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FREQUENCY RESPONSE FUNCTIONS OF RANDOM LINEAR MECHANICAL SYSTEMS AND PROPAGATION OF UNCERTAINTIES

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Abstract. In the modeling of dynamical systems, uncertainties are present and they must be taken into account to improve the prediction of the models. It is very important to understand how they propagate and how random systems behave. The aim of this work is to discuss the probability distribution function (PDF) of the amplitude and phase of the response of random linear mechanical systems when the stiffness are random. The novelty of the paper is that the computations are done analytically whenever possible. The propagation of uncertainties is then characterized. The PDF of the response of a system with random stiffness near the resonant frequency of the mean system has a complex structure and can presents multimodality in certain conditions. In Statistics a mode is a maximum of the PDF, and the modes describe the most probable values of the random variable. This multimodality makes approximations of the statistics, the mean for example, very difficult and sometimes meaningless since the behavior of the mean system can be quite different of the mean of the realizations. More complex systems, discrete and continuous, are also discussed and they show similar behaviour.

1 INTRODUCTION

In the modelling of dynamical systems, uncertainties are present and must be taken into account to improve the prediction furnished by the models. Namely, it is essential to understand how uncertainties propagate and how random systems behave.

The aim of this work is to discuss the Frequency Response Function (FRF) of random linear mechanical systems when uncertainty is considered in the stiffness to see for what conditions one has multimodal behaviour. In order to make understanding easier, we consider in the sequel the situation of a one degree of freedom system where damping is assumed deterministic and stiffness is random. This simplification allows analytical developments which widely simplify the analysis and clarify the use of the concepts, but is not a limitation: the situation where damping is also random may be studied into an analogous way. Our main objective is to show that a FRF (represented in amplitude and phase) of a random linear mass-spring-damper system is *frequently* - in a sense that will be precisely defined in the sequel - multimodal for a fixed frequency near a peak of the mean system. This work will explain and mathematically justify such a multimodal behaviour for dynamical systems and will characterise the conditions for its appearance, explaining thus the meaning of *frequently*. In addition, it will be shown that the multimodal behaviour is also found for more complex linear systems, discrete or continuous.

This multimodal behaviour of the frequency response function, to the best of our knowledge, has been few discussed in the literature. In Udwadia (1987a) the case of a single degree of freedom system with three random variables - the mass, the damping and the stiffness - is investigated regarding the probability distribution function of the natural frequency of this system. In Udwadia (1987b), the results of the response of a random system subjected to harmonic excitations, deterministic transient excitations, and random stationary excitations are presented for the system considered in Udwadia (1987a), but only for parameters having uniform distributions. In this work, multimodality appears, but expressions of analytical probability density functions are given only in an integral form (they are not explicit) due to the complexity involving with the three random parameters. In Heinkelé et al. (2006), the frequency response function of a single degree of freedom having a single random variable - the damping - is investigated using different tools and no statistical multimodality appears. In Pagnacco et al. (2009), a similar approach than that presented here is used, but for systems having both random stiffness and random damping and for only one particular frequency: the resonant frequency of the mean system; after the completion of this work it became clear that the multimodal behaviour comes only from the randomness of the stiffness, and not from the damping. In facts, the damping smooths the possible discontinuities in the probability distribution function (PDF), in particular at the extrema, but, by itself, causes no multimodal behaviour. Hence this work studies only the influence of the stiffness.

The idea of this work appeared when we tried to approach the mean FRF of discrete systems with chaos polynomial expansion and Monte Carlo simulations. The results were excellent except in small regions near the peaks of the mean system. The approximation did not improve increasing the number of terms and order of randomness. This strange behaviour claimed for explanation and this paper is our answer to that.

Indeed, to roughly understand what happens, let us consider an one DOF system with random stiffness, say homogeneously distributed. For each realisation the stiffness changes and so does the peak of the FRF. Near the peak of the mean system the distribution of values of the FRF of the different realisations is rather complex and the FRF of the mean system and the mean FRF are quite different, as one sees in Figure 3. Near the peak of the mean system the mean value is not representative of the behaviour, as is shown in this work, and tentatives to approach it by using polynomials chaos approximations (PCA) are doomed to failure.

Given the PDF of the stiffness we compute analytically the PDF of the amplitude and phase of the FRF - for the one DOF system and we give the envelopes of the FRF of the realisations. These envelopes will give the domain of definition of the PDF of the amplitude and phase of the FRF for a fixed frequency. When it becomes impossible to obtain an analytical result, we may get an approximation instead, but the analytical results facilitate the construction of a benchmark, that will be useful to test the approximations obtained, for example, by PCA of the PDF.

We will now shortly describe the content of each section. Section 2 describes how the amplitude and the phase of the FRF of the SDOF system relate with the stiffness for a fixed, deterministic frequency, that can be any non-negative real number. It also gives a general result to show how the PDF of the stiffness relates to the PDF of the amplitude and the phase. A special case is computed analytically. Section 3 discusses the case where the stiffness has uniform distribution. The envelopes of the FRF are defined and the mean and variance of the FRF for a fixed frequency are computed. The statistical modes of the amplitude and phase are computed analytically. Section 4 discuss the case where the stiffness has a Gamma distribution and presents the same results as in the last section. Section 5 discusses MDOFs and continuous systems and the results are computed with Monte Carlo simulations, in order to show that multimodality appears again for these cases. Section 6 presents some conclusions.

2 UNCERTAINTY IN THE SINGLE DEGREE OF FREEDOM SYSTEM

We consider the following Single Degree Of Freedom (SDOF) linear oscillator subject to an external harmonic forcing q in the frequency domain (Lin, 1967):

$$\left(k-\omega^2+\mathrm{j}c\omega\right)u\left(\omega\right)=q\left(\omega\right) \tag{1}$$

ω being the circular frequency. In this equation, the mass is normalised to unity and k and c are the stiffness and damping system parameters. Since the response $u\left(\omega\right)$ is a complex quantity, we need two real functions to characterise it. We choose the amplitude $|u|$ and phase θ since it is the most used representation[1]. Thus, the system response amplitude and phase are given by:

$$|u|\left(\omega\right)=\frac{q\left(\omega\right)}{\sqrt{\left(k-\omega^2\right)^2+4\eta^2k\omega^2}} \qquad \text{and} \qquad \tan\left(\theta\left(\omega\right)\right)=-\frac{2\eta\sqrt{k}\omega}{k-\omega^2} \tag{2}$$

where we assume a damping ratio $\eta=\frac{c}{2\sqrt{k}}$ such that $0\leq\eta<1$. But with respect to the definition of the FRF, a unit external forcing should be chosen for the sequel (*i.e.* $q\left(\omega\right)=1$). This system is sketched on Figure 1. FRFs examples obtained for five stiffness values varying ($\pm 20\sqrt{3}$ %) around a $10,000$ N/m central stiffness and a 3 % damping ratio are presented on Figure 2.

This system becomes stochastic if the stiffness parameter or the external forcing (or both) are random quantities.

[1]However, it is not more difficult to study the real and imaginary parts. In such case it can be demonstrate that statistical multimodality occur also frequently with generally four statistical modes for the real part and two statistical modes for the imaginary part.

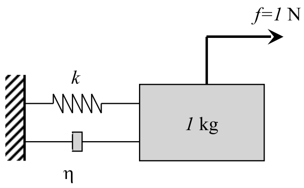
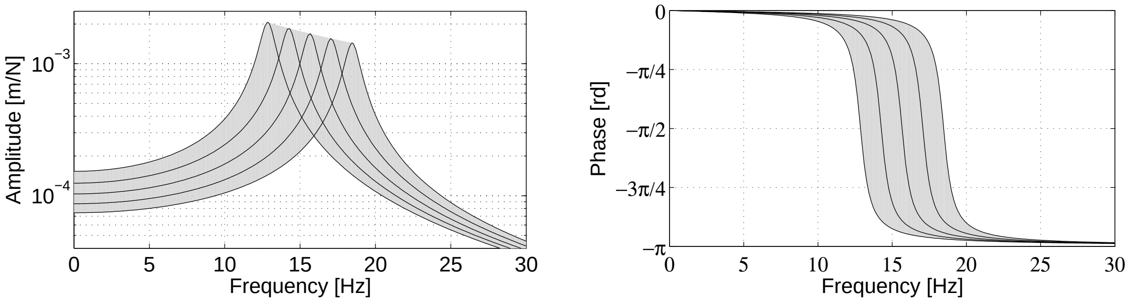


Figure 1: SDOF system



(a) (b)

Figure 2: FRFs of a SDOF system; Five stiffness values are considered; (a) response amplitude; (b) response phase;

2.1 Amplitude of the response: a general result

Let us consider the situation when only the SDOF stiffness constant k is random. Random variables will be denoted capitalising the letter that represents the deterministic variable, hence K in this case. First, in this section, we will give a general result that is valid for a general probability distribution function (PDF), and then, in the following sections, we will establish more detailed results for two particular PDF of the stiffness: the ones given by the Maximal Entropy Principle (Udwadia, 1989; Kapur and Kesavan, 1992) are: 1) the uniform distribution when the domain is fixed to be a bounded interval; and 2) the Gamma distribution if the domain is supposed to be the positive real.

Let p_K the PDF of K having $\overline{k}$ as the mean and σ_k as the standard deviation. Without loss of generality, the domain of K is considered to be an interval having the boundaries k_{inf} and k_{sup} that may or may not belong to the interval. Then we will denote $(k_{\text{inf}}, k_{\text{sup}})$ the domain of definition for p_K. For this domain, the parenthesis imply that it is not known if the boundaries belong or not to the interval of definition. For the left parenthesis, if the boundary belongs, the parenthesis is changed to a $]$, if not to $[$. For the right parenthesis it is similar. For example, for an uniform PDF $k \in [k_{\text{inf}}, k_{\text{sup}}]$ with $k_{\text{inf}} = \overline{k} - \sqrt{3}\sigma_k$ and $k_{\text{sup}} = \overline{k} + \sqrt{3}\sigma_k$, and the interval is closed, that is, the boundaries belong to the interval. On the other hand, for a Gamma PDF, $k \in]k_{\text{inf}}, k_{\text{sup}}[$ with $k_{\text{inf}} = 0$, $k_{\text{sup}} = +\infty$, and the interval is open, the boundaries do not belong to the interval.

Let us compute the PDF of the response of the system for a fixed $\omega \neq 0$ (the special extremal -static- case $\omega = 0$ is not considered in this work). Its random system response amplitude, denoted by $U(\omega)$, without the bars to simplify the notation, and phase $\Theta(\omega)$, are given by:

$$U(\omega) = |U|(\omega) = \frac{f(\omega)}{\sqrt{(K-\omega^2)^2 + 4\eta^2\overline{k}\omega^2}} \qquad \text{and} \qquad \tan(\Theta(\omega)) = -\frac{2\eta\sqrt{\overline{k}}\omega}{K-\omega^2} \tag{3}$$

where we assume a deterministic unit external forcing, $f(\omega) = 1$. In this expression, the damping ratio does not need to have the same expression as for deterministic system and is now chosen to be redefined as $\eta = \frac{c}{2\sqrt{k}}$. Since K is the single random variable of the problem, the PDF p_U is evaluated by the formula (Zwillinger and Kokoska, 2000):

$$p_U(u) = \frac{1}{\left|\frac{\mathrm{d}|U|}{\mathrm{d}K}\right|_{k_1}} p_K(k_1) + \cdots + \frac{1}{\left|\frac{\mathrm{d}|U|}{\mathrm{d}K}\right|_{k_n}} p_K(k_n) \tag{4}$$

where k_j for $j = 1, \ldots, n$ denotes the roots of the algebraic equation $u(k, \omega) = u$, for ω fixed (notice that $n = 1$ for a bijective function). That is, one seeks all the stiffnesses, k, that give a fixed amplitude u.

For ω fixed in the interval $\omega^2 < k_{\text{inf}}$ there is only one root, $k_1 = \frac{1}{u}\sqrt{1 - 4\eta^2\overline{k}\omega^2u^2} + \omega^2$. For ω fixed in the interval $\omega^2 > k_{\text{sup}}$ there is also only one root $k_1 = -\frac{1}{u}\sqrt{1 - 4\eta^2\overline{k}\omega^2u^2} + \omega^2$. In either case one can write, from relation (4):

$$p_U(u, \omega) = \frac{1}{u^2\sqrt{1 - 4\eta^2\overline{k}\omega^2u^2}} \times p_K(k_1(u, \omega)) \tag{5}$$

If $k_{\text{inf}} \leq \omega^2 \leq k_{\text{sup}}$, two roots could occur: $k_{1,2}(u) = \mp\frac{1}{u}\sqrt{1 - 4\eta^2\overline{k}\omega^2u^2} + \omega^2$, depending on some conditions for u given by equation (6). We will have:

$$p_U(u, \omega) = \begin{cases} \frac{p_K(k_1)}{u^2\sqrt{1-4\eta^2\overline{k}\omega^2u^2}} & \text{if} \quad k_1(u) > k_{\text{inf}} \text{ and } k_2(u) \geq k_{\text{sup}} \\ \frac{p_K(k_1)+p_K(k_2)}{u^2\sqrt{1-4\eta^2\overline{k}\omega^2u^2}} & \text{if} \quad k_1(u) > k_{\text{inf}} \text{ and } k_2(u) < k_{\text{sup}} \\ \frac{p_K(k_2)}{u^2\sqrt{1-4\eta^2\overline{k}\omega^2u^2}} & \text{if} \quad k_1(u) \leq k_{\text{inf}} \text{ and } k_2(u) < k_{\text{sup}} \end{cases}. \tag{6}$$

Thus, the PDF p_U may have a discontinuity if the fixed frequency $\omega^2 \in (k_{\text{inf}}, k_{\text{sup}})$ and if the stiffness distribution p_K does not vanish at the boundaries of its domain. In fact, the sum of stiffness densities in the PDF p_U comes from an aliasing of the distribution when taking the square of the apparent stiffness $K - \omega^2$ since it is distributed -at least partially- around the zero value ($k_{\text{inf}} - \omega^2 \leq 0 \leq k_{\text{sup}} - \omega^2$).

Moreover we will denote $(u_{\text{inf}}, u_{\text{sup}})$ the domain of definition for p_U with the same convention as above for the parenthesis. Hence, if the bounds belongs or not to the domain has to be evaluated separately for each given PDF. Moreover, since $u_{\text{inf}}(\omega)$ and $u_{\text{sup}}(\omega)$ are functions of the circular frequency ω, they corresponds to the lower and upper envelopes of the amplitude responses. The bounds of the domain of definition of the PDF are obtained from these envelopes when the frequency ω is fixed.

Statistical modes of the PDF p_U are also of great interest. To assess if there is more than one mode in an open interval, one can search for roots of the first derivative of p_U in its domain of definition to see if there are interior points of minimum. If there is one, this indicate that there are two maxima, either interior or in the boundaries. If the domain is closed, the extrema have to be analysed independently. Thus, without the choice of a special form for the stiffness PDF, it is not possible to decide this question. We will see that in our case of interval, p_U can be written as the product of two functions as

$$p_U(u, \omega) = g(u, \omega) \times p_K(u) \tag{7}$$

and the analysis of the generic function g shows an answer. Indeed, since we have

$$\frac{\partial p_U}{\partial u} = \frac{\partial g}{\partial u} p_K + g \frac{\partial p_K}{\partial u} \tag{8}$$

we could say that p_U would be at its minimum close to the minimum of g if the term $g\frac{\partial p_K}{\partial u}$ could be neglected. Hence, finding $\frac{\partial g}{\partial u}(u,\omega) = 0$ leads to a minimum for the function g at the value $u_0(\omega) = \frac{\sigma_k}{\sqrt{2\bar{k}}\eta\omega}$. This leads also to a minimum of $p_U(u)$ if u_0 belongs to the domain of p_U, *i.e.* if $u_0(\omega) \in [u_{\text{inf}},\, u_{\text{sup}}]$ and if $g(u_0(\omega),\omega)\,\frac{\partial p_K}{\partial u}(u_0(\omega))$ can be neglected. Thus, we can conclude that the SDOF system PDF amplitude could have more than one statistical mode for a variety of stiffness distribution, depending on the domain of p_U, or more precisely, on the fixed frequency ω, the damping ratio η, the mean $\bar{k}$ and the coefficient of variation $\frac{\sigma_k}{\bar{k}}$.

2.2 Phase of the response: a general result

Phase PDF $p_\Theta(\theta)$ can be evaluated following the same strategy as given by formula (4). We have

$$\frac{\mathrm{d}k}{\mathrm{d}\theta}(u,\omega) = -\frac{2\sqrt{\bar{k}}\eta\omega}{\sin(\theta)^2} \tag{9}$$

which leads to

$$p_\Theta(\theta,\omega) = \frac{2\sqrt{\bar{k}}\eta\omega}{\sin(\theta)^2} \times p_K(\theta) \tag{10}$$

for[2] $\omega \neq 0$ and on the domain $(\theta_{\text{inf}},\, \theta_{\text{sup}}) = \left(\tan^{-1}\left(-\frac{2\eta\sqrt{\bar{k}}\omega}{k_{\text{inf}}-\omega^2}\right),\, \tan^{-1}\left(-\frac{2\eta\sqrt{\bar{k}}\omega}{k_{\text{sup}}-\omega^2}\right)\right)$ with the same convention as above for the parenthesis.

To study the existence of more than one statistical mode, we follow the same reasoning as before. If we neglect the term $\frac{\partial p_K}{\partial\theta} = 0$, finding $\frac{\partial p_\Theta}{\partial\theta} = 0$ leads to seek for the root $-2p_\Theta\cot(\theta) = 0$ which is $\theta_0 = -\frac{\pi}{2}$. Thus, we can conclude that the SDOF system phase PDF could have more than one statistical mode for a variety of stiffness distributions, depending on the frequency excitation ω, and the stiffness distribution parameters.

3 SDOF SYSTEM WITH UNIFORM RANDOM STIFFNESS

Let us consider that K has an uniform distribution, say

$$p_K(k) = \begin{cases} \frac{1}{2\sqrt{3}\sigma_k} & \text{if } \mathrm{k} \in \left[\bar{\mathrm{k}} - \sqrt{3}\sigma_{\mathrm{k}}, \bar{\mathrm{k}} + \sqrt{3}\sigma_{\mathrm{k}}\right] \\ 0 & \text{if not} \end{cases} \tag{11}$$

For such a system, one can note that F, the natural frequency, is also a random variable. It is $F = \frac{1}{2\pi}\sqrt{K}$ and has a distribution $p_F(f) = 8\pi^2 f \times p_K(k(f)) = \frac{4\pi^2 f}{\sqrt{3}\sigma_k}$, according to the relation (4), which is defined over $(f_{\text{inf}}, f_{\text{sup}}) = \left[\frac{\sqrt{\bar{k}-\sqrt{3}\sigma_k}}{2\pi}, \frac{\sqrt{\bar{k}+\sqrt{3}\sigma_k}}{2\pi}\right]$. Thus, the mean of this random variable

$$\mathrm{E}[F] = \frac{4\pi^2}{3\sqrt{3}\sigma_k}\left(f_{\text{sup}}^3 - f_{\text{inf}}^3\right) \tag{12}$$

[2]For $\omega = 0$, the phase remains deterministic ($\theta = 0$) whatever the stiffness is random or not.

is not equal to the natural frequency of the mean system which is $f_0 = \frac{1}{2\pi}\sqrt{\overline{k}}$. At last, to complete this statistic, note that

$$\mathrm{E}\left[F^2\right] = \frac{\pi^2}{\sqrt{3}\sigma_k}\left(f_{\mathrm{sup}}^4 - f_{\mathrm{inf}}^4\right) \tag{13}$$

while the statistical mode is located at f_{sup}.

3.1 Amplitude of the response

Relations (5), (6) and (11) give the PDF of the response amplitude.

3.1.1 Upper and lower envelopes for the amplitude

The upper envelope of the system response amplitude, denoted $u_{\mathrm{sup}}(\omega)$, is the function

$$u_{\mathrm{sup}}(\omega) = \begin{cases} \frac{1}{\sqrt{\left(\overline{k}-\sqrt{3}\sigma_k-\omega^2\right)^2+4\eta^2\overline{k}\omega^2}} & \text{for } \omega^2 \in \left[0, \overline{k}-\sqrt{3}\sigma_k\right] \\ \frac{1}{2\eta\omega\sqrt{\overline{k}}} & \text{for } \omega^2 \in \left[\overline{k}-\sqrt{3}\sigma_k, \overline{k}+\sqrt{3}\sigma_k\right] \\ \frac{1}{\sqrt{\left(\overline{k}+\sqrt{3}\sigma_k-\omega^2\right)^2+4\eta^2\overline{k}\omega^2}} & \text{for } \omega^2 \in \left[\overline{k}+\sqrt{3}\sigma_k, +\infty\right[\end{cases} \tag{14}$$

Thus, the system response $U(\omega)$ is unbounded if $\eta \to 0$ for all $\omega^2 \in \left[\overline{k}-\sqrt{3}\sigma_k, \overline{k}+\sqrt{3}\sigma_k\right]$.

The lower envelope , denoted $u_{\mathrm{inf}}(\omega)$, is

$$u_{\mathrm{inf}}(\omega) = \begin{cases} \frac{1}{\sqrt{\left(\overline{k}+\sqrt{3}\sigma_k-\omega^2\right)^2+4\eta^2\overline{k}\omega^2}} & \text{for } \omega^2 \in \left[0, \overline{k}\right] \\ \frac{1}{\sqrt{\left(\overline{k}-\sqrt{3}\sigma_k-\omega^2\right)^2+4\eta^2\overline{k}\omega^2}} & \text{for } \omega^2 \in \left[\overline{k}, +\infty\right[\end{cases} \tag{15}$$

Thus, bounds of the amplitude response PDF are given by the domain $[u_{\mathrm{inf}}(\omega), u_{\mathrm{sup}}(\omega)]$ for a fixed frequency ω.

3.1.2 Statistic for fixed frequency of the PDF of the amplitude

We now compute analytically the first and second moments for the amplitude system response. These results are important for comparison with numerical simulations. They are given by

$$\mathrm{E}[U](\omega) = \int_{u_{\mathrm{inf}}(\omega)}^{u_{\mathrm{sup}}(\omega)} u p_U(u,\omega)\,\mathrm{d}u \qquad \text{and} \qquad \mathrm{E}[U^2](\omega) = \int_{u_{\mathrm{inf}}(\omega)}^{u_{\mathrm{sup}}(\omega)} u^2 p_U(u,\omega)\,\mathrm{d}u \tag{16}$$

They could be evaluated analytically easily if we neglect the damping (this hypothesis is valid for a small damping when $\omega^2 \notin \left[\overline{k}-\sqrt{3}\sigma_k, \overline{k}+\sqrt{3}\sigma_k\right]$ or for medium damping at low and high frequency):

$$\mathrm{E}[U](\omega) \approx \int_{u_{\mathrm{inf}}(\omega)}^{u_{\mathrm{sup}}(\omega)} \frac{1}{\sqrt{3}\sigma_k u}\mathrm{d}u = \frac{1}{\sqrt{3}\sigma_k}\ln\left(\frac{u_{\mathrm{sup}}(\omega)}{u_{\mathrm{inf}}(\omega)}\right)$$

$$\mathrm{E}\left[U^2\right](\omega) \approx \int_{u_{\inf}(\omega)}^{u_{\sup}(\omega)} \frac{1}{\sqrt{3}\sigma_k}\mathrm{d}u = \frac{1}{\sqrt{3}\sigma_k}\left(u_{\sup}(\omega) - u_{\inf}(\omega)\right)$$

But to obtain an exact estimation of these expectations over the full frequency range, it is necessary to consider both the damping and the discontinuity which appears in the PDF (Eq. (6)) for $\omega^2 \in \left[\overline{k} - \sqrt{3}\sigma_k, \overline{k} + \sqrt{3}\sigma_k\right]$. Solving equations (6) for the variable u leads to locate the discontinuity for the PDF at the value $u_1(\omega) = \frac{1}{\sqrt{\left(\overline{k}-\sqrt{3}\sigma_k-\omega^2\right)^2+4\eta^2\overline{k}\omega^2}}$ for the frequency range $\left[\overline{k} - \sqrt{3}\sigma_k, \overline{k}\right]$ while it is located at $u_1(\omega) = \frac{1}{\sqrt{\left(\overline{k}+\sqrt{3}\sigma_k-\omega^2\right)^2+4\eta^2\overline{k}\omega^2}}$ for the range $\left[\overline{k}, \overline{k} + \sqrt{3}\sigma_k\right]$. Thus, we have

$$\begin{aligned}\mathrm{E}[U](\omega) &= \int_{u_{\inf}(\omega)}^{u_{\sup}(\omega)} \frac{1}{2\sqrt{3}\sigma_k}\frac{1}{u\sqrt{1-4\eta^2\overline{k}\omega^2u^2}}\mathrm{d}u \\ &= \frac{1}{2\sqrt{3}\sigma_k}\left(\operatorname{arccot}\sqrt{4\eta^2\overline{k}\omega^2u_{\sup}(\omega)-1} - \operatorname{arccot}\sqrt{4\eta^2\overline{k}\omega^2u_{\inf}(\omega)-1}\right)\end{aligned}$$

for the range $\omega^2 \in \left[0, \overline{k} - \sqrt{3}\sigma_k\right] \cup \left[\overline{k} + \sqrt{3}\sigma_k, +\infty\right[$, while we have

$$\mathrm{E}[U](\omega) = \int_{u_{\inf}(\omega)}^{u_1(\omega)} \frac{1}{2\sqrt{3}\sigma_k}\frac{1}{u\sqrt{1-4\eta^2\overline{k}\omega^2u^2}}\mathrm{d}u + \int_{u_1(\omega)}^{u_{\sup}(\omega)} \frac{1}{2\sqrt{3}\sigma_k}\frac{2}{u\sqrt{1-4\eta^2\overline{k}\omega^2u^2}}\mathrm{d}u \qquad (17)$$

or

$$\begin{aligned}\mathrm{E}[U](\omega) &= \frac{1}{\sqrt{3}\sigma_k}\operatorname{arccot}\sqrt{4\eta^2\overline{k}\omega^2u_{\sup}(\omega)-1} \\ &\quad \frac{1}{2\sqrt{3}\sigma_k}\left(\operatorname{arccot}\sqrt{4\eta^2\overline{k}\omega^2u_1(\omega)-1} + \operatorname{arccot}\sqrt{4\eta^2\overline{k}\omega^2u_{\inf}(\omega)-1}\right)\end{aligned}$$

for $\omega^2 \in \left[\overline{k} - \sqrt{3}\sigma_k, \overline{k} + \sqrt{3}\sigma_k\right]$. Similarly, we have

$$\mathrm{E}\left[U^2\right](\omega) = \frac{1}{4\sqrt{3}\eta\omega\sqrt{\overline{k}}\sigma_k}\left(\arcsin\left(2\eta\omega\sqrt{\overline{k}}u_{\sup}(\omega)\right) - \arcsin\left(2\eta\omega\sqrt{\overline{k}}u_{\inf}(\omega)\right)\right) \qquad (18)$$

for the range $\omega^2 \in \left[0, \overline{k} - \sqrt{3}\sigma_k\right] \cup \left[\overline{k} + \sqrt{3}\sigma_k, +\infty\right[$ and

$$\begin{aligned}\mathrm{E}\left[U^2\right](\omega) &= \frac{1}{2\sqrt{3}\eta\omega\sqrt{\overline{k}}\sigma_k}\arcsin\left(2\eta\omega\sqrt{\overline{k}}u_{\sup}(\omega)\right) - \\ &\quad \frac{1}{4\sqrt{3}\eta\omega\sqrt{\overline{k}}\sigma_k}\left(\arcsin\left(2\eta\omega\sqrt{\overline{k}}u_1(\omega)\right) + \arcsin\left(2\eta\omega\sqrt{\overline{k}}u_{\inf}(\omega)\right)\right)\end{aligned}$$

for $\omega^2 \in \left[\overline{k} - \sqrt{3}\sigma_k, \overline{k} + \sqrt{3}\sigma_k\right]$. From these two results, it is possible to evaluate the standard deviation of the response, σ_U, by $\sigma_U^2 = \mathrm{E}\left[U^2\right] - \mathrm{E}[U]^2$.

The Figure 3 shows the system response amplitude, considering an uniform stiffness with a $10,000$ N/m mean stiffness, for two stiffness coefficients of variation ($\frac{\sigma_k}{\overline{k}}$) and two damping

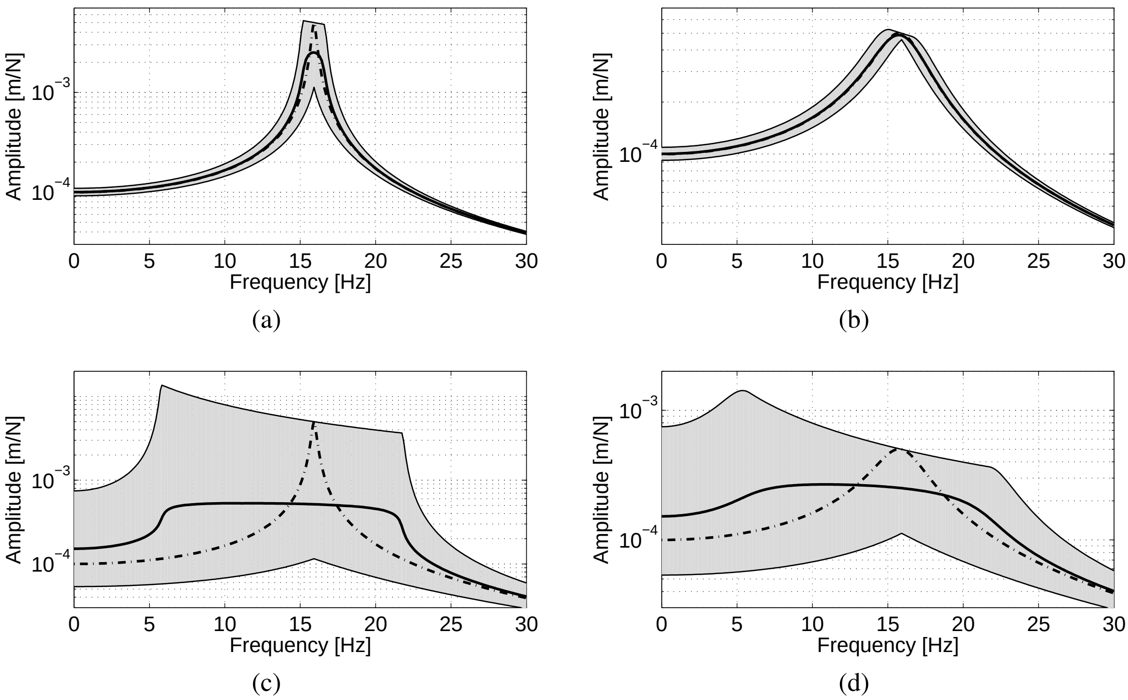


Figure 3: Amplitude of the response for a SDOF having a uniform stiffness (grey area shows the region of variation), mean of the system responses (solid thick line) and response of the mean system (dashed line); (a) 1% damping ratio, 5% coefficient of variation; (b) 10% damping ratio, 5% coefficient of variation; (c) 1% damping ratio, 50% coefficient of variation; (d) 10% damping ratio, 50% coefficient of variation;

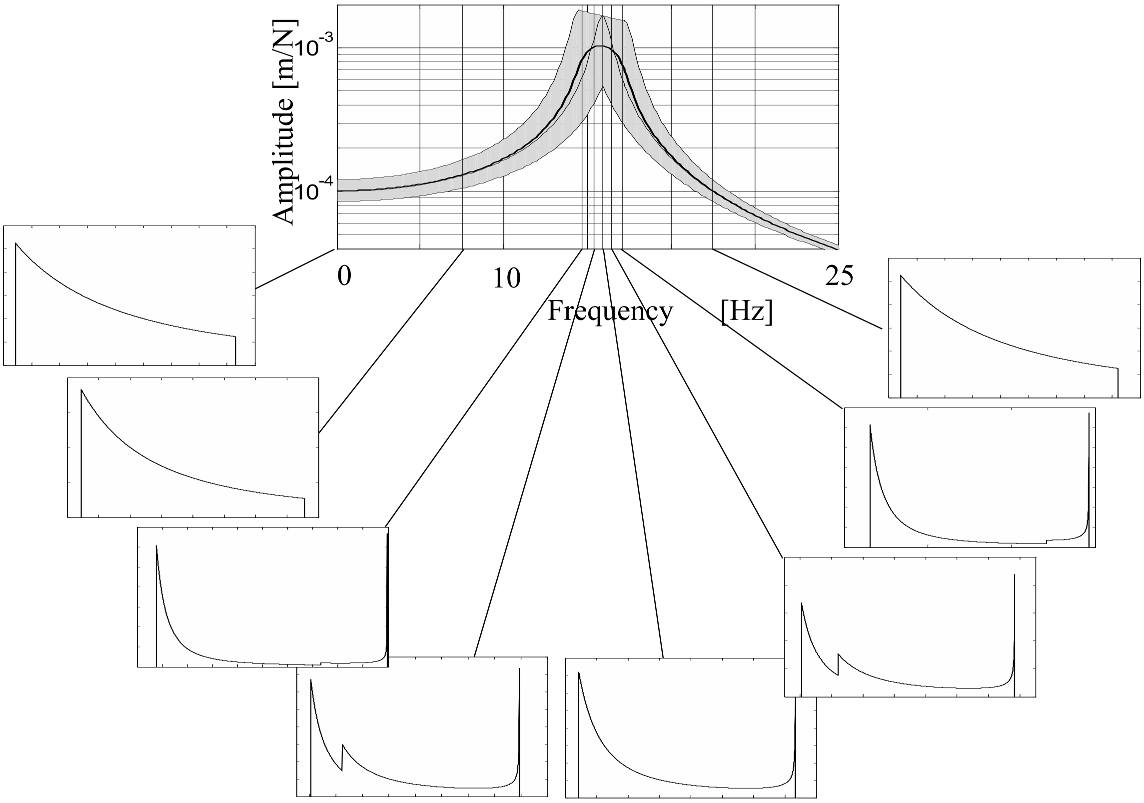


Figure 4: PDF of the amplitude response at various frequencies for a SDOF system having an uniform stiffness (with a $3\,\%$ damping ratio and a $20\,\%$ coefficient of variation)

ratios. The region of variation is represented by the grey area. The mean of the system responses and the response of the mean system are also represented.

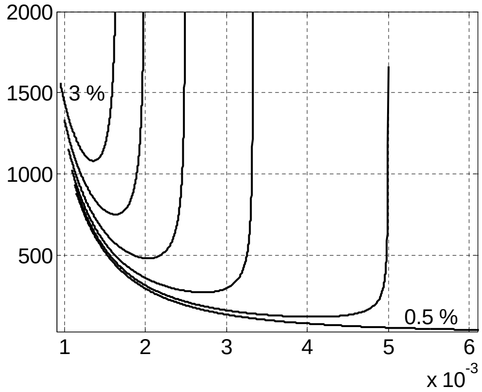
Then, considering the case of a $20\,\%$ stiffness coefficient of variation and a $3\,\%$ damping ratio, the Figure 4 shows the PDF of the amplitude at several frequencies. It is seen on this figure that various PDF shapes are obtained.

3.1.3 Statistical modes of the amplitude response

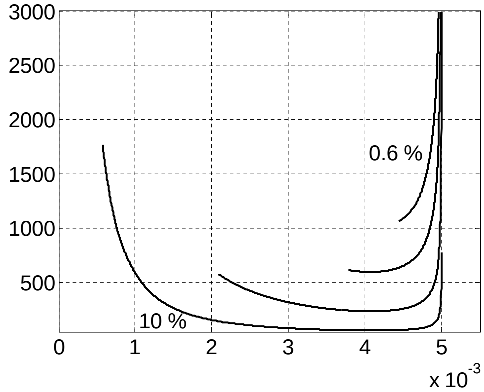
At low ($\omega \to 0$) or at high ($\omega \to \infty$) frequencies, the PDF $p_U(\bullet, \omega)$ is monotonically decreasing, thus the lower envelope $u_{\inf}(\omega)$ is a statistical mode. But for intermediate frequency, the shape of this PDF could be more complex. It can have a minimum when its first derivative vanishes, at $u_0(\omega) = \frac{1}{\sqrt{6\bar{k}\eta\omega}}$, if this value is in the domain definition of $p_U(\bullet, \omega)$, *i.e.* if $u_0(\omega) \in [u_{\inf},\, u_{\sup}]$. For example, if we consider the simplest situation when $\omega = \sqrt{\bar{k}}$ (which is the natural circular frequency of the mean system having a unit mass), if $u_0\left(\sqrt{\bar{k}}\right) \leq \frac{1}{\sqrt{\left(2\eta\bar{k}\right)^2 + 3\sigma_k^2}}$, or equivalently, if

$$\frac{\sigma_k}{\bar{k}} \leq \sqrt{\frac{2}{3}}\eta \tag{19}$$

the amplitude system response U will have only one statistical mode, since $u_0\left(\sqrt{\bar{k}}\right) \notin \left[u_{\inf}\left(\sqrt{\bar{k}}\right),\, u_{\sup}\left(\sqrt{\bar{k}}\right)\right]$. On the other hand, *i.e.* when $u_0\left(\sqrt{\bar{k}}\right) > \frac{1}{\sqrt{\left(2\eta\bar{k}\right)^2 + 3\sigma_k^2}}$ or $\frac{\sigma_k}{\bar{k}} > \sqrt{\frac{2}{3}}\eta$



(a)



(b)

Figure 5: PDF of the amplitude response for a SDOF system having an uniform stiffness with various densities and damping ratio at frequency $\omega^2 = \overline{k}$; (a) amplitude PDF for $\frac{\sigma_k}{\overline{k}} = 5\,\%$ and $\eta = \{0.5\,\%,\, 1\,\%,\, 1.5\,\%,\, 2\,\%,\, 2.5\,\%,\, 3\,\%\}$; (b) amplitude PDF for $\eta = 1\,\%$ and $\frac{\sigma_k}{\overline{k}} = \{0.6\,\%,\, 1\,\%,\, 2.5\,\%,\, 10\,\%\}$

there are two statistical modes since $u_0\left(\sqrt{\overline{k}}\right) \in \left[u_{\text{inf}}\left(\sqrt{\overline{k}}\right),\, u_{\text{sup}}\left(\sqrt{\overline{k}}\right)\right]$. Multiple statistical mode existence condition depends thus on the fixed frequency ω, the damping ratio η, the mean $\overline{k}$, and the coefficient of variation $\frac{\sigma_k}{\overline{k}}$. Figure 5 shows examples of amplitude PDF for various damping ratio and various stiffness coefficient of variation when the frequency is fixed at the natural frequency of the mean system. All these examples have two statistical modes, except in the case of a very low stiffness coefficient of variation, *i.e.* the $0.6\,\%$ curve on the right.

Considering now the frequency range $\left]0,\, \overline{k} - \sqrt{3}\sigma_k\right[$, the existence condition for multimodality, which occurs again when $u_0(\omega) \in [u_{\text{inf}}(\omega)\,,\, u_{\text{sup}}(\omega)]$, is evaluated to be

$$\frac{\left(\overline{k} - \sqrt{3}\sigma_k\right) - \omega^2}{\sqrt{2\overline{k}}\omega} < \eta < \frac{\left(\overline{k} + \sqrt{3}\sigma_k\right) - \omega^2}{\sqrt{2\overline{k}}\omega} \tag{20}$$

while, for the frequency $\omega^2 \geq \overline{k} + \sqrt{3}\sigma_k$, we would have

$$\frac{\omega^2 - \left(\overline{k} + \sqrt{3}\sigma_k\right)}{\sqrt{2\overline{k}}\omega} < \eta < \frac{\omega^2 - \left(\overline{k} - \sqrt{3}\sigma_k\right)}{\sqrt{2\overline{k}}\omega} \tag{21}$$

These conditions are less restrictive for the frequency range $\left[\overline{k} - \sqrt{3}\sigma_k, \overline{k} + \sqrt{3}\sigma_k\right]$, since only the lower bound has to be considered (indeed, $u_0(\omega)$ is always less than u_{sup} for this frequency range). Thus, the existence condition for $u_0(\omega) \in [u_{\text{inf}},\, u_{\text{sup}}]$ in the frequency range $\left[\overline{k} - \sqrt{3}\sigma_k, \overline{k}\right]$ is given by

$$\eta < \frac{\left(\overline{k} + \sqrt{3}\sigma_k\right) - \omega^2}{\sqrt{2\overline{k}}\omega} \tag{22}$$

while it is given by

$$\eta < \frac{\omega^2 - \left(\overline{k} - \sqrt{3}\sigma_k\right)}{\sqrt{2\overline{k}}\omega} \tag{23}$$

for the frequency range $\left[\overline{k}, \overline{k} + \sqrt{3}\sigma_k\right]$.

Having the existence condition of multiple statistical modes, one can see that they could appear frequently in the vicinity of the resonant frequency of the mean system ($\omega = \sqrt{\overline{k}}$) since the damping ratio is generally small for real systems.

From another point of view, an interesting question is "What range of frequency leads to multimodality ?". It is sufficient for this to solve conditions (20) and (21) for ω^2 to obtain:

$$\overline{k}\,(1+\eta^2)-\sqrt{3}\sigma_k-\overline{k}\eta\sqrt{2-2\sqrt{3}\frac{\sigma_k}{\overline{k}}+\eta^2} < \omega^2 < \overline{k}\,(1+\eta^2)+\sqrt{3}\sigma_k+\overline{k}\eta\sqrt{2+2\sqrt{3}\frac{\sigma_k}{\overline{k}}+\eta^2}$$

To conclude about the number of statistical modes for the frequency range $\left[\overline{k}-\sqrt{3}\sigma_k,\overline{k}\right[\cup\left]\overline{k},\overline{k}+\sqrt{3}\sigma_k\right]$, we have to consider also the discontinuity of the PDF. Then, it will exist only one statistical mode if the previous conditions (22) or (23) are not fulfilled, whereas there are two or three statistical modes if they are fulfilled. Figure 4 shows examples of response amplitude PDF at four frequencies taken in the range $\left[\overline{k}-\sqrt{3}\sigma_k,\overline{k}\right[\cup\left]\overline{k},\overline{k}+\sqrt{3}\sigma_k\right]$. They are the third, fourth, sixth and seventh ones PDF (the fifth one being the PDF at $\omega^2=\overline{k}$, the resonant frequency of the mean system). All of them exhibit a discontinuity. The fourth and sixth ones have three statistical modes while the others have only two statistical modes. In fact, three statistical modes arise when the discontinuity is located before the PDF minimum, *i.e.* when $u_1\,(\omega) < u_0\,(\omega)$ or

$$\frac{1}{\sqrt{\left(\overline{k}-\sqrt{3}\sigma_k-\omega^2\right)^2+4\eta^2\overline{k}\omega^2}} < \frac{1}{\sqrt{6\overline{k}}\eta\omega} \tag{24}$$

for the frequency range $\left[\overline{k}-\sqrt{3}\sigma_k,\overline{k}\right[$ or

$$\frac{1}{\sqrt{\left(\overline{k}+\sqrt{3}\sigma_k-\omega^2\right)^2+4\eta^2\overline{k}\omega^2}} < \frac{1}{\sqrt{6\overline{k}}\eta\omega} \tag{25}$$

for the frequency range $\left]\overline{k},\overline{k}+\sqrt{3}\sigma_k\right]$.

Thus, statistical modes can be located on the envelopes $u_{\text{inf}}\,(\omega)$, $u_{\text{sup}}\,(\omega)$ and the discontinuity $u_1\,(\omega)$ (depending on the system properties, the random parameters, and the excitation frequency ω). Another comment concerns the upper envelope which could tend to infinite for the (ideal) system having no damping. Thus, the rightmost mode of the PDF diminishes its value when the damping decreases and the domain of definition increases. From this behaviour, we conclude that this mode will be difficult to detect if it has to be found by Monte Carlo simulations.

Finally, this multimodal behaviour of the response implies that the mean and the standard deviation of the system response given in the previous section are insufficient to characterise the response distribution, at least in the vicinity of the resonant frequency of the mean system.

3.2 Phase of the response

From the relations (10) and (11) we have

$$p_\Theta\,(\theta,\omega) = \frac{\sqrt{\overline{k}}\eta\omega}{\sqrt{3}\sigma_k\sin\,(\theta)^2} \tag{26}$$

which is defined on the domain $\left[\tan^{-1}\left(-\frac{2\eta\sqrt{\overline{k}}\omega}{\overline{k}-\sqrt{3}\sigma_k-\omega^2}\right),\tan^{-1}\left(-\frac{2\eta\sqrt{\overline{k}}\omega}{\overline{k}+\sqrt{3}\sigma_k-\omega^2}\right)\right]$.

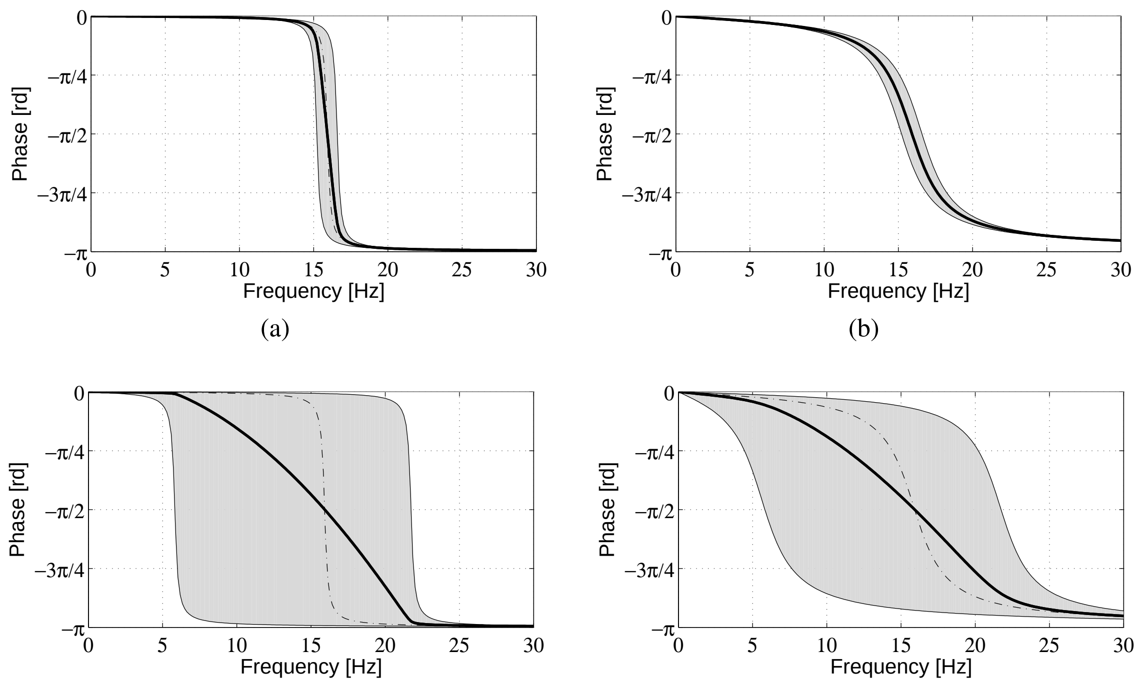


Figure 6: Phase of the response of a SDOF system having a uniform random stiffness; Mean phase of the system response (thick solid line) and phase response of the mean system (dashed line); grey area shows the region of variation; (a) 1% damping ratio, 5% coefficient of variation; (b) 10% damping ratio, 5% coefficient of variation; (c) 1% damping ratio, 50% coefficient of variation; (d) 10% damping ratio, 50% coefficient of variation;

The Figure 6 illustrate this phase response, considering an uniform stiffness with a $10,000$ $\mathrm{N/m}$ mean stiffness, two stiffness coefficients of variation ($\frac{\sigma_k}{\overline{k}}$) and two damping ratios. The region of variation is represented by the grey area. The mean of the system responses and the response of the mean system are also represented.

Then, considering the case of a $20\,\%$ stiffness coefficient of variation and a $3\,\%$ damping ratio, the Figure 7 shows the PDF response phase at several frequencies: the first one located at a very low frequency range, the second one a little before $\omega^2 = \overline{k}$, the third one at $\omega^2 = \overline{k}$, and the last one a little after $\omega^2 = \overline{k}$. It is seen on this figure that various PDF shapes are obtained. For low frequencies $(\omega \to 0)$, the PDF $p_\Theta(\bullet, \omega)$ is monotonically increasing, thus the upper envelope $\theta_{\text{sup}}(\omega)$ is a statistical mode, while at high frequencies p_Θ is monotonically decreasing leading to a statistical mode at the lower envelope.

For intermediate frequency, this PDF could have a minimum when its first derivative vanishes, at $\theta_0(\omega) = -\frac{\pi}{2}$, if this value is on the domain definition of p_Θ, *i.e.* if $\theta_{\text{inf}}(\omega) < \theta_0(\omega) < \theta_{\text{sup}}(\omega)$ or, by taking the cosines of this inequality, if

$$\overline{k} - \sqrt{3}\sigma_k < \omega^2 < \overline{k} + \sqrt{3}\sigma_k \tag{27}$$

Thus, there is one statistical mode at $\theta_{\text{sup}}(\omega)$ if $\omega^2 \leq \overline{k} - \sqrt{3}\sigma_k$, one statistical mode at $\theta_{\text{inf}}(\omega)$ if $\omega^2 \geq \overline{k} + \sqrt{3}\sigma_k$, and two statistical modes at $\theta_{\text{sup}}(\omega)$ and $\theta_{\text{inf}}(\omega)$ otherwise.

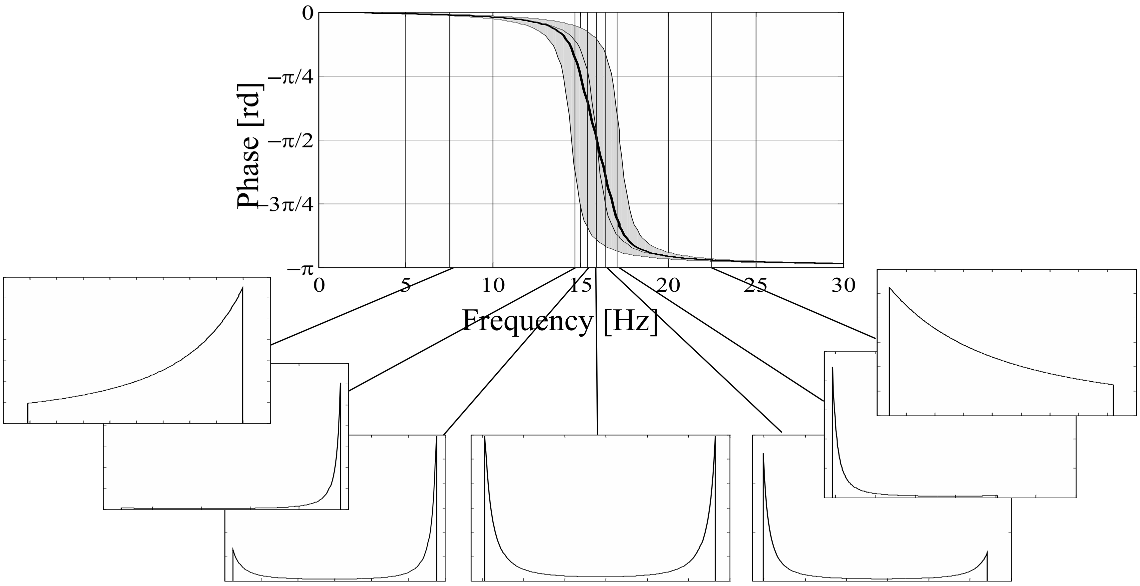


Figure 7: PDF of the phase response at various frequencies for a SDOF system having an uniform stiffness (with a $3\,\%$ damping ratio and a $20\,\%$ coefficient of variation)

4 SDOF SYSTEM WITH GAMMA RANDOM STIFFNESS

Let us assume now a Gamma distribution for the stiffness, having the PDF

$$p_K(k;a,b) = \begin{cases} \left(\frac{k}{b}\right)^{a-1} \frac{\exp\left(-\frac{k}{b}\right)}{b\Gamma(a)} & \text{if } k > 0 \\ 0 & \text{if not} \end{cases} \tag{28}$$

where $a = \left(\frac{\overline{k}}{\sigma_k}\right)^2$ and $b = \frac{\sigma_k^2}{\overline{k}}$ are positive values.

For such a system, one can note that the natural frequency F has the bell shape distribution given by

$$p_F(\varphi) = \frac{2}{\varphi\Gamma(a)} \left(\frac{4\pi^2\varphi^2}{b}\right)^a \exp\left(-\frac{4\pi^2\varphi^2}{b}\right) \tag{29}$$

which is defined over $]0,+\infty[$. We have

$$\mathrm{E}[F] = \frac{\sqrt{b}\Gamma\left(a+\frac{1}{2}\right)}{2\pi\Gamma(a)} \qquad \text{and} \qquad \mathrm{E}\left[F^2\right] = \frac{\overline{k}}{4\pi^2} \tag{30}$$

and a mode located at the value $\frac{\sqrt{b\left(a-\frac{1}{2}\right)}}{2\pi}$.

4.1 Amplitude of the response

From the domain definition of this random stiffness, we deduce that the system response PDF given by the relation (5) and (6) can be simplified to

$$p_U(u,\omega) = \frac{p_K(k_1;a,b) + p_K(k_2;a,b)}{u^2\sqrt{1-\left(2\sqrt{\overline{k}}\eta u\omega\right)^2}} \tag{31}$$

since $p_K(k_1;a,b)$ will vanish when the root k_1 becomes negative.

To find the PDF support, we consider the relation (3). The idea is the following: since $K \in]0,+\infty[$, $K-\omega^2 \in \left]-\overline{k},+\infty\right[$, $\left(K-\omega^2\right)^2 \in]0,+\infty[$, $\sqrt{\left(K-\omega^2\right)^2+\left(2\sqrt{\overline{k}}\eta\omega\right)^2} \in \left]2\eta\sqrt{\overline{k}}\omega,+\infty\right[$, and $U(\omega) \in \left]0,\frac{1}{2\eta\sqrt{\overline{k}}\omega}\right[$. Then, we have

$$u_{\mathrm{inf}} = 0 \tag{32}$$

and

$$u_{\mathrm{sup}}(\omega) = \frac{1}{2\eta\sqrt{\overline{k}}\omega} \tag{33}$$

The Figure 8 shows the system response amplitude, considering a Gamma stiffness with a $10,000\ \mathrm{N/m}$ mean stiffness, for two stiffness coefficients of variation and two damping ratios. The upper envelope is represented by a thin solid line and the response of the mean system is given by the dashed line. But to give more insights on this system, it is possible to represent the mean of the system response and confidence intervals by using numerical tools to evaluate them. Then, confidence intervals are represented by two grey areas: the dark grey area corresponds to the $95\,\%$ confidence interval, while the light grey corresponds to the $99\,\%$ confidence interval and the mean of the system response is represented by a thick solid line.

However, it is possible to give some analytical results by considering a Normal approximation of the Gamma distribution. It is the subject of the appendix, where expectations (equations (39) and (40)) and statistical modes are determined for the frequency $\omega^2 = \overline{k}$ with the condition on the system parameters to give multimodality.

Considering now the case of a $20\,\%$ stiffness coefficient of variation and a $3\,\%$ damping ratio, the Figure 9 shows the PDF of the amplitude at several frequencies. It is seen on this figure that various PDF shapes are obtained: a bell shape at low frequencies and various bimodal shapes otherwise. From these graphs, it is clear that the mean value is not a representative value of the system responses when the frequency is not low. Thus, knowing the mean and standard deviation of the system response is insufficient to characterise its response distribution.

The Figure 10 shows more examples of PDF when stiffness coefficient of variation and damping ratios are varied. On this figure, two statistical modes are exhibiting only when the random parameters satisfies the condition (38). Moreover, the Figure 11 shows a situation where the PDF exhibits three statistical modes for a 60% stiffness coefficient of variation with a 5% damping ratio. This behaviour is even found more pronounced for a higher coefficient of variation, but it is not expected by the normal approximation of the Gamma law since this approximation does not hold for such high values of the coefficient of variation.

4.2 Phase of the response

The phase PDF $p_\Theta(\theta,\omega)$ for a Gamma distribution is obtained from the relations (10) and (28) with $k(\theta) = \omega^2 - \frac{2\sqrt{\overline{k}}\eta\omega}{\tan(\theta)}$. It is defined on the domain $\left]\tan^{-1}\left(\frac{-2\eta\sqrt{\overline{k}}}{-\omega}\right),0\right[$.

The Figure 12 shows the system response phase for two damping ratios when considering Gamma random stiffnesses of $10,000\ \mathrm{N/m}$ mean for two coefficients of variation. As for the amplitude, the region of variation are represented by grey areas, and the mean of the system responses and the response of the mean system are also represented.

The behaviour of the phase PDF when the stiffness follows a Gamma distribution is shown on Figure 13. Comparing these PDFs with the uniform ones given on Figure 6, we can see that they vanish smoothly at theirs bounds.

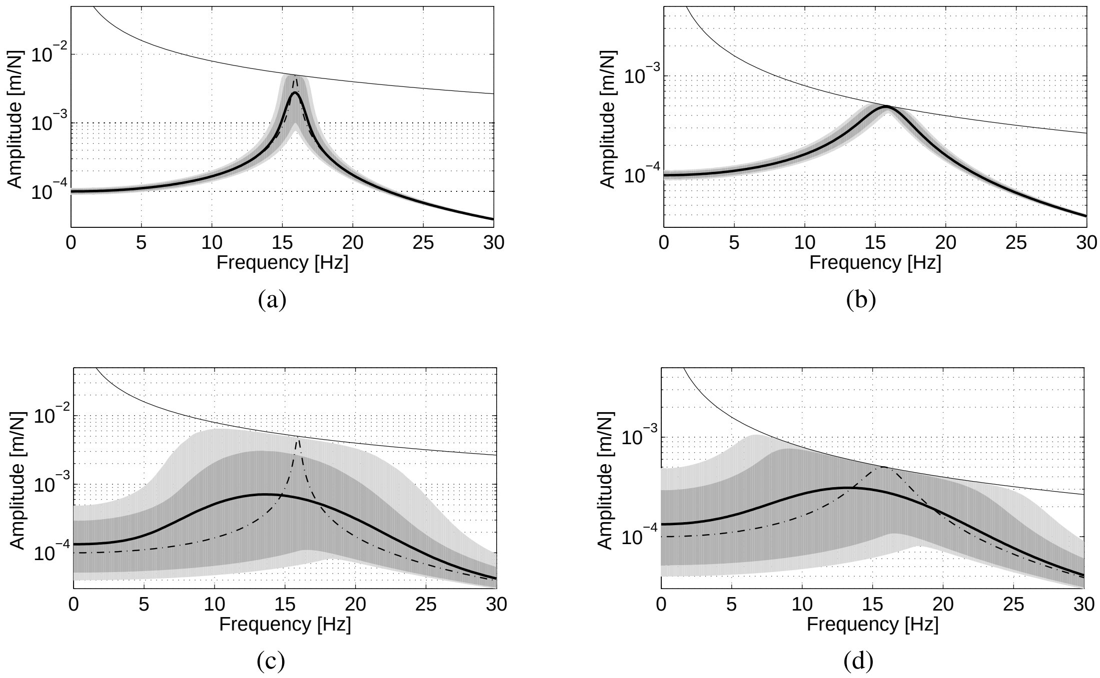


Figure 8: Amplitude response for a Gamma stiffness with the upper envelope (thin solid line), the mean of the system responses (thick solid line) and the response of the mean system (dashed line); dark grey and light grey areas show the 95 % and 99 % confidence region (respectively); (a) 1% damping ratio, 5% coefficient of variation; (b) 10% damping ratio, 5% coefficient of variation; (c) 1% damping ratio, 50% coefficient of variation; (d) 10% damping ratio, 50% coefficient of variation;

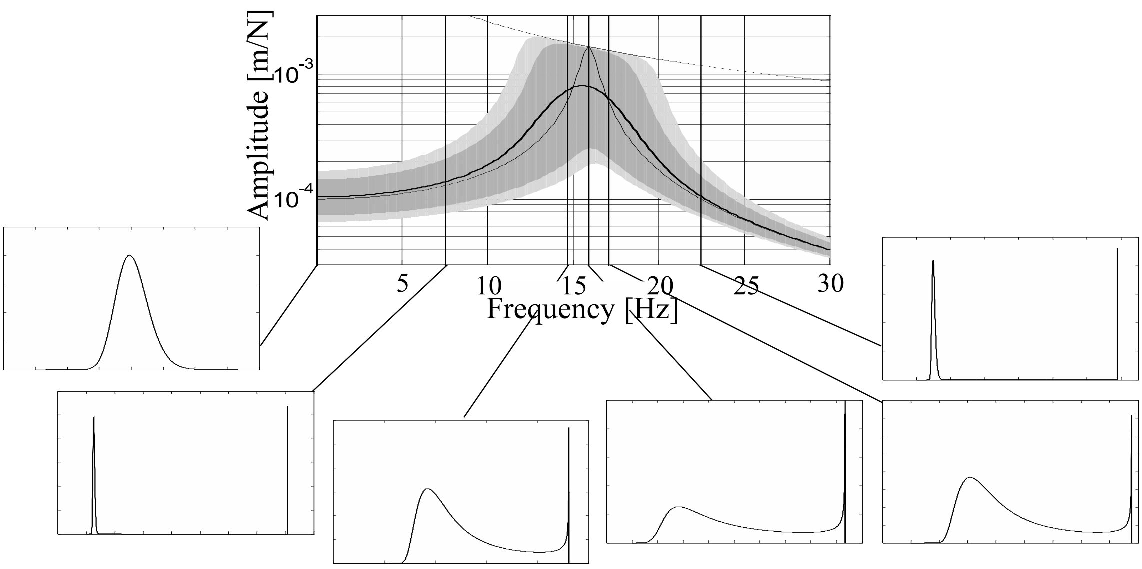
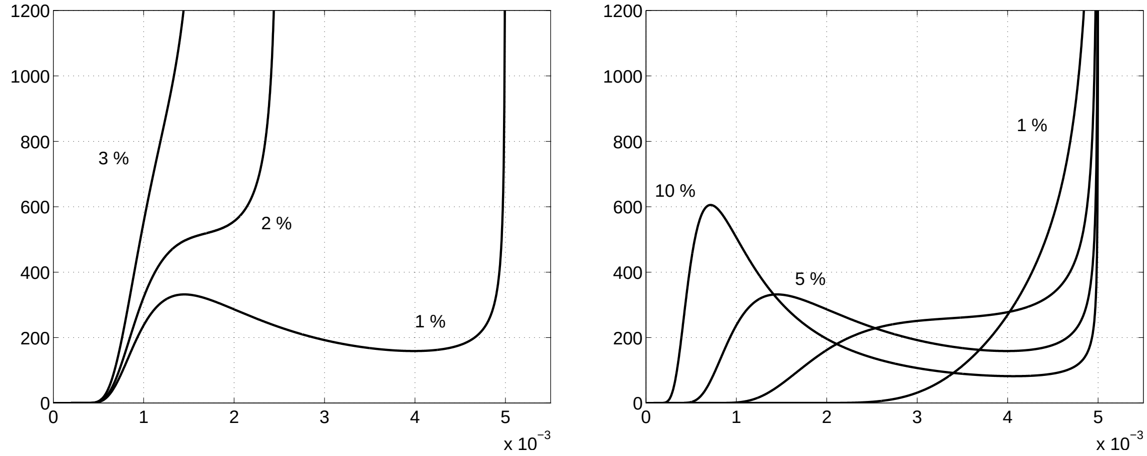
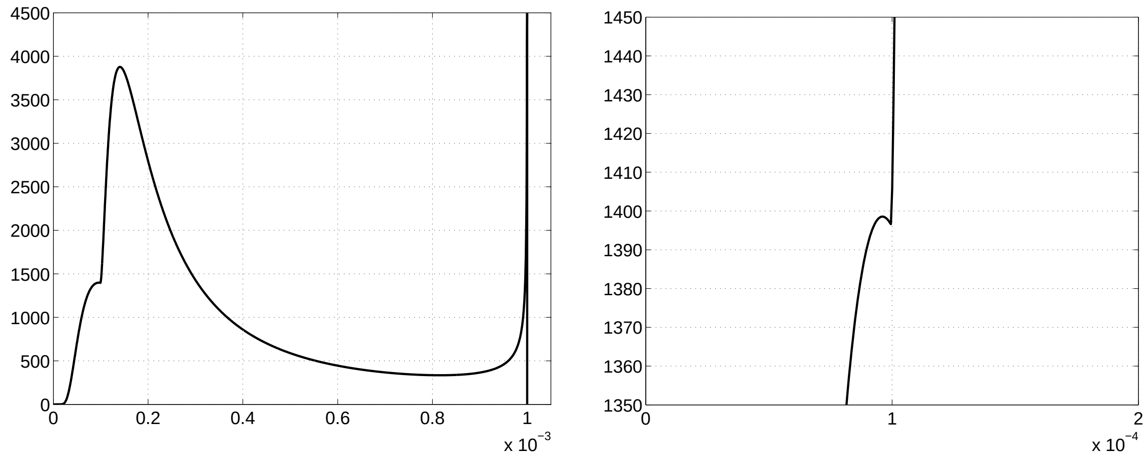


Figure 9: PDF of the amplitude response at various frequencies for a SDOF system having a Gamma stiffness (with a $3\,\%$ damping ratio and a $20\,\%$ coefficient of variation)



(a) (b)

Figure 10: Amplitude response PDF for a Gamma stiffness at $\omega^2 = \overline{k}$; (a) $\frac{\sigma_k}{\overline{k}} = 5\,\%$ and $\eta = \{1\,\%,\ 2\,\%,\ 3\,\%\}$ (b) $\frac{\sigma_k}{\overline{k}} = \{1\,\%,\ 2.5\,\%,\ 5\,\%,\ 10\,\%\}$ and $\eta = 1\,\%$



(a) (b)

Figure 11: Amplitude response PDF for a Gamma stiffness having a $\frac{\sigma_k}{\overline{k}} = 60\,\%$ coefficient of variation; (a) complete domain; (b) zoom of the first mode

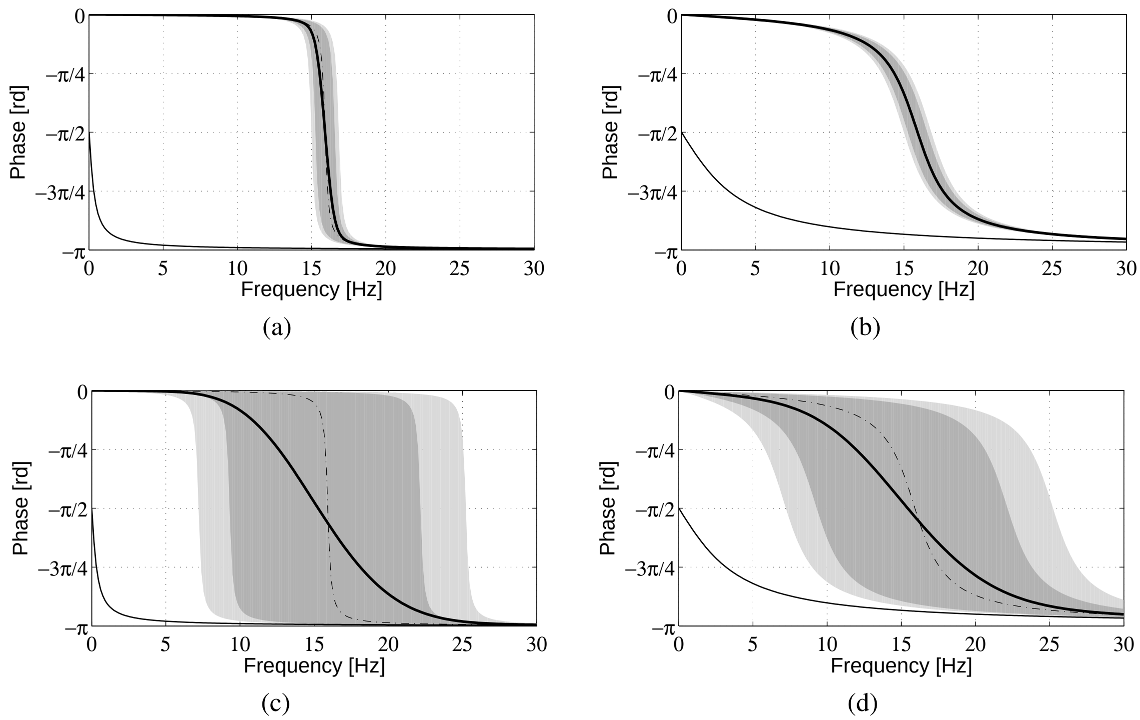


Figure 12: Phase response for a Gamma stiffness with the lower envelope (thin solid line), the mean of the system responses (thick solid line) and the response of the mean system (dashed line); dark grey and light grey areas show the 95 % and 99 % confidence region (respectively); (a) 1% damping ratio, 5% coefficient of variation; (b) 10% damping ratio, 5% coefficient of variation; (c) 1% damping ratio, 50% coefficient of variation; (d) 10% damping ratio, 50% coefficient of variation;

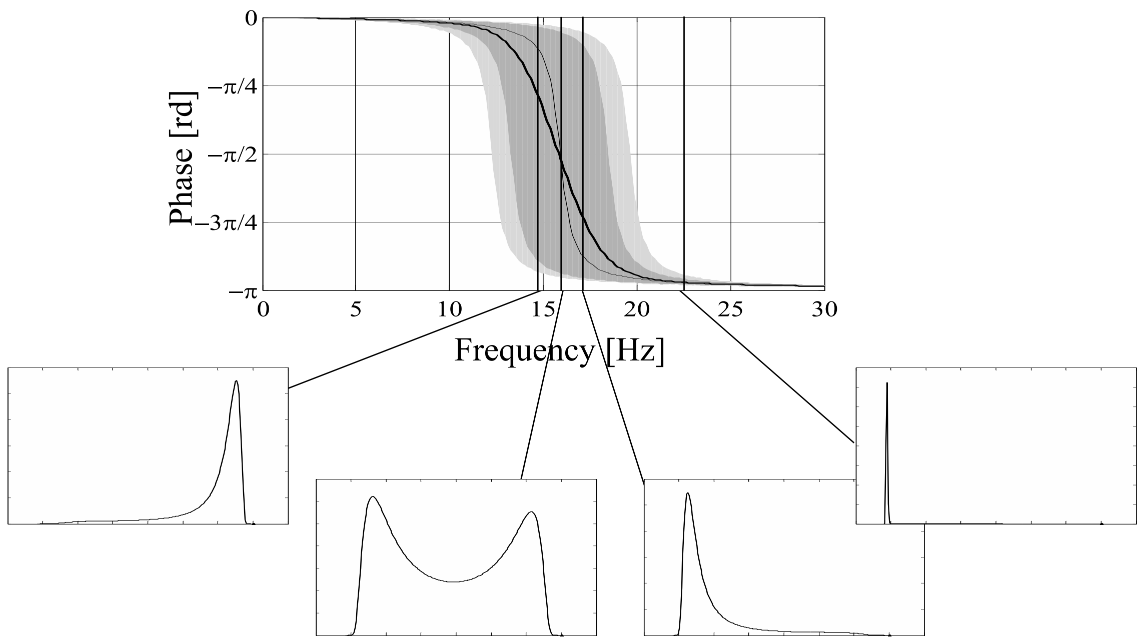


Figure 13: PDF of the phase response at various frequencies for a SDOF system having a Gamma stiffness (with a $3\,\%$ damping ratio and a $20\,\%$ coefficient of variation)

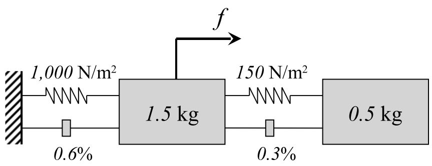


Figure 14: Two degrees of freedom system

Table 1: Natural frequencies and damping ratio of the mean aluminium plate

Normal mode #	1	2	3	4
Natural frequency [Hz]	4.40	10.7	11.3	17.6
Damping ratio [%]	1.81	0.74	0.70	0.45

5 MULTIPLE DOFS SYSTEMS AND CONTINUOUS STRUCTURES

Continuous linear systems or systems with MDOFs having uncertainties could exhibit similar behaviour, that is showing multiple statistical modes around resonant frequencies for the amplitude PDF. To illustrate these assertion, numerical experiments conducted by Monte-Carlo simulation (Rubinstein and Kroese (Rubinstein and Kroese, 2008)) are presented on two examples: a two DOFs system and a plate.

5.1 Two DOFs system

A two DOFs system sketched in Figure 14 having 1.5 and 0.5 kg masses and a Gamma distribution with a $1,000$ and 150 $\mathrm{N/m}$ mean stiffness and a 5 % coefficient of variation is considered. Its damping ratios are 0.3 and 0.6 %. Consequently, the mean system has two resonant frequencies located at 2.05 Hz and 4.50 Hz.

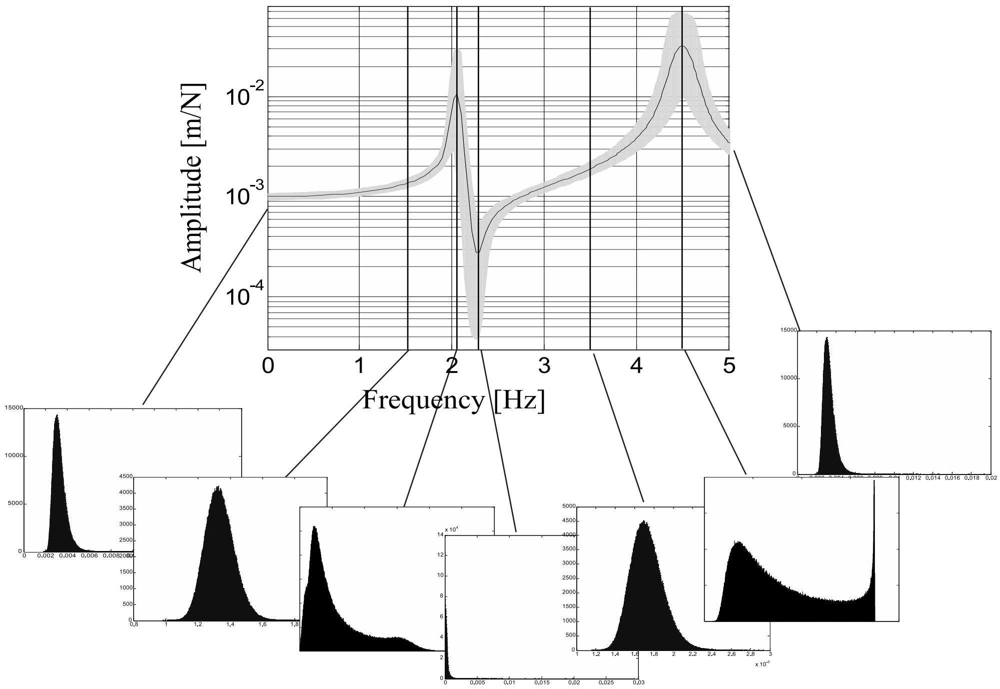
Its amplitude responses are presented on Figure 15 for each degree of freedom when a unit force is applied on the first DOF. On this figure, the grey area depicts the 95 % confidence interval obtained by Monte Carlo simulations involving a 10^6 sample size.

The Figure 15 shows also the histograms obtained for the two DOFs at various frequencies. They exhibit again two statistical modes around the resonant frequencies. However, multi-modality is more pronounced for the second DOF (the unexcited one).

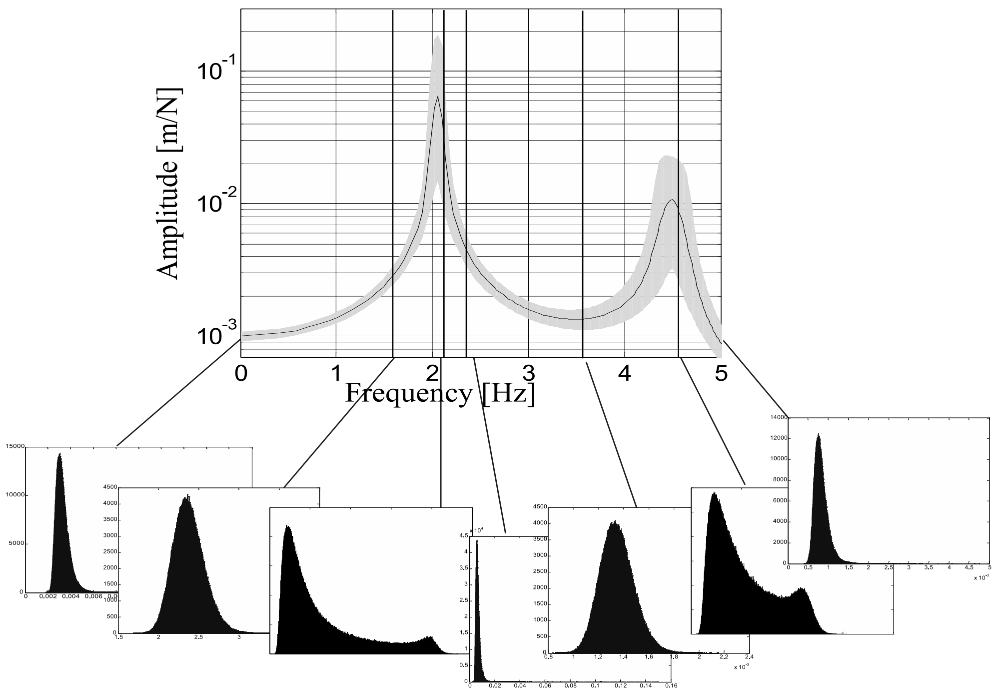
Moreover, the histogram close to the second resonant frequency for the driving point has a shape similar to the one obtained in this study for the SDOF system near its resonant frequency. On the contrary, the shape of the others histograms closed to the resonant frequencies looks like the one obtained for the SDOF system of Pagnacco et al. (2009) when the stiffness and the damping are random: they vanish smoothly at their maximal bound.

5.2 Continuous plate

We consider a continuous plate. This enable to investigate situations where normal modes are coupled by choosing a quasi-square geometry. In our example, we have chosen a 1×1.05 m^2 plate in order to couple the second and the third normal modes of the plate having the mean stiffness. It is made of aluminium alloy with a 1 mm thickness. Boundary conditions are simple supports, leading to a simple analytical expression for the frequency response. The Table 1 gives natural frequencies obtained for the mean stiffness and the damping ratio chosen. The Figure 16 shows the location of the measurements points retained for the following illustrations. The point 1 is chosen as the driving point.



(a)



(b)

Figure 15: Two degrees of freedom system amplitude response and PDF at various frequencies; Mean system responses amplitude (solid line) and 95 % confidence region (grey area); (top) Driving point FRF; (down) Transfer FRF

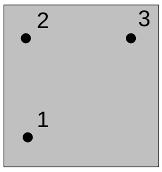


Figure 16: Location of measurements points for the plate

In this example, the Young modulus is considered uncertain and follows a Gamma distribution with a 5% coefficient of variation. Figure 17 shows the amplitude response of the mean plate (solid line) with the 95 % confidence region (grey area) and the mean of amplitude responses (dashed line) for the three measurements points (see the drawing located at the left up corner of each FRF). Stochastic results for this plate are obtained through Monte Carlo numerical simulations.

Figure 17-up shows histograms of the plate driving point displacement response (*i.e.* at point 1) at various frequencies. Many of them are chosen close to the normal modes frequencies of the mean plate. Analysis of these graphs indicates again a bi-modal statistical behaviour in case of uncoupled normal modes, while multi-modal behaviour appeared in the coupled case. The uncoupled behaviour exhibited here is thus very similar to the one found for the SDOF system studied in this work. But for the coupled case, many statistical modes appeared with ragged histogram shapes around the frequencies of interest.

Figure 17-down shows the two cross-displacement responses between the point 1 and the two other chosen points. This indicates that a multi-modal statistical behaviour is also obtained, but with dissimilar histogram shapes.

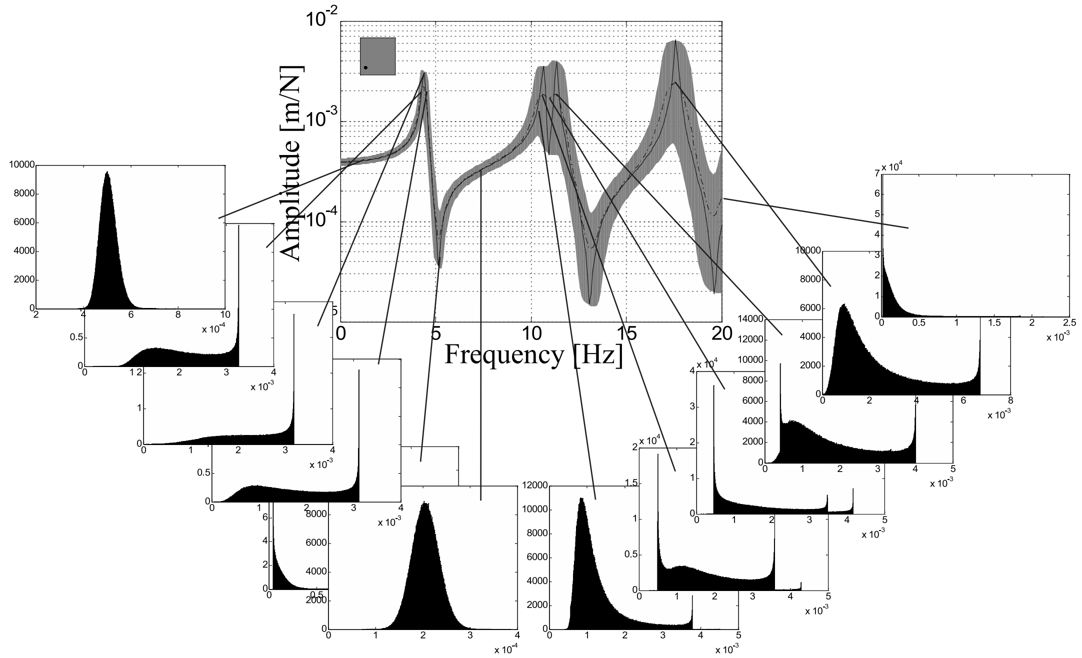
6 CONCLUSIONS

The PDF of the amplitude and the phase of the response of a random linear single-degree-of-freedom mass-spring-damper system when the stiffness are random was discussed for a general PDF of the stiffness. Then, to get more precise results, these PDF was discussed for the uniform and the Gamma distributions, that are the PDF that maximise the uncertainty (entropy) if the stiffness is bounded or unbounded, respectively.

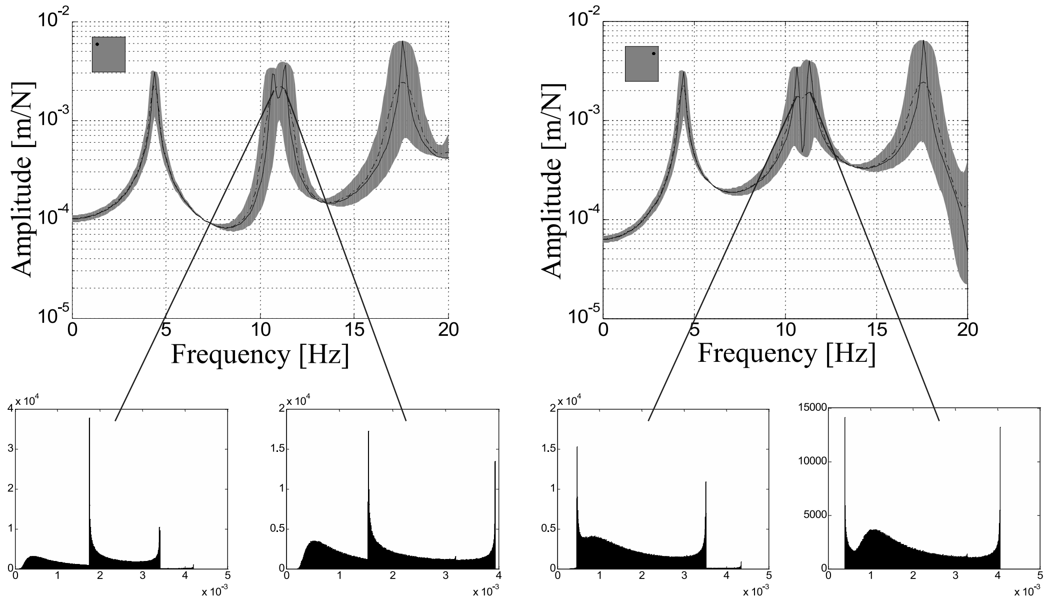
These PDF were studied for a fixed, deterministic, frequency. They were completely characterised with their envelopes, and, when possible, statistics were derived analytically. Moreover, the conditions to have multimodes were described. From these conditions, it is concluded that multimodality occurs very frequently in the vicinity of the resonant frequency of the mean system since the damping ratio is generally small for real systems. Hence, in the vicinity of these frequencies, it is deduced that the mean and the standard deviation of the system response are not representative of the random system response distribution. However, analytical statistics remains useful to provided benchmark tests for numerical computations.

But concerning the rightmost mode of the PDF, the analytical result indicate that it diminishes its value when the damping decreases, while the domain of definition increases. Knowing this behaviour, we have concluded that this mode, found analytically, would be difficult to detect for some random system parameters if it has to be found by Monte Carlo simulations.

Some complex systems, discrete and continuous, were also discussed and they show similar behaviour. This multimodal behaviour of the PDF of the response of random linear systems, to the best of our knowledge, has not been previously discussed in the literature.



(a)



(b)

Figure 17: Plate amplitude responses and PDF at various frequencies; FRFs of the mean plate (solid line), mean FRFs (dashed line) and the 95 % confidence region (grey area) ; measurement locations are indicated on the grey rectangular area representing the plate (left up corner of the FRFs graphs)

Acknowledgements

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APPENDIX A. APPROXIMATION OF THE AMPLITUDE PDF BY A NORMAL DISTRIBUTION

Analysis of the amplitude PDF for statistical modes analytically is rather difficult when the stiffness follows a Gamma distribution. An approximate way consists in considering a normal approximation of the gamma distribution for the stiffness $K \sim N\left(\overline{k}, \sigma_k^2\right)$, which is valid if $\frac{\sigma_k}{\overline{k}}$ tends towards zero:

$$p_K\left(k; \overline{k}, \sigma_k\right) = \frac{1}{\sigma_k\sqrt{2\pi}} \exp\left(-\frac{\left(k-\overline{k}\right)^2}{2\sigma_k^2}\right) \tag{34}$$

In this case $k_{\text{inf}} = -\infty$ and $k_{\text{sup}} = \infty$, but we have to notice that the probability of k became negative tends towards zero if $\frac{\sigma_k}{\overline{k}}$ tends towards zero. Then $k_{\text{inf}} < \omega^2 < k_{\text{sup}}$ and there are always two roots for the algebraic equation $u\left(k, \omega\right) = u$ when ω is fixed and is strictly positive. In this case, the relation (6) leads to the system response PDF

$$p_U\left(u, \omega\right) = \frac{p_K\left(k_1\right) + p_K\left(k_2\right)}{u^2\sqrt{1 - 4\eta^2\overline{k}\omega^2 u^2}} \tag{35}$$

which is defined over $\left]0, \frac{1}{2\eta\sqrt{\overline{k}}\omega}\right[$. Thus, for a fixed frequency $\omega = \sqrt{\overline{k}}$, this last equation simplifies to

$$p_U\left(u, \omega\right) = \frac{1}{u^2\sqrt{1 - 4\eta^2\overline{k}^2 u^2}} \times \frac{2}{\sigma_k\sqrt{2\pi}} \exp\left(-\frac{\left(\frac{1}{u}\sqrt{1 - 4\eta^2\overline{k}\omega^2 u^2}\right)^2}{2\sigma_k^2}\right) \tag{36}$$

Analysis of this result gives a first statistical mode at $\frac{1}{2\eta\overline{k}}$. Moreover, having define $\eta' = \frac{2\eta\overline{k}}{\sigma_k}$, roots of this PDF first derivative are

$$u_{1,2} = \frac{1}{\sigma_k}\sqrt{\frac{1}{6} + \frac{1}{3\eta'^2} \pm \frac{\sqrt{4 - 8\eta'^2 + \eta'^4}}{6\eta'^2}} \tag{37}$$

which lead to a maximum and a minimum for the first and the second value (respectively), when $4 - 8\eta'^2 + \eta'^4 \geq 0$. This is an existence condition for another statistical mode. Thus, when

$$\frac{\sigma_k}{\overline{k}} > \frac{2}{1 + \sqrt{3}}\eta \tag{38}$$

there is two statistical modes for the response PDF, the first one being located at $\frac{1}{\sigma_k}\sqrt{\frac{1}{6} + \frac{1}{3\eta'^2} + \frac{\sqrt{4-8\eta'^2+\eta'^4}}{6\eta'^2}}$ and the second at $\frac{1}{2\eta\overline{k}}$, while only one mode exist at $\frac{1}{2\eta\overline{k}}$. Note that in the no damping case, the first mode is at $\frac{1}{\sqrt{2}\sigma_k}$, while the second tends to infinite.

Having the response amplitude PDF given by equation (36), it is possible to evaluate some expectations such as

$$\mathrm{E}\left[U\right]\left(\sqrt{k}\right)=\frac{1}{\sigma_k\sqrt{2\pi}}\exp\left(\frac{\eta^2\overline{k}^2}{\sigma_k^2}\right)K_0\left(\frac{\eta^2\overline{k}^2}{\sigma_k^2}\right) \tag{39}$$

and

$$\mathrm{E}\left[U^2\right]\left(\sqrt{k}\right)=\frac{1}{2\eta\overline{k}\sigma_k}\sqrt{\frac{\pi}{2}}\exp\left(\frac{2\eta^2\overline{k}^2}{\sigma_k^2}\right)\mathrm{erfc}\left(\frac{\sqrt{2}\eta\overline{k}}{\sigma_k}\right) \tag{40}$$

where K_0 is the modified Bessel function of the second kind of order 0.

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