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COUPLING H(DIV) AND H1 FINITE ELEMENT APPROXIMATIONS FOR A POISSON PROBLEM

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Abstract. The main purpose of this article is to approximate an elliptic problem coupling classical Galerkin and H(div) formulations.

As a model problem we consider the Laplace equation on two or three dimensional domain. The domain is split into two non-overlapping subdomains. On the first one, the problem is approximated using classical Galerkin method. On the other one, the mixed formulation is applied. On the interface, the continuity of flux and pressure is imposed strongly using the transmission condition. The resulting formulation is a saddle point problem which is analysed for stability, existence and uniqueness using Brezzi's theory.

1 INTRODUCTION

In the present paper we consider a technique for the combination of different finite element formulations in different parts of the domain. As a test case we consider the Darcy problem which basically consists of mass conservation equation augmented with Darcy's law relating the average velocity of the fluid in a porous medium with the gradient of a potential field through the hydraulic conductivity tensor.

The basic idea is to split the domain into two non overlapping sub-domains and approximate the problem on the first one using the classical Galerkin method and, on the other one, apply a mixed formulation.

2 VARIATIONAL FORMULATIONS

2.1 Basic notation

Let Ω be a domain with Lipschitz boundary $\partial\Omega$, whose outer unit normal vector is denoted by $\boldsymbol{\eta}$. We shall used the following vector spaces and norms.

$$L^2(\Omega) = \{f : \int_\Omega |f(x)|^2 < \infty\}, \quad \|f\|_2 = \left(\int_\Omega |f|^2 dx\right)^{\frac{1}{2}} \tag{1}$$

$$H^1(\Omega) = \{f \in L^2(\Omega) : \partial^\alpha f \in L^2(\Omega)\}, \quad \|f\|_{H^1(\Omega)} = \left(\sum_{|\alpha|\leq m} \int_\Omega |\partial^\alpha f(x)|^2 dx\right)^{\frac{1}{2}} \tag{2}$$

with

$$\partial^\alpha f = \frac{\partial^\alpha f}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}, |\alpha| = \alpha_1 + \cdots + \alpha_n$$

$$H_0^1(\Omega) = \left\{\varphi \in H^1(\Omega) : \varphi|_{\partial\Omega} = 0\right\} \tag{3}$$

$$H\left(div; \Omega\right) = \left\{\boldsymbol{v} \in L^2\left(\Omega\right)^n : div\left(\boldsymbol{v}\right) \in L^2\left(\Omega\right)\right\}, \tag{4}$$

and norm

$$\|\boldsymbol{v}\|^2_{H(div;\Omega)} = \|\boldsymbol{v}\|^2_{L^2(\Omega)} + \|div(\boldsymbol{v})\|^2_{L^2(\Omega)} \tag{5}$$

$$H_0\left(div; \Omega\right) = \{\boldsymbol{v} \in H\left(div; \Omega\right) : \boldsymbol{v} \cdot \boldsymbol{\eta}|_{\partial\Omega} = 0\} \tag{6}$$

Consider a partition of the domain Ω into two non-overlapping sub-domains Ω_1 and Ω_2, and let $\Gamma = \overline{\Omega}_1 \cap \overline{\Omega}_2$, as described the Figure 1. We introduce the spaces

$$H^1_{0,\partial\Omega\cap\partial\Omega_i}(\Omega_i) = \left\{q \in H^1(\Omega_i) : q|_{\partial\Omega\cap\partial\Omega_i} = 0\right\}, \quad i = 1, 2 \tag{7}$$

$$H_{0,\partial\Omega\cap\partial\Omega_i}\left(div; \Omega_i\right) = \{\boldsymbol{v} \in H\left(div; \Omega_i\right) : \boldsymbol{v} \cdot \boldsymbol{\eta}|_{\partial\Omega\cap\Omega_i} = 0\}, \quad i = 1, 2. \tag{8}$$

$$H^{\frac{1}{2}}_{00}(\Gamma) = \{u \in H^{\frac{1}{2}}(\Gamma) : \mathcal{R}u \in H^{\frac{1}{2}}(\partial\Omega)\} \tag{9}$$

where $\mathcal{R}u$ denotes an extension of u to $\partial\Omega$.

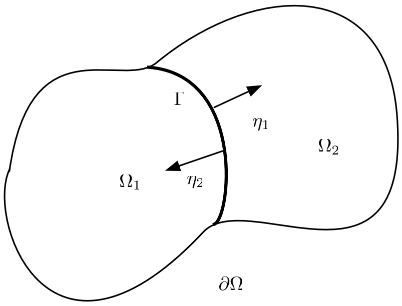

Figure 1: partition of domain Ω

For each function $q \in H^{\frac{1}{2}}_{00}(\Gamma)$ and $\psi \in (H^{\frac{1}{2}}_{00}(\Gamma))'$, $\langle \psi, q \rangle = \int_\Gamma \psi q ds$ denotes the duality pairing between $H^{\frac{1}{2}}_{00}(\Gamma)$ and $(H^{\frac{1}{2}}_{00}(\Gamma))'$. Moreover, if $\tilde{u} \in H^{\frac{1}{2}}(\partial\Omega)$ is a extension by zero to $\partial\Omega$ of the $u \in H^{\frac{1}{2}}_{00}(\Gamma)$ then

$$\|\tilde{u}\|_{H^{\frac{1}{2}}(\partial\Omega)} \leq \|u\|_{H^{\frac{1}{2}}_{00}(\partial\Omega)} \tag{10}$$

2.2 The model problem

Consider the model problem

$$\begin{cases} \boldsymbol{u} &= \mathbf{K}\nabla p \text{ in } \Omega \\ -div(\boldsymbol{u}) &= f \text{ in } \Omega \\ \boldsymbol{u}\cdot\boldsymbol{\eta} &= 0 \text{ in } \partial\Omega_N \\ p &= \overline{p} \text{ in } \partial\Omega_D \end{cases} \tag{11}$$

where $\mathbf{K}$ is the hydraulic conductivity tensor, p is the hydraulic potential (or pressure), and $\boldsymbol{u}$ is the velocity field of the fluid.

Consider a partition of Ω as describe in Figure 1, and suppose that $\partial\Omega_D \subset \partial\Omega_1$, $\partial\Omega_N \subset \partial\Omega_2$. We reformulate the problem (11) in the multi-domain decomposition (see Quarteroni and Valli (1999)) as:

$$-\, div(\mathbf{K}\nabla p_1) = f \text{ in } \Omega_1 \tag{12}$$
$$p_1 = \overline{p} \text{ in } \partial\Omega_D \tag{13}$$
$$\boldsymbol{u}_1\cdot\boldsymbol{\eta}_1 = -\, \mathbf{K}\nabla p_2\cdot\boldsymbol{\eta}_2 \text{ in } \Gamma \tag{14}$$

and

$$\boldsymbol{u}_2 = \mathbf{K}\nabla p_2 \text{ in } \Omega_2 \tag{15}$$
$$-div(\boldsymbol{u}_2) = f \text{ in } \Omega_2 \tag{16}$$
$$\boldsymbol{u}_2\cdot\boldsymbol{\eta}_2 = 0 \text{ in } \partial\Omega_N \tag{17}$$
$$p_1 = p_2 \text{ in } \Gamma \tag{18}$$

Equations (14) and (18) indicate the transmission condition on Γ, expressing the continuity of the pressure and mass conservation.

2.3 Weak Formulation

The classical weak formulation for (12-14) reads

$$\int_{\Omega_1} f\, q_1 dx = \int_{\Omega_1} \mathbf{K}\nabla p_1 \cdot \nabla q_1 dx - \int_{\Gamma} q_1(\boldsymbol{u}_2 \cdot \boldsymbol{\eta}_2) ds \quad \forall q_1 \in H^1_{0,\partial\Omega\cap\partial\Omega_1}(\Omega_1) \tag{19}$$

Using the mixed formulation for (15-18) we obtain

$$\int_{\Omega_2} \Lambda \boldsymbol{u}_2 \cdot \boldsymbol{v}_2 dx - \int_{\Omega_2} p_2 div(\boldsymbol{v}_2) dx + \int_{\Gamma} p_1(\boldsymbol{v}_2 \cdot \boldsymbol{\eta}_2) ds = 0,\ \forall\, \boldsymbol{u}_2 \in H_{0,\partial\Omega\cap\partial\Omega_2}(div;\Omega_2) \tag{20}$$

where $\Lambda = K^{-1}$.

Defining the bilinear and linear forms,

1. $c(\cdot,\cdot) : H^1(\Omega_1) \times H^1(\Omega_1) \to I\!R$, $(p,q) \mapsto \int_{\Omega_1} \mathbf{K}\nabla p \cdot \nabla q\, dx$;
2. $\mathbf{f}_1 : L^2(\Omega_1) \to I\!R$, $q \mapsto \int_{\Omega_1} f q\, dx$ for all $q \in L^2(\Omega_1)$;
3. $c_\Gamma : H^{\frac{1}{2}}_{00}(\Gamma) \times H^{\frac{1}{2}}_{00}(\Gamma)' \to I\!R$, $(q,\psi) \mapsto \int_\Gamma q\psi\, ds$;
4. $a(\cdot,\cdot) : H(div;\Omega_2) \times H(div;\Omega_2) \to I\!R$, $(\boldsymbol{u},\boldsymbol{v}) \mapsto \int_{\Omega_2} \Lambda \boldsymbol{u} \cdot \boldsymbol{v} dx$;
5. $b(\cdot,\cdot) : H(div,\Omega_2) \times L^2(\Omega_2)$, $(\boldsymbol{u}, p) \mapsto \int_{\Omega_2} p div(\boldsymbol{u}) dx$;
6. $\mathbf{f}_2 : L^2(\Omega_2) \to I\!R$, $q \mapsto \int_{\Omega_2} f q\, dx$ for all $q \in L^2(\Omega_2)$;

the problem reduces to: Find $p_1 \in H^1(\Omega_1)$ and $(\boldsymbol{u_2}, p_2) \in H_{0,\partial\Omega\cap\partial\Omega_2}(div,\Omega_2) \times L^2(\Omega_2)$ such that

$$\left\{ \begin{array}{ccl} c(p_1,q_1) - c_\Gamma(q_1,\boldsymbol{u_2}) & = & \mathbf{f}_1(q_1) \quad \forall\, q_1 \in H^1_{0,\partial\Omega\cap\partial\Omega_1}(\Omega_1) \\ a(\boldsymbol{u_2},\boldsymbol{v_2}) + c_\Gamma(p_1,\boldsymbol{v_2}) - b(\boldsymbol{v_2},p_2) & = & 0 \quad \forall\, \boldsymbol{v_2} \in H_{0,\partial\Omega\cap\partial\Omega_2}(div;\Omega_2) \\ -b(q_2,\boldsymbol{u_2}) & = & \mathbf{f}_2(q_2) \quad \forall\, q_2 \in L^2(\Omega_2) \end{array} \right. \tag{21}$$

The next step is to prove the existence of solution for (21). Let us introduce vectorial space $\mathbf{M} = H^1(\Omega_1) \times H(div;\Omega_2)$ with the graph norm

$$|||(q,\boldsymbol{u})|||^2 := \|q\|^2_{H^1(\Omega_1)} + \|\boldsymbol{u}\|^2_{H(div;\Omega_2)}. \tag{22}$$

Let $\tilde{\boldsymbol{w}} = (p_1, \boldsymbol{u}_2)$, $\tilde{\boldsymbol{v}} = (q_1, \boldsymbol{v}_2)$, and define

$$\begin{array}{rcl} \tilde{a} : \mathbf{M} \times \mathbf{M} & \to & I\!R \\ (\tilde{\boldsymbol{w}}, \tilde{\boldsymbol{v}}) & \mapsto & c(p_1,q_1) + a(\boldsymbol{u}_2,\boldsymbol{v}_2) - c_\Gamma(q_1,\boldsymbol{u}_2) + c_\Gamma(p_1,\boldsymbol{v}_2) \end{array} \tag{23}$$

$$\begin{array}{rcl} \tilde{\mathbf{f}} : L^2(\Omega_1) \times H(div;\Omega_2) \times L^2(\Omega_2) & \to & I\!R \\ (q_1, (\boldsymbol{v}_2, q_2)) & \mapsto & \int_{\Omega_1} f\, q_1 dx - \int_{\Omega_2} f\, q_2 dx \end{array}$$

$$\begin{array}{rcl} \tilde{b}(\cdot,\cdot) : \mathbf{M} \times L^2(\Omega_2) & \to & I\!R \\ (\tilde{\boldsymbol{w}}, q_2) & \mapsto & -\int_{\Omega_2} q_2\, div(\boldsymbol{u}_2) dx \end{array}$$

Thus the problem (21) can be written as: Find $\tilde{\boldsymbol{w}} \in \mathbf{M}$ and $p_2 \in L^2(\Omega)$ such that

$$\left\{ \begin{array}{c} \tilde{a}(\tilde{\boldsymbol{w}}, \tilde{\boldsymbol{v}}) + \tilde{b}(\tilde{\boldsymbol{v}}, p_2) = 0 \quad \forall \tilde{\boldsymbol{v}} \in \mathbf{M} \\ b(\tilde{\boldsymbol{w}}, q) = \tilde{\mathbf{f}}(q) \quad \forall\, q \in L^2(\Omega_2) \end{array} \right. \tag{24}$$

Lemma 2.1 *The bilinear form $\tilde{b}(\cdot,\cdot)$ is continuous and satisfies the inf-sup condition. That is, for all $p \in L^2(\Omega)$ exist $(\boldsymbol{u}, q) \in \boldsymbol{M} \times L^2(\Omega_2)$ and $\beta > 0$ such that*

$$\tilde{b}(p, (\boldsymbol{u}, q)) \geq \beta \|\|(\boldsymbol{u}, q)\|\|_{\tilde{\boldsymbol{M}}} \|p\|_{L^2(\Omega)} \tag{25}$$

Proof: We begin showing that $\tilde{b}$ is continuous.

$$\begin{aligned} |\tilde{b}((\boldsymbol{u}, q), p)| &= |\int_\Omega p\, div(\boldsymbol{u}) dx| \\ &\leq \|p\|_{L^2(\Omega)} \|\|div(\boldsymbol{u})\|\|_{L^2(\Omega)} \\ &\leq \|p\|_{L^2(\Omega)} \|\boldsymbol{u}\|_{H(div;\Omega)} \\ &\leq \|p\|_{L^2(\Omega)} \|\|(\boldsymbol{u}, q)\|\|_{\mathbf{M} \times L^2(\Omega_2)} \end{aligned} \tag{26}$$

Now, for $p \in L^2(\Omega)$ let φ be the solution of the problem

$$\left\{ \begin{array}{rcl} -\triangle\varphi &=& p \quad \text{on } \Omega_2 \\ \frac{\partial\varphi}{\partial\eta} &=& 0 \quad \text{on } \partial\Omega \cap \partial\Omega_2 \\ \varphi &=& 0 \quad \text{on } \Gamma \end{array} \right. \tag{27}$$

If $\boldsymbol{u} = -\nabla\varphi$, thus $div(\boldsymbol{u}) = p$. That is, $\boldsymbol{u} \in H_{0,\partial\Omega_2 \cap \partial\Omega}(\Omega_2)$ and

$$\|\boldsymbol{u}\|_{L^2(\Omega_2)^2} = \|\nabla\varphi\|_{L^2(\Omega_2)} \leq C\|p\|_{L^2(\Omega_2)}.$$

Consequently $\|\boldsymbol{u}\|_{Hdiv(\Omega_2)} \leq C_2\|p\|_{L^2(\Omega_2)}$. Setting $q = 0$ we have that

$$\begin{aligned} \tilde{b}((\boldsymbol{u}, q), p) &= \int_{\Omega_2} p\, div(\boldsymbol{u}) dx \\ &= \|p\|^2_{L^2(\Omega_2)} \\ &\geq C_2^{-1} \|\boldsymbol{u}\|_{H(div;\Omega)} \|p\|_{L^2(\Omega_2)} \\ &= C_2^{-1} \|(\boldsymbol{u}, q)\|_{\boldsymbol{M} \times L^2(\Omega)} \|p\|_{L^2(\Omega_2)}. \end{aligned}$$

Thus the *inf-sup condition* is verified with $\beta = C_2^{-1}$.

$\square$

For each of the bilinear forms $\tilde{a}$ and $\tilde{b}$ we associate the linear operators,

i) $\tilde{A} : \mathbf{M} \to \mathbf{M}'$

$$\langle A\tilde{\boldsymbol{w}}, \tilde{\boldsymbol{v}} \rangle = a(\tilde{\boldsymbol{w}}, \tilde{\boldsymbol{v}})$$

ii) $\tilde{B} : \mathbf{M} \to L^2(\Omega_2)'$ and $\tilde{B}^T : L^2(\Omega_2) \to \mathbf{M}'$ such that

$$\langle \tilde{B}\tilde{\boldsymbol{w}}, q_2 \rangle = \langle \tilde{\boldsymbol{w}}, \tilde{B}^T q_2 \rangle_{\mathbf{M} \times \mathbf{M}'} = \tilde{b}(\tilde{\boldsymbol{w}}, q_2)$$

Thus, setting $\tilde{\boldsymbol{w}} = (\boldsymbol{u}_2, p_1) \in \mathbf{M}$ and $p_2 \in L^2(\Omega_2)$, the system (24) can also be written as

$$\begin{pmatrix} \tilde{A} & \tilde{B}^T \\ \tilde{B} & 0 \end{pmatrix} \begin{pmatrix} \tilde{\boldsymbol{w}} \\ p_2 \end{pmatrix} = \begin{pmatrix} 0 \\ \tilde{\mathbf{f}} \end{pmatrix} \tag{28}$$

Lemma 2.2 *The bilinear form $\tilde{a}(\cdot,\cdot)$ defined by (23) is coercive on Ker($\tilde{B}$).*

Proof: Let $\tilde{\boldsymbol{w}} = (\boldsymbol{u}_2, p_1) \in Ker(\tilde{B})$, such that $\int_\Omega q_1 div(\boldsymbol{u}_2)dx = 0$ for all $q_1 \in L^2(\Omega)$. Getting $q_1 = div(\boldsymbol{u}_2)$ it follows that $div(\boldsymbol{u}_2) = 0$. Therefore, using Poincaré inequality we have that

$$\begin{aligned} \tilde{a}(\tilde{\boldsymbol{w}}, \tilde{\boldsymbol{w}}) &= c(p_1, p_1) + a(\boldsymbol{u}_2, \boldsymbol{u}_2) \\ &= \int_\Omega |\nabla p_1|^2 dx + \|\boldsymbol{u}\|^2_{H(div;\Omega)} \\ &\geq C_3 \|p_1\|_{H^1(\Omega_1)} + \|\boldsymbol{u}\|_{Hdiv(\Omega_2)} \\ &\geq min\{C_3, 1\} \|\tilde{\boldsymbol{w}}\|_{\mathbf{M}} \end{aligned}$$

Thus the coercive property holds with $\alpha = min\{C_3, 1\}$.

□

As a consequence of Lemma 2.1 and Lemma 2.2 and the application of classical results from Brezzi and Fortin (1991), we obtain

Proposition 2.1 *Given $\tilde{\boldsymbol{f}} \in Im\tilde{B}$, there exist unique $\tilde{\boldsymbol{w}} \in \boldsymbol{M}$, and $\tilde{p} \in L^2(\Omega_2)$ solution of the problem (24).*

3 CONCLUSIONS

In this present paper we present analytical aspects about coupling classical Galerkin and Mixed formulation for a specific model problem. On the interface between continuous formulation and mixed formulation a transmission condition is defined resulting in a well posed saddle point problem. In the future the formulation will be integrated in a finite element program and numerical tests performed.

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