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REGULARIZATION OF INVERSE ILL-POSED PROBLEMS WITH GENERAL PENALIZING TERMS: APPLICATIONS TO IMAGE RESTAURATION.

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Abstract. The Tikhonov-Phillips method is widely used for regularizing ill-posed problems due to the simplicity of its formulation as an optimization problem. The use of different penalizers in the functionals associated to the corresponding optimization problems has originated a few other methods which can be considered as "variants" of the traditional Tikhonov-Phillips method of order zero. Such is the case for instance of the Tikhonov-Phillips method of order one, the total variation regularization method, etc. The purpose of this article is twofold. First we study the problem of determining general sufficient conditions on the penalizers in generalized Tikhonov-Phillips functionals which guarantee existence and uniqueness of minimizers, in such a way that finding such minimizers constitutes a regularization method, that is, in such a way that these minimizers approximate, as the regularization parameter tend to 0^+, a least squares solution of the problem. Secondly we also study the problem of characterizing those limiting least square solutions in terms of properties of the penalizers and find conditions which guarantee that the regularization method thus defined is stable under different types of perturbations. Finally, several examples with different penalizers are presented and a few numerical results in an image restoration problem are shown which better illustrate the results.

1 INTRODUCTION

In a quite general framework an inverse problem can be formulated as the need for determining x in an equation of the form

$$Tx = y, \tag{1}$$

where T is a linear bounded operator between two infinite dimensional Hilbert spaces X and Y (in general these will be function spaces), the range of T, $\mathcal{R}(T)$, is non-closed and y is the data, supposed to be known, perhaps with a certain degree of error. It is well known that under these hypotheses problem (1) is ill-posed in the sense of Hadamard (Hadamard (1902)). In this case the ill-posedness is a result of the unboundedness of $T^{\dagger}$, the Moore-Penrose generalized inverse of T. The Moore-Penrose generalized inverse is a fundamental tool in the treatment of inverse ill-posed problems and their regularized solutions, mainly due to the fact that this operator is strongly related to the least squares solutions of problem (1). In fact, it is well known that the least squares solution of minimum norm of problem (1), known as the best approximate solution, is precisely given by $x^{\dagger} \doteq T^{\dagger}y$. Moreover, the set of all least squares solutions of problem (1) is given by $x^{\dagger} + \mathcal{N}(T)$, where $\mathcal{N}(T)$ denotes the kernel of the operator T.

The unboundedness of $T^{\dagger}$ has as an undesired consequence the fact that small errors or noise in the data y can result in arbitrarily large errors in the corresponding approximated solutions, turning unstable all standard numerical approximation methods, making them unsuitable for most applications and inappropriate from any practical point of view. The so called "regularization methods" are mathematical tools designed to restore stability to the inversion process and consist essentially of parametric families of continuous linear operators approximating $T^{\dagger}$. The mathematical theory of regularization methods is very wide (a comprehensive treatise on the subject can be found in the book by Engl, Hanke and Neubauer, (Engl et al. (1996))) and it is of great interest in a broad variety of applications in many areas such as Medicine, Geology, Geophysics, Biology, image restauration and processing, etc.

There exist numerous ways of regularizing an inverse ill-posed problem. Among the most standar and traditional methods we mention the Tikhonov-Phillips method, truncated singular value decomposition, Showalter's method, total variation regularization (Acar and Vogel (1994)), etc. Among all regularization methods, probably the best known and most commonly and widely used is the Tikhonov-Phillips regularization method, which was originally proposed by Tikhonov and Phillips in 1962 and 1963 (Phillips (1962), Tikhonov (1963)). Although this method can be rigorously formalized within a very general framework by means of spectral theory (Dautray and Lions (1990)), the widespread of its use is undoubtedly due to the fact that it can also be formulated in a very simple way as an optimization problem. In fact, the regularized solution of problem (1) obtained by applying Tikhonov-Phillips method is the minimizer x_{α} of the functional

$$J_{\alpha}(x) \doteq \|Tx - y\|^2 + \alpha \|x\|^2, \tag{2}$$

where α is a positive constant known as the regularization parameter.

The penalizing term $\alpha \|x\|^2$ in (2) not only induces stability but it also determines certain regularity properties of the approximating regularized solutions x_{α} and of the corresponding least squares solution which they approximate as $\alpha \to 0^+$. Thus, for instance, it is well known that minimizers of (2) are always "smooth" and, for $\alpha \to 0^+$, they approximate the least squares solution of minimum norm of (1), that is $\lim\limits_{\alpha \to 0^+} x_{\alpha} = T^{\dagger}y$. This method is more precisely known as the Tikhonov-Phillips method of order zero. Choosing other penalizing terms gives rise to different approximations with different properties, and approximating different least squares solutions of (1). Thus, for instance, the use of $\|\nabla x\|^2$ as penalizer instead of $\|x\|^2$ in (2) originates the so called Tikhonov-Phillips method of order one, the penalizer $\|x\|_{\text{BV}}$ (where $\|\cdot\|_{\text{BV}}$ denotes the bounded variation norm) gives rise to the so called bounded variation regularization method introduced by Acar and Vogel in 1994 (Acar and Vogel (1994)), etc. In particular, in the latter case, the approximating solutions are only forced to be of bounded variation rather than smooth and they approximate, for $\alpha \to 0^+$, the least squares solution of problem (1) of minimum $\|\cdot\|_{\text{BV}}$-norm (see Acar and Vogel (1994)). This method has been proved to be a much better choice for instance in image restoration problems in which it is highly desirable preserve and detect sharp edges and discontinuities of the original image.

Hence, the penalizing term in (2) is used not only to stabilize the inversion of the ill-posed problem but also to enforce certain characteristics on the approximating solutions and on the particular limiting least squares solution that they approximate. As a consequence, it is reasonable to assume that an adequate choice of the penalizing term, based on certain *"a-priori"* knowledge about the exact solution of problem (1), will lead to approximated "regularized" solutions which will appropriately reflect such characteristics.

With the above considerations in mind, we shall consider functionals of the form

$$J_{W,\alpha}(x) \doteq \|Tx - y\|^2 + \alpha W(x), \qquad x \in \mathcal{D}, \tag{3}$$

where $W(\cdot)$ is an arbitrary functional with domain $\mathcal{D} \subset \mathcal{X}$ and α is a positive constant.

The purpose of this article is mainly twofold. First we study the problem of determining general sufficient conditions on the penalizer $W(\cdot)$ which guarantee existence and uniqueness of minimizers of the generalized Tikhonov-Phillips functional (3), in such a way that finding such minimizers constitutes a regularization method for (1), that is, in such a way that these minimizers approximate, for $\alpha \to 0^+$, a least squares solution of (1).

Secondly we shall also study the problem of characterizing those limiting least square solutions in terms of properties of $W(\cdot)$ and find conditions on the penalizer which guarantee that the regularization method thus defined is stable under different types of perturbations.

Finally, several examples with different penalizers will be presented and a few numerical results in an image restoration problem will be shown in order to better illustrate the results.

2 EXISTENCE, UNIQUENESS AND STABILITY FOR GENERAL PENALIZING TERMS

In this section we shall consider the problem of finding conditions on the penalizer $W(\cdot)$ which guarantee existence, uniqueness and stability of the minimizers of (3). Previously we will need to introduce a few definitions.

Definition 2.1 *Let $\mathcal{X}$ be a vector space, W a functional defined over a set $\mathcal{D} \subset \mathcal{X}$ and A a subset of $\mathcal{D}$. We say that A is W-bounded if there exists a constant $k < \infty$ such that $|W(a)| \leq k$ for every $a \in A$.*

Definition 2.2 *(W-coercivity) Let $\mathcal{X}$ be a vector space and W, F functionals defined on a set $\mathcal{D} \subset \mathcal{X}$. We will say that the functional F is W-coercive if $\lim_{n\to\infty} F(x_n) = +\infty$ for every sequence $\{x_n\} \subset \mathcal{D}$ for which $\lim_{n\to\infty} W(x_n) = +\infty$.*

In the following theorem, sufficient conditions on the operator T and the functional W guaranteeing the existence and uniqueness of the minimizer of the functional (3) are established.

Theorem 2.3 *(Existence and uniqueness) Let $\mathcal{X}$ be a normed vector space, $\mathcal{Y}$ an inner product space, $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$, $y \in \mathcal{Y}$, $\mathcal{D} \subset \mathcal{X}$ a convex set and $W : \mathcal{D} \longrightarrow \mathbb{R}$ satisfying the following hypotheses:*

- *(H1): $\exists\, \gamma \geq 0$ such that $W(x) \geq -\gamma \quad \forall\, x \in \mathcal{D}$.*
- *(H2): for every W-bounded sequence $\{x_n\} \subset \mathcal{D}$ such that $x_n \overset{w}{\to} x \in \mathcal{D}$, there exists a subsequence $\{x_{n_j}\} \subset \{x_n\}$ such that $W(x) \leq \liminf_{j \to \infty} W(x_{n_j})$.*
- *(H3): for every W-bounded sequence $\{x_n\} \subset \mathcal{D}$ there exist a subsequence $\{x_{n_j}\} \subset \{x_n\}$ and $x \in \mathcal{D}$ such that $x_{n_j} \overset{w}{\to} x$ in $\mathcal{X}$.*

Then the functional (3) has a global minimizer. If moreover the operator T is injective or W is strictly convex, then such a minimizer is unique.

As it was previously mentioned, inverse ill-posed problems appear in a wide variety of applications in diverse areas. Solving these problems usually involves several steps going from modeling, through measurements and data acquisition for the experiment under study, to the discretization of the mathematical model and the derivation of numerical approximations for the regularized solutions. All these steps involve intrinsic errors, many of which are unavoidable. For this reason, in the context of the study of inverse ill-posed problems from the optic of Tikhonov-Phillips methods with general penalizing terms, it is of particular interest the analysis of the stability of the minimizers of the functional given in (3) under different types of perturbations. To proceed in this direction we shall need the following definitions.

Definition 2.4 *(uniform W-coercivity) Let W, F_n, $n = 1, 2, \ldots,$ be functionals defined on a set $\mathcal{D}$. We will say that the sequence $\{F_n\}$ is uniformly W-coercive if $\lim_{n \to \infty} F_n(x_n) = +\infty$ for every sequence $\{x_n\} \subset \mathcal{D}$ for which $\lim_{n \to \infty} W(x_n) = +\infty$.*

Definition 2.5 *(W-consistency) Let $W, F, F_n, n = 1, 2, ...,$ be functionals defined on a set $\mathcal{D}$. We will say that the sequence $\{F_n\}$ is W-consistent for F if $F_n \to F$ uniformly on every W-bounded set, that is if for any given $c > 0$ and $\epsilon > 0$, there exists $N = N(c, \epsilon)$ such that $|F_n(x) - F(x)| < \epsilon$ for every $n \geq N$ and every x such that $|W(x)| \leq c$.*

The following theorem is a weak stability result for the minimizers of the functional (3).

Theorem 2.6 *(Weak stability) Let $\mathcal{X}$ be a normed space, $\mathcal{D}$ a subset of $\mathcal{X}$, $W : \mathcal{D} \subset \mathcal{X} \longrightarrow \mathbb{R}$ a functional satisfying the hypotheses (H1) and (H3) of Theorem 2.3 (i.e. there exists $\gamma > 0$ such that $W(x) \geq -\gamma$ for every $x \in \mathcal{D}$ and every W-bounded sequence contains a weakly convergent subsequence in X with limit in $\mathcal{D}$), J, J_n, $n = 1, 2, \ldots,$ functionals on $\mathcal{D}$ such that J is weakly lower semicontinuous in $\mathcal{D}$ and $\{J_n\}$ is uniformly W-coercive and W-consistent for J. Assume further that there exists a unique global minimizer $\bar{x} \in \mathcal{D}$ of J and that each functional J_n also possesses on $\mathcal{D}$ a global minimizer x_n (not necessarily unique). Then $x_n \overset{w}{\to} \bar{x}$ in $\mathcal{X}$.*

Next we present a result on strong stability of minimizers of the functional (3).

Theorem 2.7 *(Strong stability) Let $\mathcal{X}$ be a normed space, $\mathcal{D}$ a subset of $\mathcal{X}$, $W : \mathcal{D} \subset \mathcal{X} \longrightarrow \mathbb{R}$ a functional satisfying hypothesis (H1) and every W-bounded sequence contains a subsequence convergent in $\mathcal{X}$ with limit in $\mathcal{D}$), J, J_n, $n = 1, 2, \ldots$, functionals on $\mathcal{D}$ such that J is lower semicontinuous in $\mathcal{D}$ and $\{J_n\}$ is uniformly W-coercive and W-consistent for J. Suppose further that there exists a unique global minimizer $\bar{x} \in \mathcal{D}$ of J and that each functional J_n also possesses on $\mathcal{D}$ a global minimizer x_n (not necessarily unique). Then $x_n \to \bar{x}$ in $\mathcal{X}$.*

3 PARTICULAR CASES

3.1 Total variation penalization

Let $d \geq 2$, $\Omega \subset \mathbb{R}^d$ a convex, bounded subset with Lipschitz continuous boundary, $1 \leq p \leq \frac{d}{d-1}$, $\mathcal{X} \doteq L^p(\Omega)$, $\mathcal{D} \doteq BV(\Omega)$, where $BV(\Omega)$ is the space of functions of bounded variations on Ω. Then the functional W defined by $W(\cdot) : \mathcal{D} \longrightarrow \mathbb{R}_0^+$, $W(x) \doteq \|x\|_{BV(\Omega)}$ satisfies the conditions *(H1)*, *(H2)* and *(H3)* of Theorem 2.3.

3.2 Penalizing with differential operators

Let $\mathcal{X}$, $\mathcal{Z}$ be reflexive Banach spaces and $L : \mathcal{D}(L) \subset \mathcal{X} \to \mathcal{Z}$ a linear operator with $\mathcal{R}(L)$ weakly closed and such that $\|Lx\| \geq \gamma \|x\|$, $\forall\, x \in \mathcal{D}(L)$ for a certain constant $\gamma > 0$. Let W_L be the functional defined by $W_L : \mathcal{D}(L) \longrightarrow \mathbb{R}_0^+$, $W_L(x) = \|Lx\|^2$. Then W_L satisfies the hypotheses *(H1), (H2)* and *(H3)*.

3.3 Penalization by linear combination of seminorms

We propose here the study of the case in which the penalizing terms are linear combinations of seminorms induced by differential operators. More precisely, we will particularized the study of generalized Tikhonov-Phillips regularization methods for which the functional $W(\cdot)$ in (3) is of the form $W(x) \doteq \sum_{i=1}^N \alpha_i \|L_i x\|^2$ where $\alpha_i > 0$ for every $1 \leq i \leq N$ and the operators L_i are appropriately chosen.

Theorem 3.1 *(Existence and uniqueness) Let $\mathcal{X}$, $\mathcal{Z}_1$, $\mathcal{Z}_2, \ldots, \mathcal{Z}_N$ be reflexive Banach spaces, $\mathcal{Y}$ an inner-product space, $T : \mathcal{X} \longrightarrow \mathcal{Y}$ a linear bounded operator, $\mathcal{D}$ a subspace of $\mathcal{X}$, $L_i : \mathcal{D} \longrightarrow \mathcal{Z}_i$, $i = 1, 2, ..., N$, linear operators with $\mathcal{R}(L_i)$ weakly closed for every $1 \leq i \leq N$ and such that $\mathcal{R}(L) = \bigotimes \mathcal{R}(L_i)$ where $L \doteq (L_1, L_2, ..., L_N)^T$. Let also $\vec{\alpha} \doteq (\alpha_1, \alpha_2, \ldots, \alpha_N)^T \in \mathbb{R}_+^N$ and $y \in \mathcal{Y}$. Suppose that there exists a constant $c < \infty$ such that $\|x\|^2 \leq c \sum_{i=1}^N \alpha_i \|L_i x\|^2$ for every $x \in \mathcal{D}$. Then the functional*

$$J_{\vec{\alpha}, L_1, L_2, \ldots, L_N}(x) \doteq \|Tx - y\|^2 + \sum_{i=1}^N \alpha_i \|L_i x\|^2, \tag{4}$$

has a global minimizer. If moreover the operator T is injective or the functional $\sum_{i=1}^N \alpha_i \|L_i x\|^2$ is strictly convex, then such a minimizer is unique.

4 NUMERICAL RESULTS

The purpose of this section is to present an inverse problem in image restoration (Hansen (2002), Hansen et al. (2006)) which we will solve using some both the classical and some of the generalized Tikhonov-Phillips regularization methods. Image restoration deals with methods allowing the recovery of an original image from degraded observations of such an image.

A very general mathematical model for blurring of images is given by the following Fredholm first class integral equation

$$K\, f(x', y') = \int_0^1 \int_0^1 k(x, y, x', y') f(x', y') dx' dy' = g(x, y), \tag{5}$$

where f and g denote the original and blurred two-dimensional images, respectively and k is the so called point-spread function (PSF). Thus the image restoration or deblurring problem consist in "solving" the integral equation (5), that is, in obtaining a "good" approximation to the original image f from a blurred and perhaps noisy version of image g. Under very general conditions on the PSF k it turns out that the integral operator K associated to (5) is compact and the corresponding inverse problem is ill-posed in the sense of Hadamard (the generalized Moore-Penrose inverse of K is unbounded).

It is well known that no regularization method can lead to the recovering of the exact solution since in this type of problems there is always intrinsic loss of information, even under exact data. However, the choice of an adequate regularization method can result in better approximations to the exact solution.

The following examples have as their main objective showing how properties of a reconstructed image can chance with different regularization methods. In the particular case of the family of generalized Tikhonov-Phillips methods, an adequate choice of the penalizer can lead to a reconstructed image which better reflect certain known characteristics of the original image.

In the case of the family of generalized Tikhonov-Phillips regularization methods and adequate choice of the penalizer can lead to a reconstructed image which better reflects certain *"a-priori"* known characteristings of the original image. We note that in the following examples the blurred image (i.e. the data of the inverse problem) was obtained by solving the direct problem in (5) with a gaussian PSF given by

$$k(x, y, x', y') = \exp\left(-\kappa \left\|(x, y) - (x', y')\right\|^2\right),$$

where the blurring parameter κ was fixed at $\kappa = 10$.

In Figures 3(d) and 3(e) the original image and a 120×120 pixels boxed which is used as the f function in the direct problem. Figure 1(a) shows the resulting blurred image, which is used as data for the inverse problem. Figures 1(b) and 1(c) show the regularized solutions obtained with the classical Tikhonov-Phillips methods of order 0 and 1, corresponding to the choice of penalizers $\|f\|^2$ and $\|\nabla f\|^2$, respectively. As expected, the image reconstructed with the method of order 0 does a relative good job at places where the image is regular but method smooth out borders while the method of order one tends to improve the reconstruction near sharp borders and edges but introduces a "pixelin" effect in regular regions.

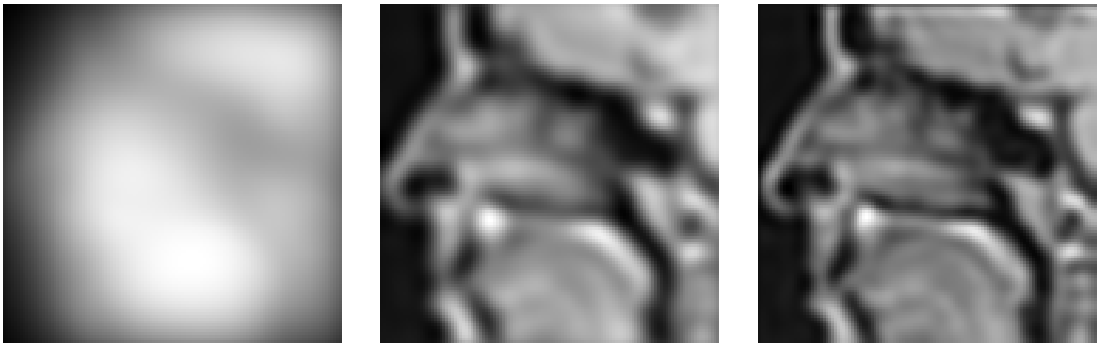
We constructed a hybrid penalizer of the form

$$W(\cdot) \doteq \alpha_1 \left\|\cdot\right\|^2 + \alpha_2 \left\|\nabla \cdot\right\|^2,$$

where the α_i's now play the role of a vector-valued regularization parameter, and serves to introduce different relative weights on both $\|\cdot\|^2$ and $\|\nabla\cdot\|^2$ functionals. It is reasonable to assume that an "adequate" choice of these relative parameters could improve the reconstructed solution.

Figure 2 show the three radial directions chosen for the penalizer W in the examples below. Figures 3(a), 3(b) and 3(c) show the regularized solutions obtained for these three different choices of weights. We observe how these choices of parameters result in reconstructions emphasizing different properties of the solution. A large relative choice of α_1 (Fig. 3(b)) results in a worse reconstruction than any of both methods separately, while a large choice of α_2 (Fig. 3(c)) seems to result in a reasonable good reconstruction both in regular areas and in regions with sharp edges.

Studies and applications of more elaborated and ad-hoc penalizers are currently underway.



(a) (b) (c)

Figure 1: 1(a): Blurred image (data for the inverse problem); 1(b): Regularized image obtained with the Tikhonov-Phillips method of order 0 (TP0); 1(c): Regularized image obtained with the Tikhonov-Phillips method of order 1 (TP1).

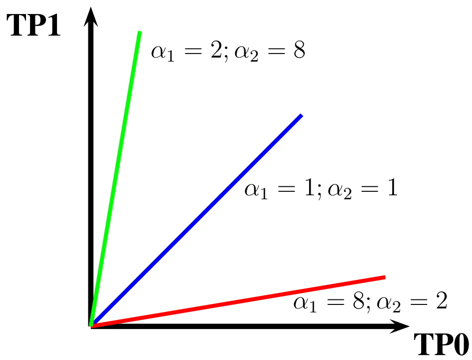
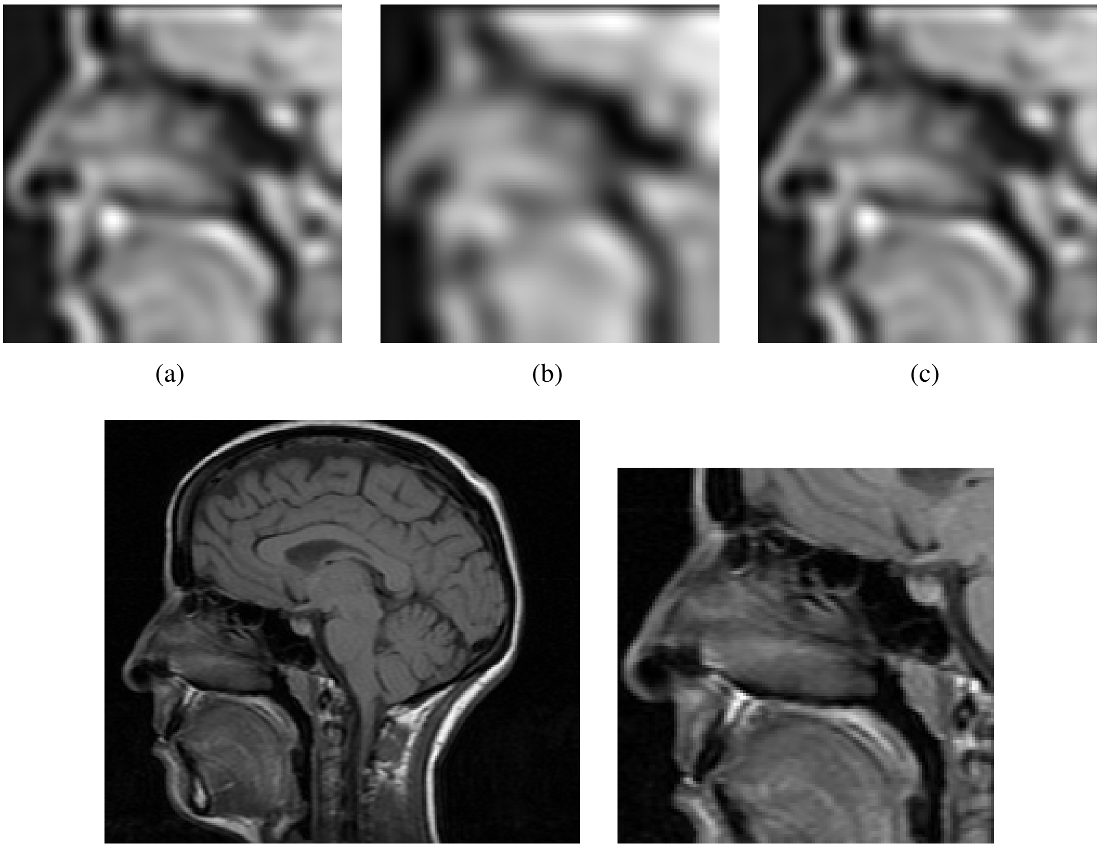


Figure 2: Radial reconstruction directions used with hybrid penalizer TP0-TP1.



(a) (b) (c) (d) (e)

Figure 3: 3(a), 3(b), 3(c): Regularized images obtained with the hybrid TP0-TP1 method with radial directions $\alpha_1 = \alpha_2 = 1$, $\alpha_1 = 8, \alpha_2 = 2$ y $\alpha_1 = 2, \alpha_2 = 8$, respectively. 3(d): Original image (256×228); 3(e): Portion of the original image (120×120).

5 CONCLUSIONS

In this article, results on existence, uniqueness and stability of the solutions of Tikhonov-Phillips methods with general penalizers were presented. Several examples of non classical penalizers where shown for which existence and uniqueness of solutions is guaranteed. Finally an example of a hybrid penalizer applied to an image restauration problem was shown to illustrate the fact that an adequate choice of the penalizer can result in restored images which better reflect certain properties of the exact solution.

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