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# COMPACTLY SUPPORTED RADIAL BASIS FUNCTIONS FOR FUNCTION INTERPOLATION: COMPARATIVE BEHAVIOUR

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**Abstract.** The use of compactly supported radial basis functions (CSRBFs) for interpolation of known functions in different methods in engineering is widespread. In this work a comprehensive comparative analysis was undertaken, using the root mean square error, for different weighed residual techniques and different Radial Basis Functions (RBFs). Interpolation was carried out in one and two dimensions by point collocation, point least-squares summation, and integral Galerkin method. The tested functions were taken from literature and cover most of the published functions. The results are presented in convergence plots which show the root mean square error against the number of RBFs taken. The results demonstrate convergence of the studied functions. It can be concluded from the results that some functions behave better than others in particular problems.

# 1 INTRODUCTION

Radial basis functions (RBFs), which return a value as a function of the radius from a given center, came into the attention of the engineering community quite a while ago. The early uses were mostly for interpolation of scattered data. The dual reciprocity formulation of the boundary element method uses RBFs for interpolation of the results (Goldberg et al., 1999). More recently, numerous approaches on meshless finite elements have been proposed using RBFs as local approximation functions. Compactly supported radial basis functions (CSRBFs) have been proposed in all realms of application of RBF interpolation, and present good accuracy for most applications. Their use is desirable for localization of the numerical procedure thereby transforming the full matrix into a banded matrix. The accuracy of such procedures, though, cannot be guaranteed *a priori* due to derivative continuity and other numerical issues which can arise with their application.

# 2 RADIAL BASIS FUNCTIONS

Let $\mathbb{R}^+ = \{x \in \mathbb{R}, x \geq 0\}$ be the non-negative half line and let $f : \mathbb{R}^+ \rightarrow \mathbb{R}$ be a continuous function with $f(0) \geq 0$. A radial basis function on $\mathbb{R}$ is a function of the form(Goldberg et al., 1999)

$$f(||\mathbf{P} - \mathbf{Q}||), \tag{1}$$

where $(\mathbf{P}, \mathbf{Q}) \in \mathbb{R}^n$ and $||.||$ denotes the Euclidean distance between $\mathbf{P}$ and $\mathbf{Q}$, i.e.

$$r = ||\mathbf{P} - \mathbf{Q}|| = [(\mathbf{P} - \mathbf{Q}) \cdot (\mathbf{P} - \mathbf{Q})]^{1/2}. \tag{2}$$

If one chooses $m$ points $\mathbf{Q}_j$, for $j = 1, m$, we can find the $m$ real coeficients $c_j$ such that the linear combination

$$g(\mathbf{P}) = \sum_{j-1}^{m} c_j f(||\mathbf{P} - \mathbf{Q}_j||) \tag{3}$$

interpolates a given function. This can also be called a radial basis function(Goldberg et al., 1999), though we prefer the terminology linear combination of radial basis functions.

Numerous propositions for function $f(r)$, taken from the literature, were tested for interpolation of different given functions in order to test their convergence properties and accuracy. The tested functions can be gathered into six groups, according to their respective references. Classical RBFs (Floater and Iske, 1996; Hickernell and Hon, 1999; de Boer et al., 2007) were tested to ensure the generality of the proposed procedures and for purpose of comparison. The tested classical RBFs are: Duchon's thin plate splines,

$$f(r) = r^2 log(r), \tag{4}$$

Hardy's multiquadrics,

$$f(r) = (c^2 + r^2)^{1/2}, \tag{5}$$

inverse multiquadrics,

$$f(r) = (c^2 + r^2)^{-1/2}, \tag{6}$$

Gaussians,

$$f(r) = e^{-r^2}, \tag{7}$$

quadric,

$$f(r) = 1 + r^2, \tag{8}$$

| Function | $C^n$ | Positive Definite |
|---|---|---|
| 12 | $C^0$ | $\mathbb{R}$ |
| 13 | $C^2$ | $\mathbb{R}$ |
| 14 | $C^4$ | $\mathbb{R}$ |
| 15 | $C^0$ | $\mathbb{R}^3$ |
| 16 | $C^2$ | $\mathbb{R}^3$ |
| 17 | $C^4$ | $\mathbb{R}^3$ |

Table 1: Wendland functions continuity (Floater and Iske, 1996).

and inverse quadric,

$$f(r) = \frac{1}{1+r^2}. \tag{9}$$

.

Also studied were the CSRBFs proposed in Buhmann (1998):

$$f(r) = \begin{cases} \frac{1}{3} + r^2 - \frac{4}{3}r^3 + 2r^2 log(r) & if \ \ 0 \leq r \leq 1, \\ 0 & otherwise, \end{cases} \tag{10}$$

and

$$f(r) = \begin{cases} \frac{1}{15} + \frac{19}{6}r^2 - \frac{16}{3}r^3 + 3r^4 - \frac{16}{15}r^5 + \frac{1}{6}r^6 + 2r^2 log(r) & if \ \ 0 \leq r \leq 1, \\ 0 & otherwise. \end{cases} \tag{11}$$

Most CSRBFs span the unit radius such as the functions 10 and 11. When the function support is needed for a different radius, a radius scaling can be performed in order to provide the needed support. All the functions described from now on will have only the support for $r \leq 1$, being zero for $r > 1$.

Were also tested a few functions proposed by Wendland (1995), taken from Floater and Iske (1996)):

$$f(r) = (1-r), \tag{12}$$
$$f(r) = (1-r)^3(3r+1), \tag{13}$$
$$f(r) = (1-r)^5(8r^2+5r+1), \tag{14}$$
$$f(r) = (1-r)^2, \tag{15}$$
$$f(r) = (1-r)^4(4r+1), \tag{16}$$
$$f(r) = (1-r)^6(35r^2+18r+3). \tag{17}$$

The functions 12 to 14 are said to be positive definite on $\mathbb{R}$ and the functions 15 to 17 positive definite on $\mathbb{R}^3$. Functions 12 and 15 present continuity $C^0$, functions 13 and 16 present continuity $C^2$, and functions 14 and 17 present continuity $C^4$. The continuity and the dimensionality of the space where the function is positive definite are shown at table 1:

The paper by de Boer et al. (2007) also mention a few compactly supported RBFs proposed

| Function | $C^n$ | Positive Definite |
|---|---|---|
| 18 | $C^0$ | $\mathbb{R}^3$ |
| 19 | $C^2$ | $\mathbb{R}^3$ |
| 20 | $C^4$ | $\mathbb{R}^3$ |
| 21 | $C^6$ | $\mathbb{R}^3$ |
| 22 | $C^0$ | |
| 23 | $C^1$ | |
| 24 | $C^2$ | |
| 25 | $C^2$ | |

Table 2: Wendland functions continuity (de Boer et al., 2007).

by Wendland (1996):

$$f(r) = (1-r)^2, \tag{18}$$
$$f(r) = (1-r)^4(4r+1), \tag{19}$$
$$f(r) = (1-r)^6(\frac{35}{3}r^2 + 6r + 1), \tag{20}$$
$$f(r) = (1-r)^8(32r^3 + 25r^2 + 8r + 1), \tag{21}$$
$$f(r) = (1-r)^5, \tag{22}$$
$$f(r) = 1 + \frac{80}{3}r^2 - 40r^3 + 15r^4 + \frac{8}{3}r^5 + 20r^2 log(r), \tag{23}$$
$$f(r) = 1 - 30r^2 - 10r^3 + 45r^4 - 6r^5 - 60r^3 log(r), \tag{24}$$
$$f(r) = 1 - 20r^2 + 80r^3 - 45r^4 - 16r^5 + 60r^4 log(r). \tag{25}$$

Functions 18 to 20 are equivalent to those described by Floater and Iske (1996), functions 15 to 17. Functions 18 to 21 are based on polynomials and present continuity $C^0$, $C^2$, $C^4$, and $C^6$, respectively. Functions 22 to 25 are based on the thin plate spline solution. Function 22 present $C^0$ continuity and function 23 present $C^1$ continuity. The are two possible $C^2$ continuous functions which are distinct, given by functions 24 and 25. The continuity and the dimensionality of the space where the function is positive definite are shown at table 2:

Wong et al. (2002) also proposed other Wendland (1995) functions:

$$f(r) = (1-r)^3, \tag{26}$$
$$f(r) = (1-r)^{10}(429r^4 + 450r^3 + 210r^2 + 50r + 5), \tag{27}$$
$$f(r) = (1-r)^{12}(2048r^5 + 2697r^4 + 1644r^3 + 566r^2 + 108r + 9), \tag{28}$$

as well as those proposed by Wu (1995):

$$f(r) = (1-r)^5(r^4 + 5r^3 + 9r^2 + 5r + 1), \tag{29}$$
$$f(r) = (1-r)^4(3r^3 + 12r^2 + 16r + 4), \tag{30}$$
$$f(r) = (1-r)^3(3r^2 + 9r + 8), \tag{31}$$
$$f(r) = (1-r)^6(5r^5 + 30r^4 + 72r^3 + 82r^2 + 36r + 6), \tag{32}$$
$$f(r) = (1-r)^7(35r^6 + 245r^5 + 720r^4 + 1120r^3 + 928r^2 + 336r + 48), \tag{33}$$
$$f(r) = (1-r)^6(35r^5 + 210r^4 + 515r^3 + 640r^2 + 384r + 64), \tag{34}$$
$$f(r) = (1-r)^5(35r^4 + 175r^3 + 345r^2 + 325r + 128). \tag{35}$$

| Function | $C^n$ | Positive Definite |
|---|---|---|
| 26 | $C^0$ | $\mathbb{R}^5$ |
| 27 | $C^8$ | $\mathbb{R}^3$ |
| 28 | $C^{10}$ | $\mathbb{R}^3$ |
| 29 | $C^4$ | $\mathbb{R}$ |
| 30 | $C^2$ | $\mathbb{R}^3$ |
| 31 | $C^0$ | $\mathbb{R}^5$ |
| 32 | $C^4$ | $\mathbb{R}^3$ |
| 33 | $C^4$ | $\mathbb{R}^5$ |
| 34 | $C^2$ | $\mathbb{R}^7$ |
| 35 | $C^0$ | $\mathbb{R}^9$ |

Table 3: Wendland and Wu functions continuity (Wong et al., 2002).

Function 26 is continuous and positive definite on $\mathbb{R}^5$, and functions 27 and 28 are two more $C^n$ continuous positive definite functions on $\mathbb{R}^3$, with $n$ respectively equal to $8$ and $10$. The same functions described on the previously mentioned references were also reported (Floater and Iske, 1996; de Boer et al., 2007). The continuity and the dimensionality of the space where the function is positive definite are shown at table 3:

Finally, a few CSRBFs used in meshless finite element Smooth Particle Hidrodynamics method (Belytschko et al., 1996) are also included, such as the exponential (still $f(r) = 0$ for $r > 1$), the cubic spline or the quartic spline functions.

The exponential function is:

$$f(r) = e^{-(r/\alpha)}, \tag{36}$$

where $\alpha$ is a scaling parameter. This exponential function is actually $C^{-1}$ continuous since it is not equal to zero at $S = 1$, but for numerical purposes, it resembles a function with $C^1$ continuity or higher ($\alpha = 0.4$ results in $f(1) = 0.002$).

The cubic spline function can be written as follows:

$$f(r) = \begin{cases} \frac{2}{3} - 4r^2 + 4r^3, & for \;\; r \leq \frac{1}{2}, \\ \frac{4}{3} - 4r + 4r^2 - \frac{4}{3}r^3, & for \;\; \frac{1}{2} < r \leq 1, \end{cases} \tag{37}$$

and the quartic spline:

$$f(r) = 1 - 6r^2 + 8r^3 - 3r^4. \tag{38}$$

## 3 WEIGHED RESIDUAL METHOD

To approximate a function $F(\overrightarrow{x})$, over the multidimensional domain $\Omega$, by a function

$$\hat{F}(\overrightarrow{x}) = G(\overrightarrow{x}) + \sum_{j=1}^{J} c_j N_j(\overrightarrow{x}), \tag{39}$$

using the weighed residual method, it is necessary first to define the residual function:

$$R(\overrightarrow{x}) = F(\overrightarrow{x}) - \hat{F}(\overrightarrow{x}). \tag{40}$$

In the equation (39) one has the known function $G(\overrightarrow{x})$ and the summation of a set of $J$ known functions $N_j(\overrightarrow{x})$, each one multiplied by a coefficient $c_j$ which shall be determined. The determination of these coefficients is performed by the weighed residual method.

The determined residual should at its best vanish at all points in the domain. The weighed residual methods take hold of this property to assert that

$$\int_\Omega P_i(\overrightarrow{x})R(\overrightarrow{x})d\Omega = 0, \tag{41}$$

where $P_i(\overrightarrow{x})$ is a set of $J$ arbitrary weigh functions. It is necessary to adopt $J$ functions $P_i(\overrightarrow{x})$, $i = 1, ..., J$, such that one obtains a system of $J$ equations and $J$ unknowns for the determination of the $c_j$ coefficients. The different sets of functions chosen will determine the specific method adopted.

Independent of the adopted method, using (40) and (41), the final equations become:

$$\int_\Omega P_i(\overrightarrow{x})\left[F(\overrightarrow{x}) - \hat{F}(\overrightarrow{x})\right] d\Omega = 0, \tag{42}$$

$$i = 1, ..., J,$$

$$\int_\Omega P_i(\overrightarrow{x})\hat{F}(\overrightarrow{x})d\Omega = \int_\Omega P_i(\overrightarrow{x})F(\overrightarrow{x})d\Omega, \tag{43}$$

$$i = 1, ..., J,$$

or, using (39):

$$\int_\Omega P_i(\overrightarrow{x})\left[G(\overrightarrow{x}) + \sum_{j=1}^{J} c_j N_j(\overrightarrow{x})\right] d\Omega = \int_\Omega P_i(\overrightarrow{x})F(\overrightarrow{x})d\Omega, \tag{44}$$

$$i = 1, ..., J,$$

or, alternativelly,

$$\int_\Omega P_i(\overrightarrow{x})\left[\sum_{j=1}^{J} c_j N_j(\overrightarrow{x})\right] d\Omega = \int_\Omega P_i(\overrightarrow{x})\left[F(\overrightarrow{x}) - G(\overrightarrow{x})\right] d\Omega, \tag{45}$$

$$i = 1, ..., J.$$

Equation (45) might be written in matrix form:

$$[A_{ij}]\{c_j\} = \{b_i\}, \tag{46}$$

where

$$A_{ij} = \int_\Omega P_i(\overrightarrow{x})N_j(\overrightarrow{x})d\Omega, \tag{47}$$

and

$$b_i = \int_\Omega P_i(\overrightarrow{x})\left[F(\overrightarrow{x}) - G(\overrightarrow{x})\right] d\Omega. \tag{48}$$

### 3.1 Point Collocation Method

The point collocation method consists in the attempt of vanishing the residual at determined points $\overrightarrow{x}_i$, which is equivalent of making function $\hat{F}$ equal to $F$ in those points.

In order to obtain the desired result one needs to find a set of functions $P_i$ such that

$$\int_{\Omega} P_i(\overrightarrow{x})R(\overrightarrow{x})d\Omega = R(\overrightarrow{x}_i), \tag{49}$$

hence, using (41), one gets

$$R(\overrightarrow{x}_i) = 0. \tag{50}$$

Functions $P_i$ are known from generalized function theory: the property desired (i.e., the integral of the product of the function itself times another function over a domain is equal to the other function evaluated at a given point) is equivalent to the translation property statement associated with the Dirac delta function. This generalized function, or measure, is represented by

$$P_i(\overrightarrow{x}) = \delta(\overrightarrow{x} - \overrightarrow{x}_i) = < \overrightarrow{x} - \overrightarrow{x}_i >^{-1}. \tag{51}$$

The actual definition of the Dirac delta function is that it evaluates to

$$\delta(x) = \begin{cases} +\infty & if \;\; x = 0, \\ \\ 0 & otherwise(x \neq 0), \end{cases} \tag{52}$$

and it also satisfies the identity

$$\int_{-\infty}^{+\infty} \delta(x)dx = 1. \tag{53}$$

Our weigh function $P_i$ takes the distance between the points $\overrightarrow{x}_i$ and $\overrightarrow{x}$ as the argument of the delta function.

So, the point collocation method can be stated as a weighed residual method with the weigh function equal to the Dirac delta function.

### 3.2 Galerkin Method

Galerkin method is particularly suited to the adoption of approximation functions $N_j$ mutually orthogonal, in the problem domain. Two functions are said to be orthogonal if the integral of their product over the domain vanishes, i.e.:

$$\int_{\Omega} N_i(\overrightarrow{x})N_j(\overrightarrow{x})d\Omega \begin{cases} = 0 & if \;\; i \neq j, \\ \neq 0 & if \;\; i = j. \end{cases} \tag{54}$$

For the Galerkin method one adopts

$$P_i(\overrightarrow{x}) = N_i(\overrightarrow{x}), \tag{55}$$

$$i = 1, ..., J.$$

The final equation of the Galerkin method becomes

$$\int_{\Omega} N_i(\overrightarrow{x})R(\overrightarrow{x})d\Omega = 0, \tag{56}$$

$$i = 1, ..., J,$$

or

$$\int_\Omega N_i(\overrightarrow{x}) \left[ F(\overrightarrow{x}) - G(\overrightarrow{x}) - \sum_{j=1}^{J} c_j N_j(\overrightarrow{x}) \right] d\Omega = 0, \tag{57}$$

$$i = 1, ..., J,$$

such that

$$\int_\Omega N_i(\overrightarrow{x}) \left[ \sum_{j=1}^{J} c_j N_j(\overrightarrow{x}) \right] d\Omega = \int_\Omega N_i(\overrightarrow{x}) \left[ F(\overrightarrow{x}) - G(\overrightarrow{x}) \right] d\Omega, \tag{58}$$

$$i = 1, ..., J,$$

thus, as the coefficients $c_j$ do not depend of the internal domain points, and the functions are continuous and limited throughout the domain:

$$\sum_{j=1}^{J} \left[ \int_\Omega N_i(\overrightarrow{x}) N_j(\overrightarrow{x}) d\Omega \right] c_j = \int_\Omega N_i(\overrightarrow{x}) \left[ F(\overrightarrow{x}) - G(\overrightarrow{x}) \right] d\Omega, \tag{59}$$

$$i = 1, ..., J,$$

which reminds us of the weighed residual statement, Eq. 46.

If the adopted functions are mutually orthogonal the matrix $A_{ij}$ will be a diagonal matrix.

### 3.3 Least Squares Method

For the least squares method it is assumed that if the function cannot be exactly approximated, at least the error can be minimized for this approximation function. In order to guarantee that the error is always positive (the above defined residual function, Eq. 40, can have negative values), the square of the residual function will be taken. Therefore, the square of the residual function will be minimized, ergo the least squares method.

The objective is, thus, to find the coefficients $c_j$ that minimize the function:

$$Q = \int_\Omega \left( R(\overrightarrow{x}) \right)^2 d\Omega, \tag{60}$$

or

$$Q = \int_\Omega \left( F(\overrightarrow{x}) - \hat{F}(\overrightarrow{x}) \right)^2 d\Omega. \tag{61}$$

The sought coefficients represent a minimum of the function $Q$, and can be found by making its partial derivatives with respect to the unknowns vanish:

$$\frac{\partial Q}{\partial c_i} = 0, \tag{62}$$

$$i = 1, ..., J.$$

Thus:

$$
\begin{aligned}
\frac{\partial Q}{\partial c_i} &= \frac{\partial}{\partial c_i}\int_\Omega \left(F(\overrightarrow{x}) - \hat{F}(\overrightarrow{x})\right)^2 d\Omega = 0, \\
&= \int_\Omega \frac{\partial}{\partial c_i}\left[\left(F(\overrightarrow{x}) - \hat{F}(\overrightarrow{x})\right)^2\right] d\Omega \\
&= 2\int_\Omega \left[F(\overrightarrow{x}) - \hat{F}(\overrightarrow{x})\right]\left(-\frac{\partial \hat{F}(\overrightarrow{x})}{\partial c_i}\right) d\Omega = 0.
\end{aligned} \tag{63}
$$

Dividing both sides by $-2$, and substituting the partial derivative by the result,

$$
\frac{\partial \hat{F}(\overrightarrow{x})}{\partial c_i} = \frac{\partial}{\partial c_i}\left[G(\overrightarrow{x}) + \sum_{j=1}^{J} c_j N_j(\overrightarrow{x})\right] = N_i(\overrightarrow{x}), \qquad i = 1, ..., J, \tag{64}
$$

and then

$$
\int_\Omega \left[F(\overrightarrow{x}) - \hat{F}(\overrightarrow{x})\right](N_i(\overrightarrow{x}))\, d\Omega = 0, \qquad i = 1, ..., J. \tag{65}
$$

The least squares method for approximation of known functions is, therefore, equivalent to the Galerkin method, since

$$
P_i(\overrightarrow{x}) = N_i(\overrightarrow{x}), \qquad i = 1, ..., J. \tag{66}
$$

#### 3.3.1 Point Least Squares Method

The most common usage of the least squares method for function approximation is to find a function that best approximates a set of data points. This application arises frequently in experimental data analysis, and can be stated as follows: to approximate a set of $N$ given points, by a function, such that the square of the error is minimal. The difference between this approach and the continuous least squares method is that the error is computed as a summation of the square of the residual function values computed at the given points.

Equation (60), rewritten in this way, becomes

$$
Q = \sum_{n=1}^{N} \left(R(x_n)\right)^2, \tag{67}
$$

or, (61),

$$
Q = \sum_{n=1}^{N} \left(F(x_n) - \hat{F}(x_n)\right)^2. \tag{68}
$$

Turning our attention back to equation 62, reproduced next:

$$
\frac{\partial Q}{\partial c_i} = 0, \tag{62}
$$

$$i = 0, ..., J,$$

one gets the point form of equation (65), which in matrix form can be expressed as

$$\begin{bmatrix} \sum N_0 N_0 & \sum N_0 N_1 & \cdots & \sum N_0 N_J \\ \sum N_1 N_0 & \sum N_1 N_1 & \cdots & \sum N_1 N_J \\ \vdots & \vdots & \ddots & \vdots \\ \sum N_J N_0 & \sum N_J N_1 & \cdots & \sum N_J N_J \end{bmatrix} \begin{Bmatrix} c_0 \\ c_1 \\ \vdots \\ c_J \end{Bmatrix} = \begin{Bmatrix} \sum f_n N_0 \\ \sum f_n N_1 \\ \vdots \\ \sum f_n N_J \end{Bmatrix}, \tag{69}$$

where the summation of the $N_j$ functions is taken for each of the $x_n$ points.

## 4 RESULTS

A series of interpolation trial functions was attempted, in order to show convergence. For all cases the error measure presented is the normalised root mean squared error. In order to obtain this error measure we need, first, to compute the mean square error (Equation 70), calculated as the second moment of the error with respect to the real value:

$$E^2 = \frac{\sum_{i=1}^{n} (\hat{F}(x) - F(x))^2}{n}. \tag{70}$$

The root mean squared error, in turn, is obtained by taking the square root of the mean square error.

$$E = \sqrt{E^2}. \tag{71}$$

The normalised root mean squared error is then calculated by normalising the root mean squared error with respect to the difference between maximum and minimum values of the interpolated function.

$$e = \frac{E}{F_{max} - F_{min}}. \tag{72}$$

All plots are made using a bi-logarithmic scale, to better present order of magnitude convergence.

### 4.1 One dimensional interpolation

Equation 73 expresses the interpolated function, which shall be called general function (for distinction from specific polynomial functions), shown at Figure 1.

$$f(x) = 2x^2 + 10e^{25(0.3-x)^2} + 3sen(4x) - 4log(x + 0.2). \tag{73}$$

The interpolation results using the different weighed residual techniques are plotted at Figures 2 to 7:

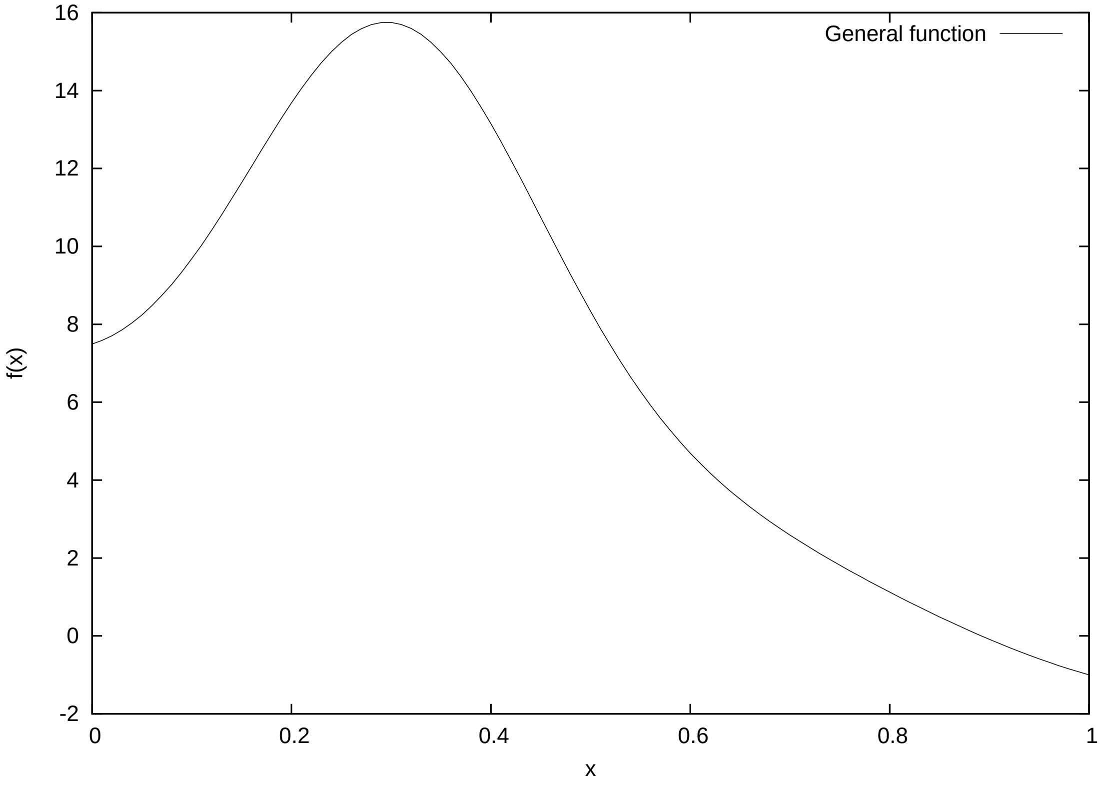


Figure 1: General Function.

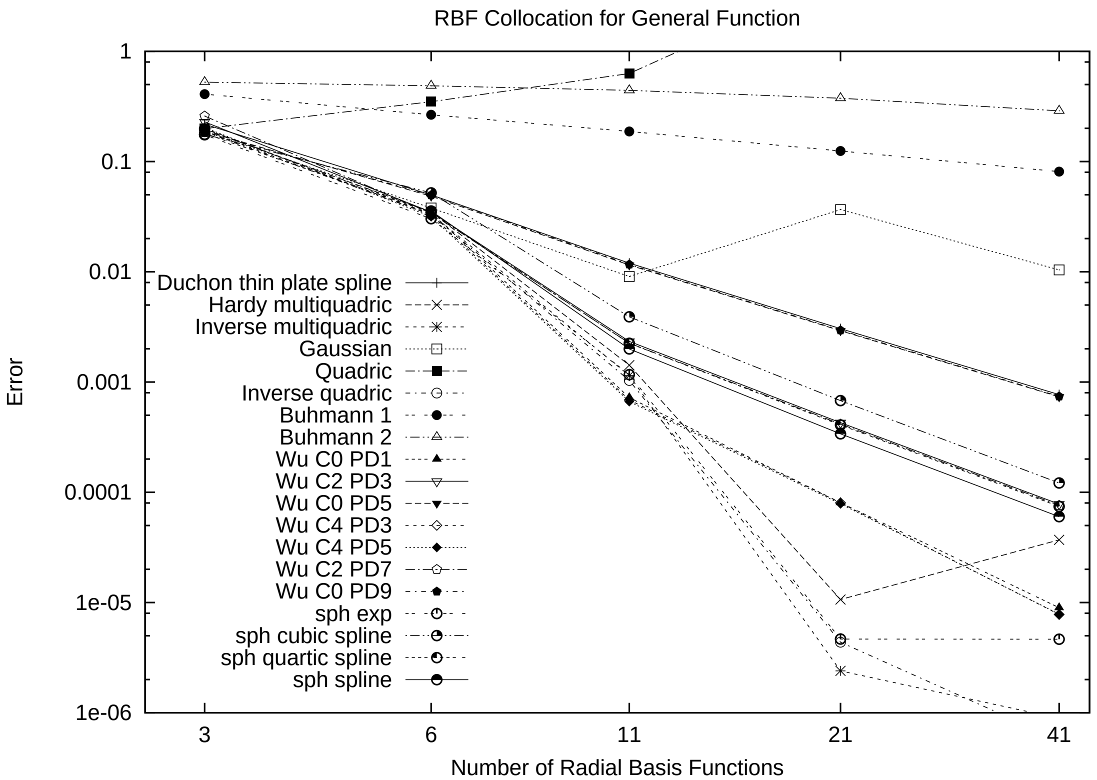


Figure 2: General function interpolation using Classical and other RBFs, Collocation method.

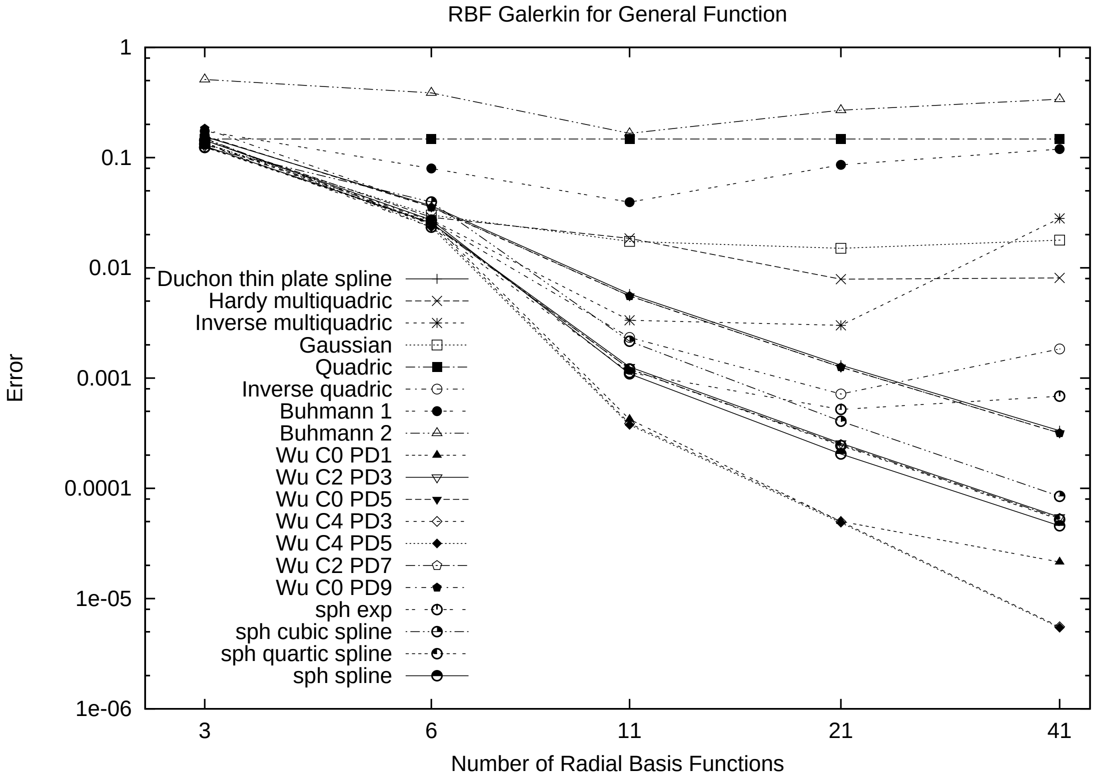


Figure 3: General function interpolation using Classical and other RBFs, Galerkin method.

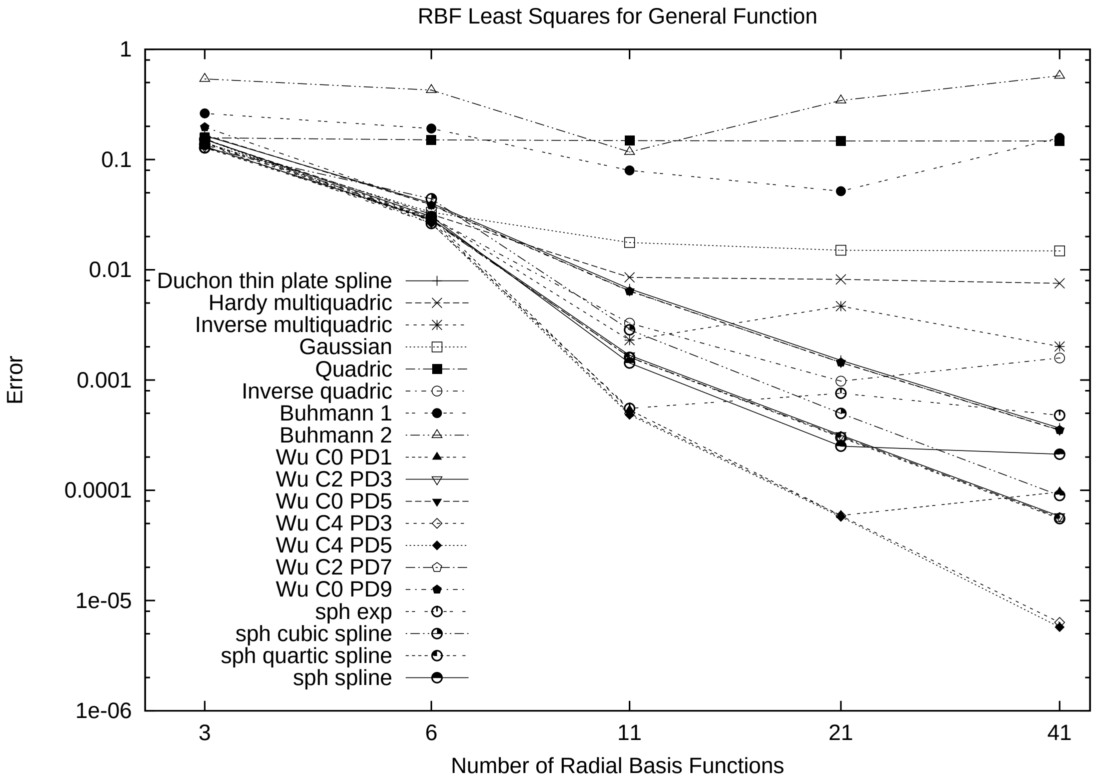


Figure 4: General function interpolation using Classical and other RBFs, Least squares method.

For comparison purpose, all plots are presented in the same scale. The Quadric function does not present convergence, regardless of the interpolation method. This function even diverges for the collocation method. Other classical functions with convergence problems are Hardy's multiquadrics (Figure 2) and Inverse multiquadrics (Figures 3 and 4). Gaussian functions do not behave better, with lower error, when an increased number of functions is used.

Wu functions presented the most satisfactory results, even when compared to some classical functions. Buhmann functions presented far superior error values than those obtained by other interpolation functions, and do not converge.

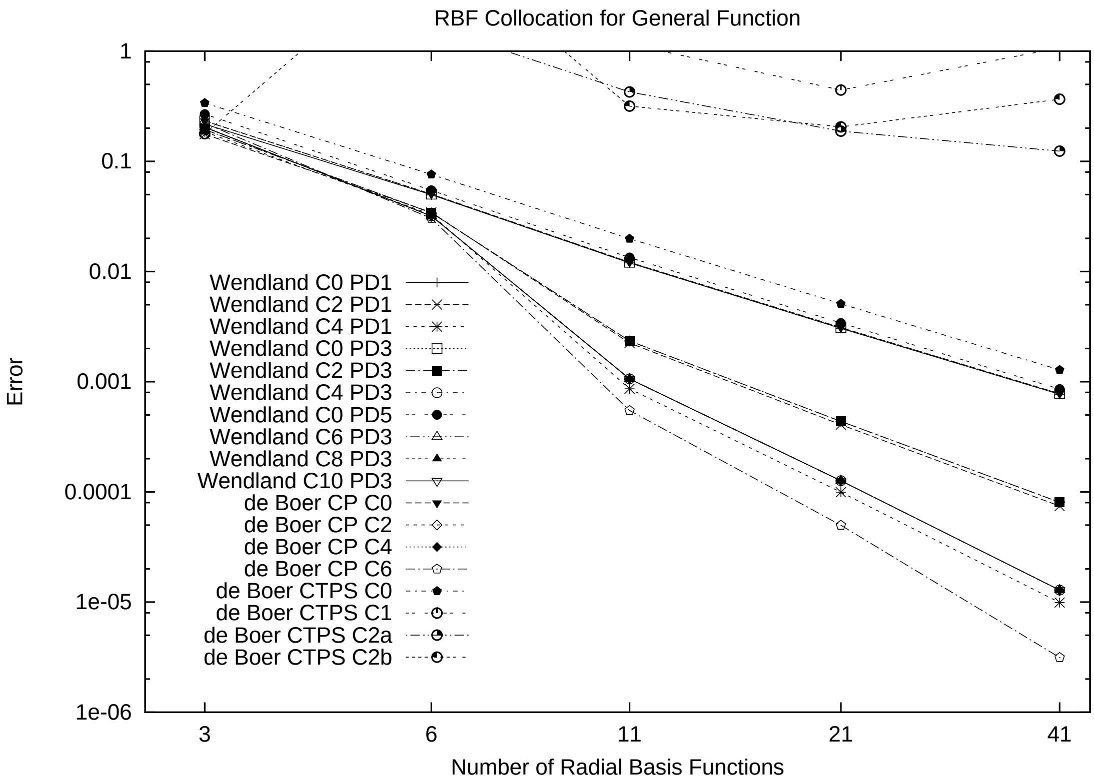


Figure 5: General function interpolation using Wendland RBFs, Collocation method.

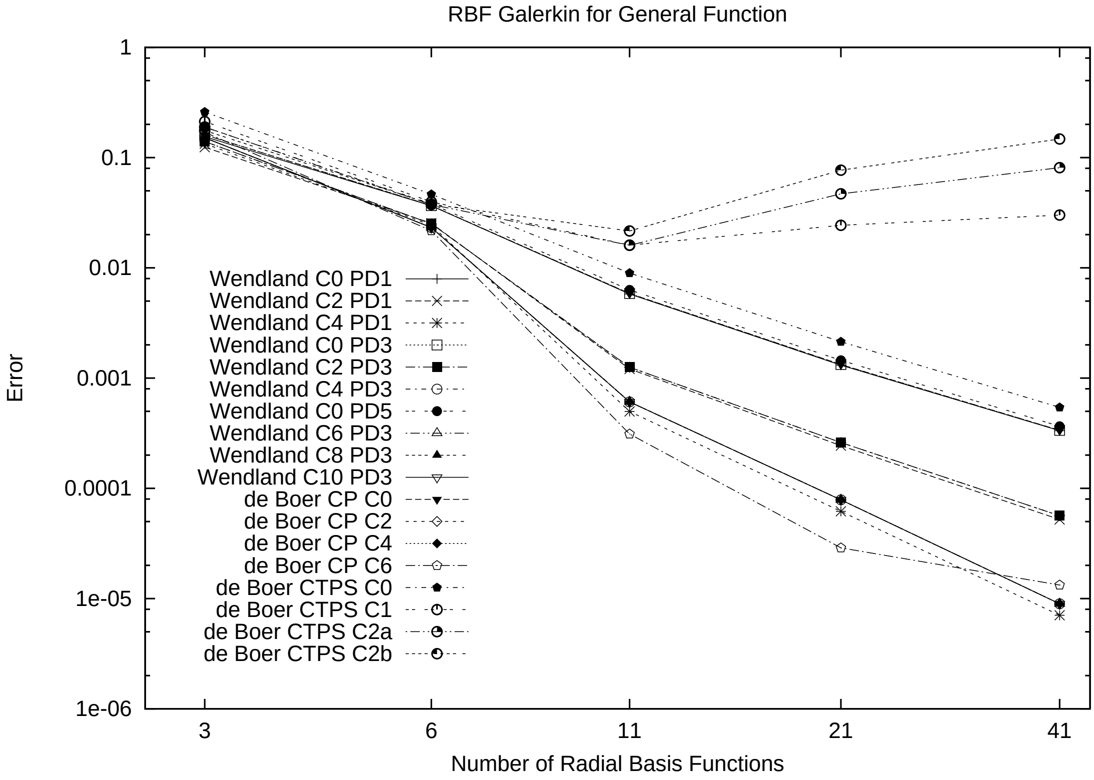


Figure 6: General function interpolation using Wendland RBFs, Galerkin method.

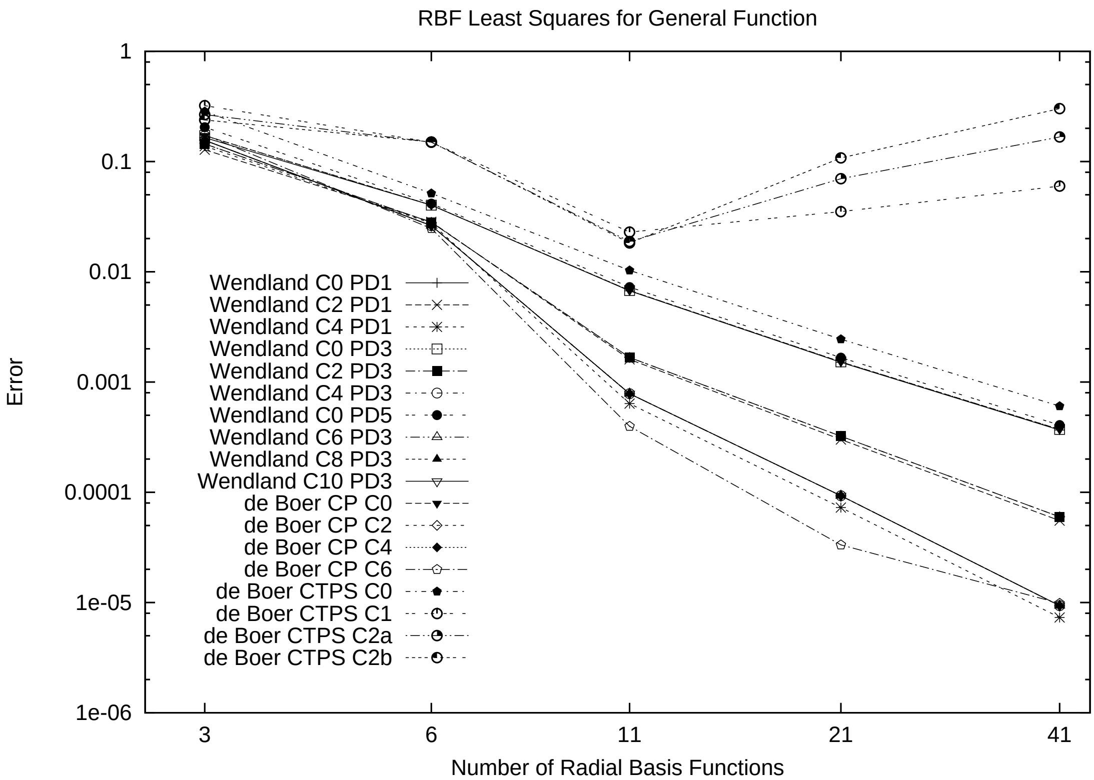


Figure 7: General function interpolation using Wendland RBFs, Least squares method.

Wendland functions, Figures 5 to 7, presented satisfactory results for one dimensional interpolation. The interpolation using Galerkin method obtained results very similar to the point least squares. Some higher order functions interpolations coincide. Some of the thin plate spline based functions proposed by de Boer et al. (2007) (Eq. 23 to 25) did not present convergence for the adopted methods, with exception to the $C^0$ continuity function (Eq. 22).

### 4.2 Two dimensional interpolation

The test function for two dimensional interpolation is Franke's function (Eq. 74, Fig. 8), taken from Floater and Iske (1996).

$$\begin{aligned} f(x) &= 0.75e^{(-\frac{(9x-2)^2}{4}-\frac{(9y-2)^2}{4})} + 0.75e^{(-\frac{(9x-2)^2}{49}-\frac{(9y-2)^2}{10})} \\ &+ 0.5e^{(-\frac{(9x-7)^2}{4}-\frac{(9y-3)^2}{4})} - 0.2e^{(-(9x-4)^2-(9y-7)^2)}. \end{aligned} \tag{74}$$

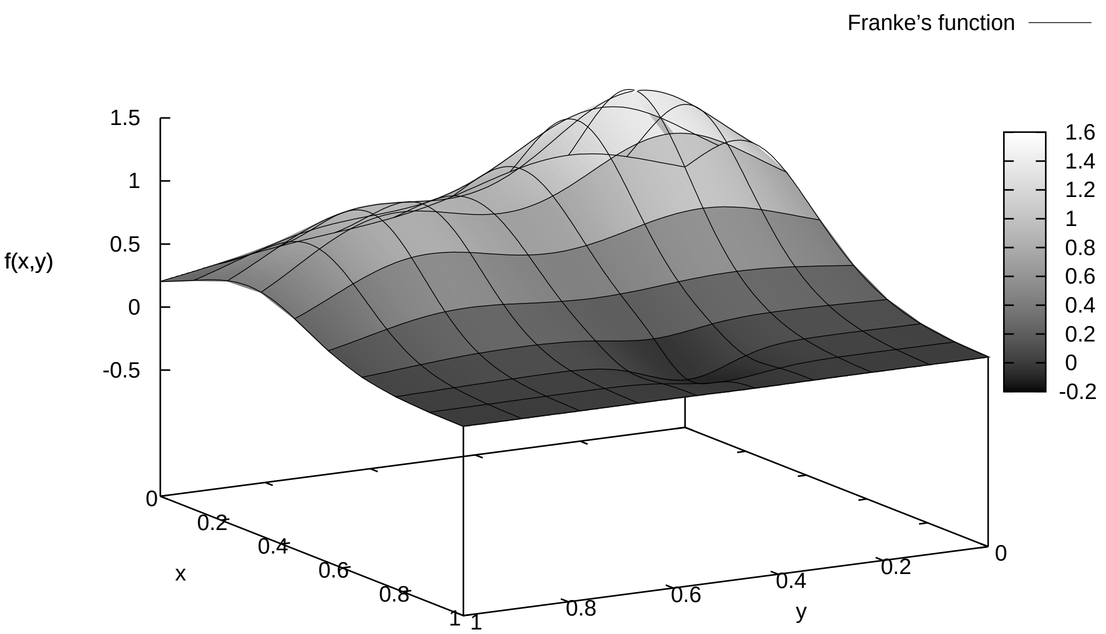


Figure 8: Franke's function.

For this test the interpolation was attempted using function sets of 4 (2x2, positioned on each vertix of the domain), 16 (4x4), and 64 (8x8) functions. The interpolation results for the different methods are shown next (Figures 9 to 14):

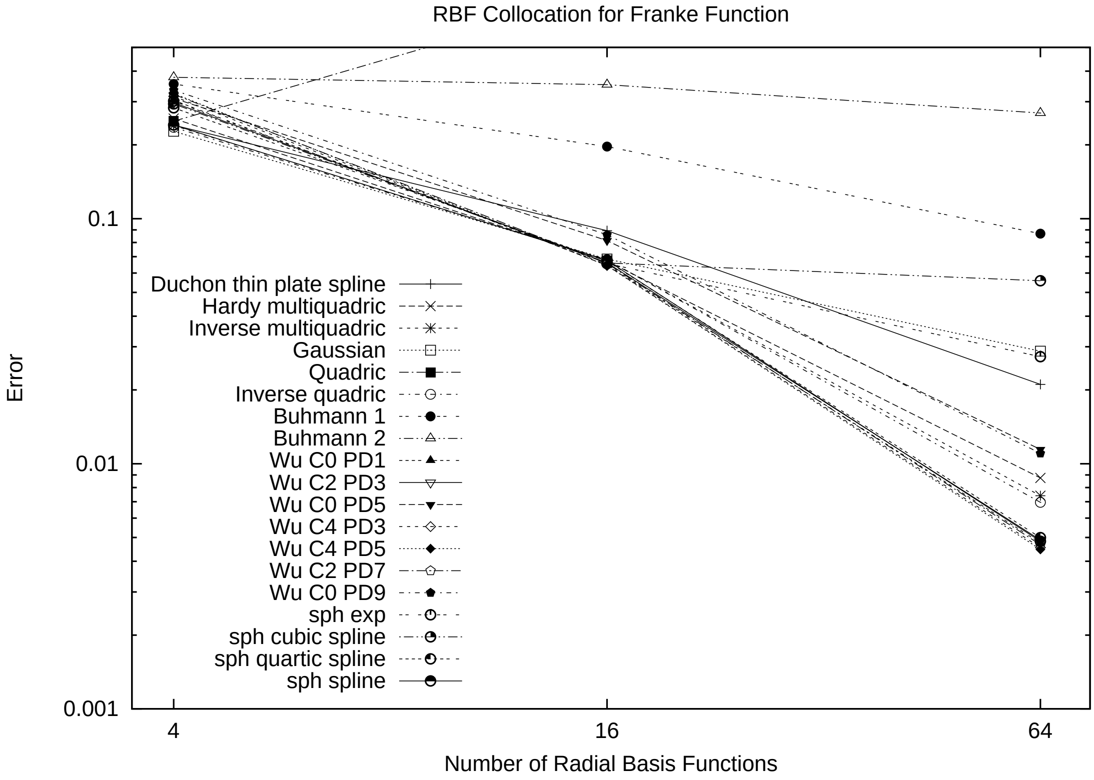


Figure 9: Franke’s function interpolation using Classical and other RBFs, Collocation method.

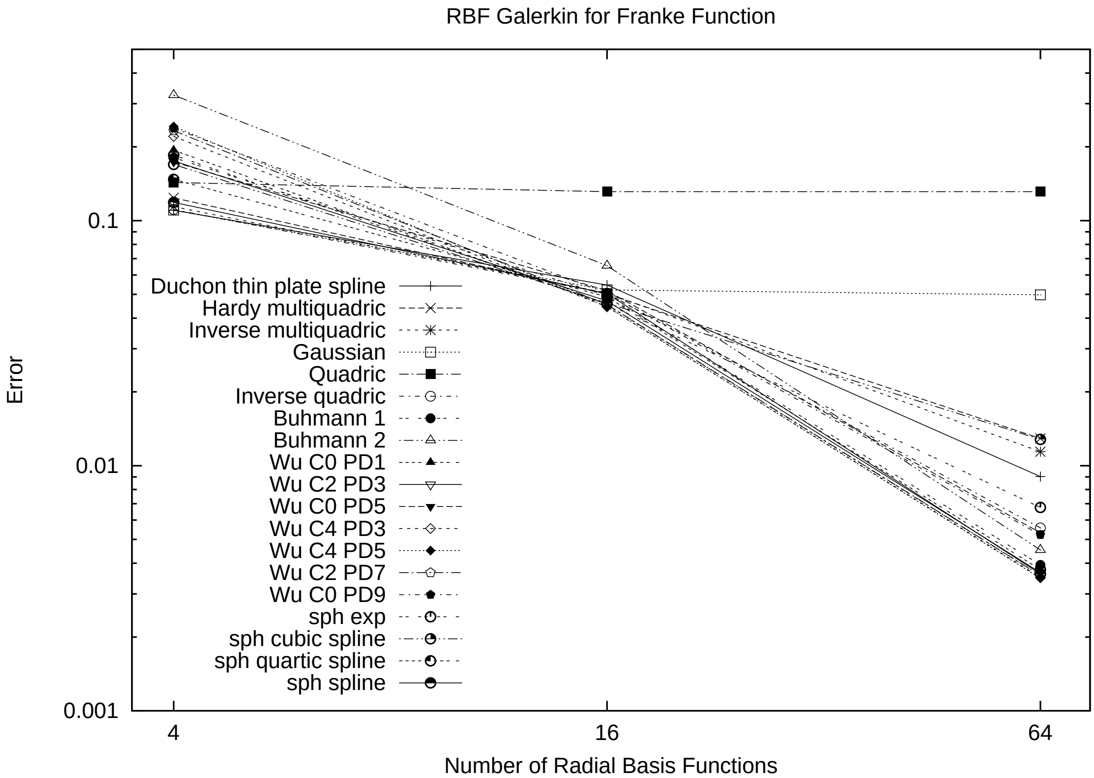


Figure 10: Franke’s function interpolation using Classical and other RBFs, Galerkin method.

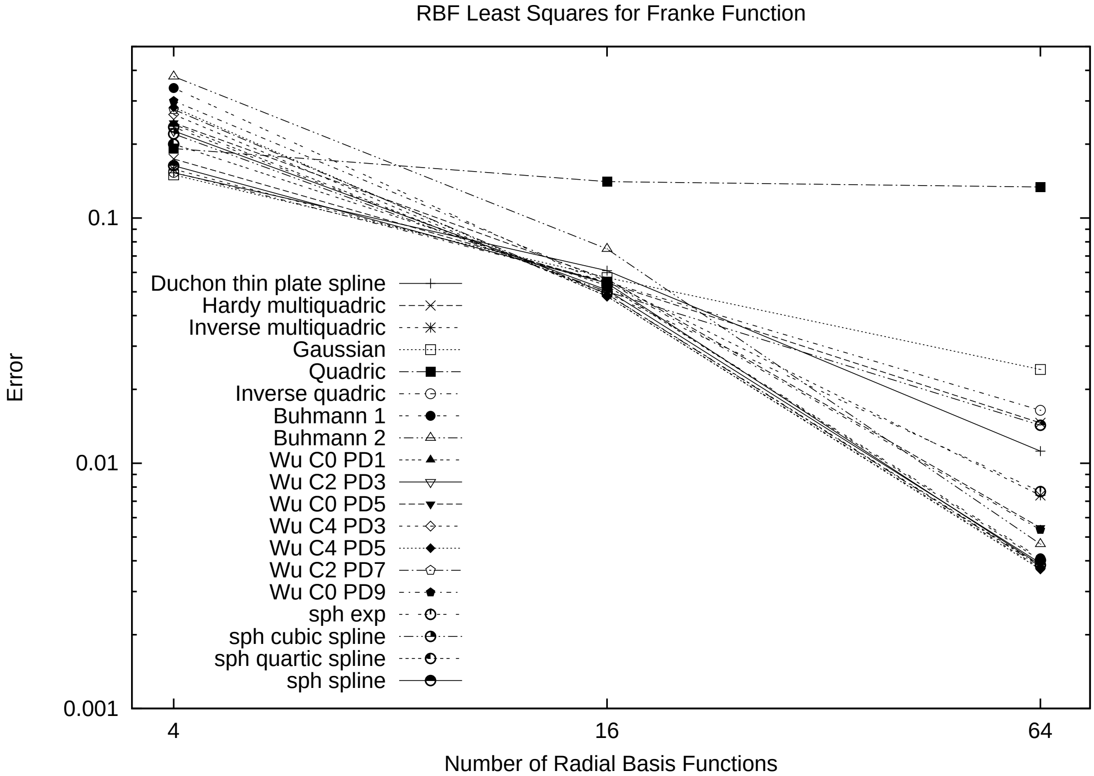


Figure 11: Franke's function interpolation using Classical and other RBFs, Least Squares method.

Classical functions behave better for the two dimensional problem tested, though the Quadric function still diverges. The Gaussian function and Inverse Quadric function also did not present convergence: the first diverges and the second does not show improvement.

Buhmann functions present a slight improvement for an increased number of functions, but the error is still quite large in comparison to the other interpolations for the Collocation method. This can be accounted by a problem of the Collocation method, which takes only the central point of the function. Wu functions always converge in a smooth fashion for the two dimensional interpolation. The Cubic Spline function from Smooth Particle Hidrodynamics (Eq. 37) presented numerical problems for the Collocation method, but did converge for the other methods.

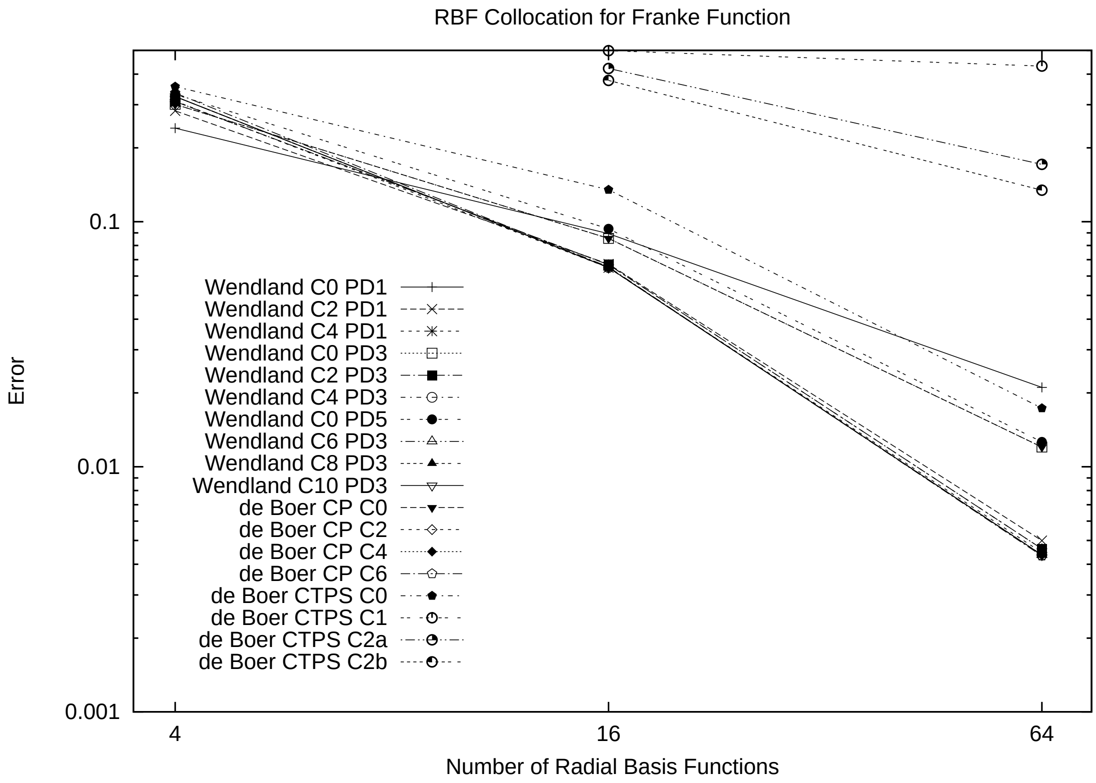


Figure 12: Franke's function interpolation using Wendland RBFs, Collocation method.

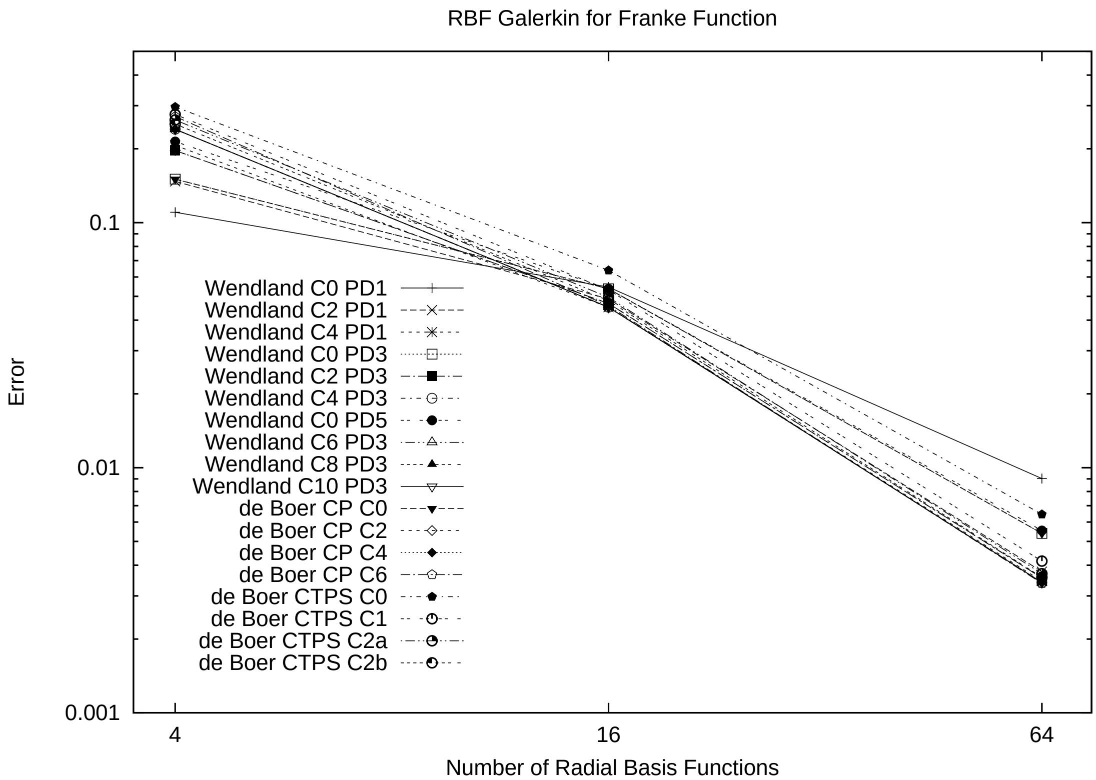


Figure 13: Franke's function interpolation using Wendland RBFs, Galerkin method.

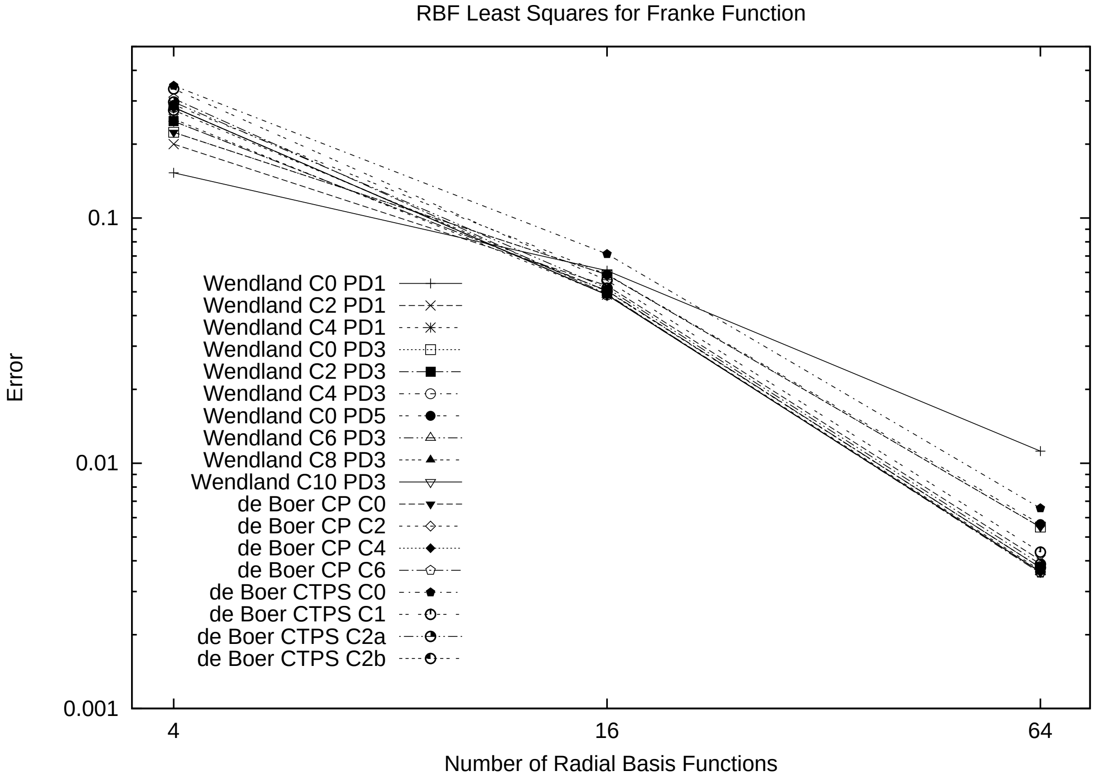


Figure 14: Franke's function interpolation using Wendland RBFs, Least Squares method.

As for the one dimensional interpolation, three of the thin plane spline based functions proposed by de Boer et al. (2007) (Eq. 23 to 25) did not present convergence for methods adopted, though for two dimensional interpolation that behaviour was only observed for the Collocation method.

## 5 FINAL REMARKS

The results demonstrate the convergence of the majority of the tested functions. A few numerical problems were detected, particularly in the case of the Quadric function and the Buhmann functions, which presented results respectively divergent and imprecise. The Gaussian function and the Inverse quadric functions also presented convergence problems.

In general, Wu functions and Wendland functions behaved well, whereas Buhmann functions and the functions based on the thin plate spline function proposed by de Boer et al. (2007) (Eq. 23 to 25) do not improve results with increasing number of functions.

The results, using a very coarse point set, highlighted a few properties of classical and compact support radial basis functions for data interpolation. The results will be implemented in a meshless finite element approach, taking advantage of the inheritance properties of C++ programming language.

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