical problem discretized by boundary elements, Eq. (4) can be taken to represent the relation between two boundary element points (nodes) $\boldsymbol{x}$ and $\boldsymbol{x_0}$. A certain load is applied at a source-point $\boldsymbol{x_0}$ in one element resulting in a displacement at a field-point $\boldsymbol{x}$ in another. For a problem discretized by N boundary element nodes, the following equation can be assembled:

$$\begin{bmatrix} u_m^1 \\ u_m^2 \\ \\ u_m^N \end{bmatrix} = \begin{bmatrix} G_{mn}^{1,2} & G_{mn}^{1,3} & \cdots & G_{mn}^{1,N} \\ G_{mn}^{2,1} & G_{mn}^{2,2} & & G_{mn}^{2,N} \\ \vdots & & \ddots & \\ G_{mn}^{N,1} & G_{mn}^{N,2} & & G_{mn}^{N,N} \end{bmatrix} \begin{bmatrix} q_n^1 \\ q_n^2 \\ \\ q_n^N \end{bmatrix} \Rightarrow \boldsymbol{u} = \boldsymbol{Gq} \tag{5}$$

The upper indices i,j (i,j=1,N) indicates the N boundary element nodes in which the fictitious stresses q^j_n are being applied and the resulting displacements u^i_m are being read. Each of the terms G^{ij}_{mn} are 3×3 submatrices of **G**.

Analogously to Eq. (4), the stress response of a physical problem can be written as:

$$\sigma_{mn}(\boldsymbol{x},\boldsymbol{x}_0) = H_{mnp}(\boldsymbol{x},\boldsymbol{x}_0)q_p(\boldsymbol{x}_0) \tag{6}$$

The response in terms of traction can also be written as Eq. (7), in which **n** is the normal vector pointing outward the domain at the point $\boldsymbol{x}$.

$$t_m(\boldsymbol{x}) = \sigma_{mn}(\boldsymbol{x},\boldsymbol{x}_0)n_n(\boldsymbol{x}) = H_{mnp}(\boldsymbol{x},\boldsymbol{x}_0)n_n(\boldsymbol{x})q_p(\boldsymbol{x}_0) = S_{mp}(\boldsymbol{x},\boldsymbol{x}_0)q_p(\boldsymbol{x}_0) \tag{7}$$

Analogously to Eq. (5), Eq. (7) can be written in matrix form as:

$$\boldsymbol{t} = \boldsymbol{Sq} \tag{8}$$

Equations (5) and (8) furnish, respectively, the displacements **u** and tractions **t** at each of the N boundary element points (nodes) due to fictitious loadings **q** applied to those nodes. These two equations represent the final expressions in the formulations of the indirect version of BEM (IBEM). They are constructed based on a set of non-singular auxiliary states expressed by the influence matrices **S** and **G**.

One way of expressing the traction-displacement relation from these equations is by putting them together as $\mathbf{u}=\mathbf{G}\cdot\mathbf{q}=\mathbf{G}(\mathbf{S}^{-1}\mathbf{t})=\mathbf{G}\cdot\mathbf{S}^{-1}\mathbf{t}$. However, this technique involves the computationally expensive inversion of the matrix **S**. In the present work, we use a different strategy presented by Labaki, Mesquita and Adolph (2008).

3.1 Diagonal terms of S

The diagonal terms of the influence matrix **S** require special attention. These terms contain results of Green's functions for the case in which the source-point equals the field-point, or $\boldsymbol{x}=\boldsymbol{x_0}$, so that R=0. In other words, the responses of stress and displacements are being measured in the same point that the load is being applied.

In this situation, three particular components of stress, σ_{XZX}, σ_{YZY} and σ_{ZZZ}, are discontinuous. They present different values if the point where they are measured approaches the loading surface from negative or positive values of the coordinate z.

The discontinuity between the values of these components right above and right below the loading surface equals the loading $p_n(\boldsymbol{x_0})$, i.e., $\sigma_{mnp}(0,0,z=0^+)-\sigma_{mnp}(0,0,z=0^-)=p_n(\boldsymbol{x_0})$.

In the implementation of IBEM, it is necessary to set up explicitly which value these discontinuous components of stress should assume in this case. In the present implementation, the values referring to $z=0^+$ were chosen, because positive values of z represent the interior portion of the domain of the problem. The intensity 0.5 is set up, because unitary load $p_n(\boldsymbol{x_0})=1$ was assumed in the formulation of the Green's functions (see Eq. (1) and (2)).

Another consequence of this discontinuity of stresses concerns the application of IBEM to dynamic problems with closed domain. Rajapakse and Shah (1987) have shown that, in that type of problems, and because of the discontinuity of stresses, a term regarding the inertia force of the domain of the problem should be included in the formulation to satisfy the equilibrium of forces. If this term is disregarded, the formulation of IBEM is constrained to problems with open domains. Static problems with closed domains can also be treated, because in that case there are no inertia forces.

The present formulation of IBEM was implemented in Fortran programming language under an ordinary procedural programming paradigm. Non-singular Green's functions based on the integrations of Eqs. (1) and (2) were used. The program was used to analyze 3D elastostatic and elastodynamic problems, and the results are shown in the next section.

4 NUMERICAL RESULTS

The present implementation was used to analyze simple but representative prismatic elastic members subjected to longitudinal loads. Nine models were analyzed, whose dimensions L_X, L_Y and L_Z are listed in Table 1 (see Fig. 3a). In all the models, the boundary conditions were: the bars were clamped at one end and loaded with a uniformly distributed load with unitary intensity (see Fig. 3b). The material parameters are: Poisson's ratio $\nu=0.25$; Lamé's constant $\mu=1.0$; Young's modulus E=2.5.

Constant rectangular boundary elements were used in different numbers of elements per face (Nf). All the elements of a given face of the bars had the same dimensions, and all the faces had the same number of elements. Figure 3a shows an example in which Nf=25 elements per face were used.

The first IBEM application in this work used a non-singular static load Green's function based on the double integration of Eq. (1) according to Eq. (3). This application is proper to analyze three-dimensional elastostatic problems. Its results are compared in this work with a classical DBEM solution implemented by Dominguez (1993) for elastostatic and elastodynamic problems discretized by constant rectangular boundary elements.

Due to the boundary conditions applied, a static reaction F'=T=1 should be observed in the clamped surface of the bars. The resulting static displacement in the loaded surface should approach the analytical solution of displacement of bars as the discretization is refined. This analytical solution is given by $u'=T \cdot L_Y/(EL_XL_Z)$, in which E is the material's Young's modulus. Table 1 also lists these expected results for the nine models analyzed.

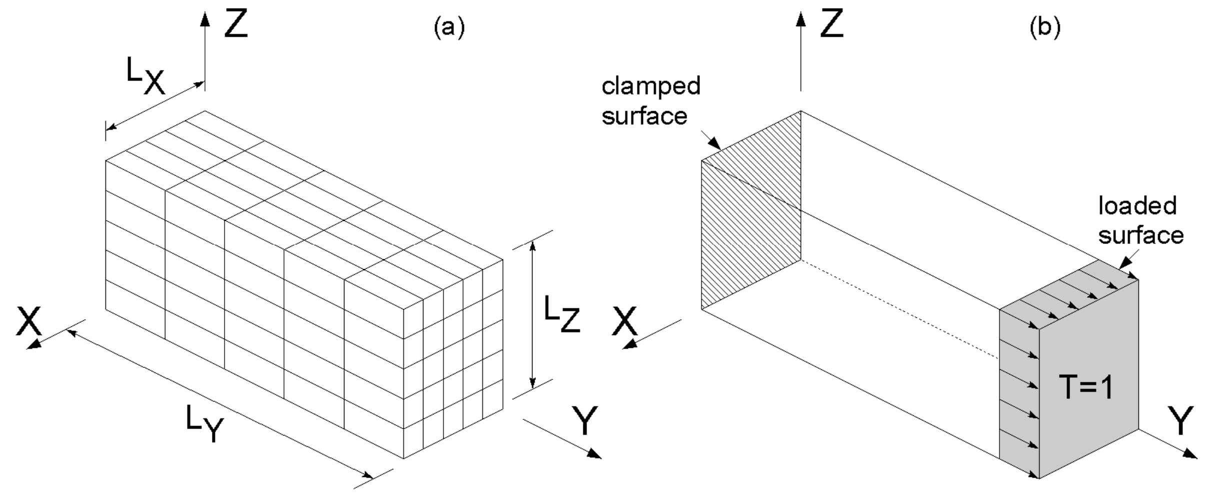


Figure 3. (a) Dimensions and example of discretization with Nf=25 elements per face of a prismatic bar and (b) boundary conditions.

Table 1. Dimensions of the models of bar.

Model	**m1**	**m2**	**m3**	**m4**	**m5**	**m6**	**m7**	**m8**	**m9**
L_X	1	2	3	1	1	1	2	2	2
L_Y	1	2	3	1	2	2	1	1	2
L_Z	1	2	3	2	1	2	1	2	1
u'	0.4	0.2	0.133	0.2	0.8	0.4	0.2	0.1	0.4

Tables 2 and 3 show, respectively, the results of the program for the static displacement u at the tip of the bars and the static reaction F at their clamped surface.

Table 2. Convergence of the solution of displacements at the tip of the bars for increasing numbers of elements.

Nf →	**1**	**4**	**9**	**16**	**25**	**36**	**49**	**64**	**81**	**100**
m1	1.2231	1.1861	1.1629	1.1450	1.1316	1.1213	1.1130	1.1063	1.1007	1.0960
m2	1.2231	1.1861	1.1629	1.1450	1.1316	1.1213	1.1130	1.1063	1.1007	1.0960
m3	1.2200	1.1831	1.1600	1.1421	1.1288	1.1185	1.1103	1.1036	1.0980	1.0932
m4	1.1845	1.1517	1.1382	1.1271	1.1182	1.1111	1.1052	1.1003	1.0961	1.0925
m5	1.8669	1.5059	1.3730	1.3071	1.2650	1.2352	1.2128	1.1953	1.1811	1.1694
m6	1.5266	1.3218	1.2523	1.2144	1.1888	1.1701	1.1558	1.1444	1.1352	1.1274
m7	1.1845	1.1517	1.1382	1.1271	1.1182	1.1111	1.1052	1.1003	1.0961	1.0925
m8	1.1318	1.1160	1.1137	1.1098	1.1057	1.1018	1.0983	1.0952	1.0925	1.0900
m9	1.5266	1.3218	1.2523	1.2144	1.1888	1.1701	1.1558	1.1444	1.1352	1.1274

These tables show the numerical results u and F divided by the expected analytical result, i.e., u/u’ and F/F’=F, so that the convergence of the results towards the analytical results can be observed as the discretization gets finer.

Table 3. Convergence of the solution of tractions at the clamped surfaces of the bars for increasing numbers of elements.

Nf →	**1**	**4**	**9**	**16**	**25**	**36**	**49**	**64**	**81**	**100**
m1	1.2070	1.1495	1.1130	1.0922	1.0787	1.0690	1.0618	1.0562	1.0516	1.0478
m2	1.2070	1.1495	1.1130	1.0922	1.0787	1.0690	1.0618	1.0562	1.0516	1.0478
m3	1.2058	1.1484	1.1119	1.0911	1.0776	1.0680	1.0608	1.0551	1.0505	1.0468
m4	1.1096	1.0995	1.0805	1.0674	1.0581	1.0513	1.0460	1.0418	1.0384	1.0356
m5	1.5657	1.3442	1.2566	1.2104	1.1796	1.1577	1.1412	1.1282	1.1179	1.1093
m6	1.3186	1.2242	1.1706	1.1403	1.1199	1.1052	1.0942	1.0856	1.0786	1.0729
m7	1.1096	1.0995	1.0805	1.0674	1.0581	1.0513	1.0460	1.0418	1.0384	1.0356
m8	1.0637	1.0757	1.0627	1.0524	1.0450	1.0396	1.0355	1.0322	1.0294	1.0272
m9	1.3186	1.2242	1.1706	1.1403	1.1199	1.1052	1.0942	1.0856	1.0786	1.0729

A very slow convergence rate is observed in these results. An unusually fine discretization (for BEM standards) of 100 elements per face is necessary to obtain an error of around 9% and 4% in the solutions of displacements and tractions, respectively. It is also observed that the models m5, m6 and m9 present an even slower convergence rate. These models are the ones in which the ratios length/(transversal area) are the largest.

Table 4 shows the results for the same problems obtained by Dominguez’ DBEM program (Dominguez, 1993). A much faster convergence rate is observed in this case.

Table 4. Convergence of the solution of tractions and displacements of the bars, for increasing numbers of elements, obtained by a classical DBEM program.

	F/F’			**u’/u**		
Nf →	**1**	**9**	**25**	**1**	**9**	**25**
m1	0.9980	1.0046	1.0021	1.0111	1.0217	1.0435
m2	0.9917	0.9988	0.9968	1.0016	1.0038	1.0130
m3	0.9804	0.9880	0.9862	0.9911	0.9911	0.9983
m4	0.9975	1.0054	1.0027	0.9890	1.0235	1.0431
m5	0.9934	0.9979	0.9970	1.4129	1.0592	1.0424
m6	0.9924	0.9993	0.9975	1.2063	1.0317	1.0292
m7	0.9984	1.0055	1.0025	0.9895	1.0236	1.0430
m8	0.9979	1.0052	1.0023	0.9629	1.0158	1.0340
m9	0.9933	0.9994	0.9973	1.2068	1.0318	1.0289

Moreover, it was observed that the IBEM application is more computationally expensive than its DBEM counterpart. For instance, Dominguez’ program obtains the solution for the cases Nf=1, Nf=9 and Nf=25 3.1, 9.4 and 12.6 times faster than the present IBEM implementation, respectively.

The second IBEM application in this work used a non-singular dynamic load Green’s function based on the double integration of Eq. (2) according to Eq. (3). This application is proper to analyze three-dimensional elastodynamic problems. Its results are compared in this work with an implementation of DBEM presented by Carrion, Mesquita and Romanini (2001) for elastostatic and elastodynamic problems discretized by rectangular boundary elements.

This second program was used to study an elastic three-dimensional half-space discretized in its flat surface by square constant boundary elements. Two discretizations of this problem were considered, with 5×5 and 11×11 elements. In both discretizations, the elements have dimensions 1×1. The central element in the mesh is loaded with a dynamic vertical load whose frequency varies. The vertical displacement of the loaded element was measured for frequencies ω=0 (static case) to ω=2 (see Fig. 4). The material parameters are: Poisson's ratio ν=0.25; Lamé's constant μ=1.0; Young's modulus E=2.5.

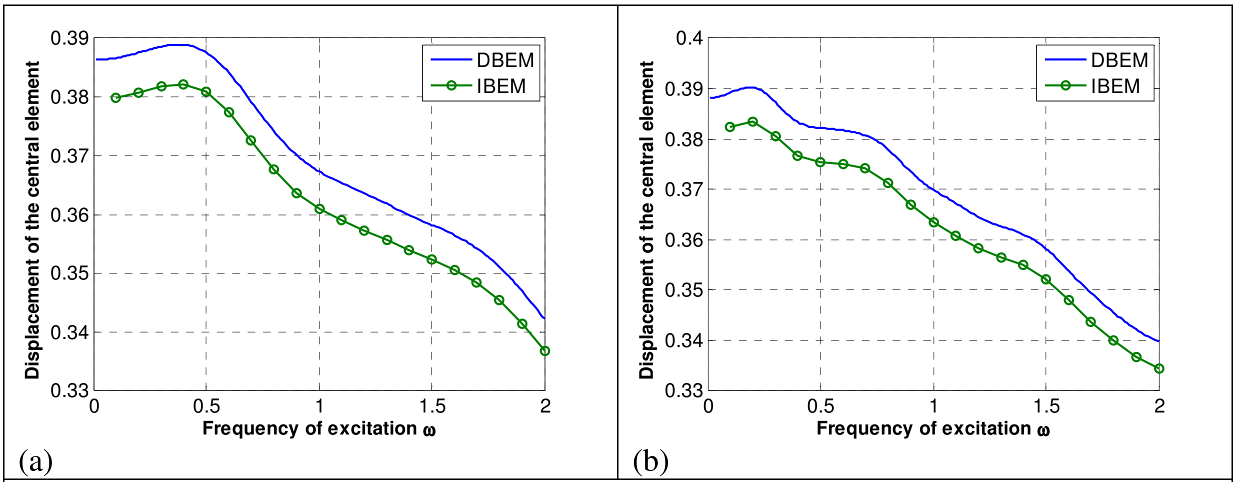


Figure 4. Vertical displacement of the central point at the surface of a half-space discretized by a (a) 5×5 and a (b) 11×11 square boundary elements mesh.

The program was also used to analyze the problem of a bounded domain shown in Figure 5a.

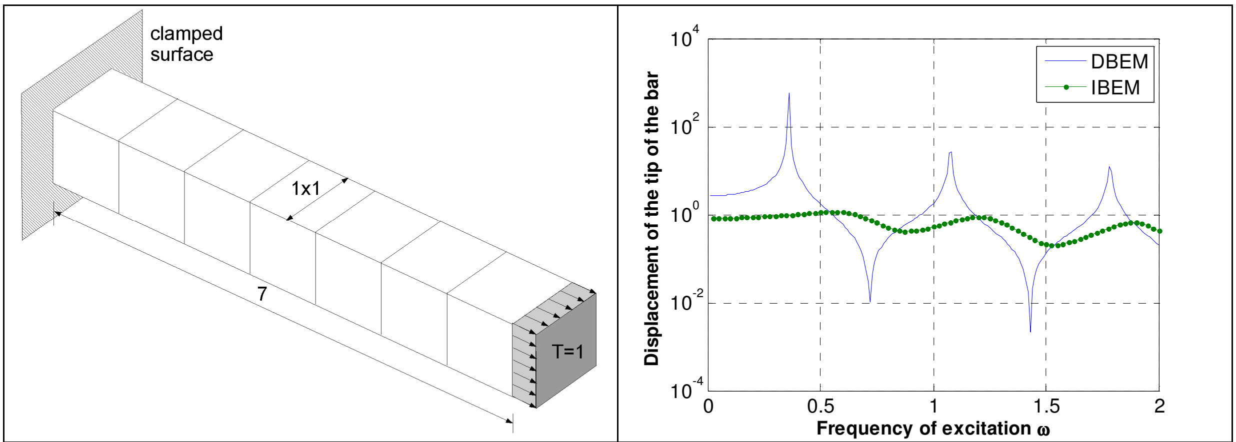


Figure 5. Dynamic analysis of a (a) problem with bounded domain; (b) Displacements at the tip of the bar.

The problem consists of a long 3D bar, discretized by 30 square boundary elements. One of its ends is clamped, and the other end is subjected to a uniformly distributed harmonic load of unitary intensity. The material parameters are: Poisson's ratio ν=0.25; Lamé's constant μ=1.0; Young's modulus E=2.5.

The results of displacement at the tip of this bar are shown in Figure 5b for different frequencies of loading. The results are compared with the DBEM program (Carrion, Mesquita and Romanini, 2001). It is observed that there is a disagreement between the two results. One explanation for this disagreement is that, in the present IBEM implementation, the inertia force of the domain of the problem was disregarded. Rajapakse and Shah (1986) pointed out that, for this reason, the present program is not capable of solving problems with bounded domain properly (see Section 3.1).

5. CONCLUDING REMARKS

The present article described a formulation of the Indirect version of the Boundary Element Method (IBEM) based on non-singular Green's functions. Two different non-singular Green's functions were obtained by double-integrating classical Green's function for concentrated loads. These Green's functions allow the present implementation of IBEM to deal with elastostatic and elastodynamic problems.

It was observed that the IBEM solution presents a slower convergence rate than a classical Direct-BEM program. It is also more computationally expensive than its DBEM counterpart.

The application of the IBEM to elastodynamic problems with bounded domain is conditioned to the inclusion of a term regarding the inertia forces of the domain. This condition arises from the discontinuity of the stress Green's functions that occur in the diagonal of the influence matrix **S**.

The IBEM has shown to be conveniently free from strong singularities. A large variety of problems with unbounded domains can be dealt with this method. Its computational cost can be reduced by using parallel computing techniques.

6. ACKNOWLEDGEMENTS

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