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NUMERICAL VALUATION ALGORITHMS FOR EXOTIC DERIVATIVES. I: THE TRINOMIAL LOOKBACK CASE

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Key words: Call Option, Put Option, Vanilla Option, Exotic Option, Lookback Option, Binomial Tree, Trinomial Tree, Backward Optimal Algorithm, Stocks, Bonds.

Abstract. *The underlying asset of an option can include, e.g., bonds, stocks, stocks indices, and commodities. The exercise price of the contract and the date of expiration (maturity) are well defined. The Exotic options are variations of the traditional ones, e.g. Asians, Binary, Barrier, Compound, Lookback. The Exotic ones are stated over-the-counter. One of the parts of these contracts has the long position; i.e. who purchases the option. The other has the short position, i.e. who sells or writes the option. Binomial trees are the basic instruments to deal with the valuation of European and American stock and bond options. A quite realistic model is one that supposes that movements in the prices of the assets are made up of a great amount of small binomial movements. The basic expected return of an asset must be greater than those provided by the risk free rate of interest, r. The standard deviation s (or volatility) of the change in the price of an asset in a small time interval (yearly based) is fundamental in the numerical valuation procedure. As the delta and other indicators change during the life of the option we need to make periodic adjustments in our portfolio. This is called hedging. The Lookback options are a class of Exotic options whose price of exercise is the maximum value (minimum for the call) reached by the price of the stock to the instant of evaluation or exercise. The valuation process traditionally uses binomial trees and simulation methods that work backwards in the binomial tree with the purpose of estimating the Lookback value. At the moment of the expiration it is observed retrospectively. Montecarlo methods are also used. In this paper we obtain results of the application of a more accurate trinomial method to this evaluation and study the performance of the special algorithms devised by us to solve this problem.*

1 INTRODUCTION

The underlying asset of an option can include e.g. stocks, stocks indices, commodities, currencies, futures contracts. Two main types of options exist: call and put. A *call* gives the owner the right, but not the obligation, to buy the underlying asset up to a certain date, at a certain price; a *put* gives the owner the right, but not the obligation, to sell the underlying asset up to a certain date, at a certain price. The options being European, the exercise date is strictly the date of expiration, but if the option, call or put, is of the American type, it can be exercised at any date up to the date of expiration.

Most of the options that negotiate in the markets are American and usually each contract is an agreement to buy or to sell one hundred of the underlying assets.

The exercise price of the contract and the date of expiration (maturity) are well defined in all explicit derivatives contracts. There are some implicit derivatives included in other contracts that may have parameters not so precisely defined. The options can be classified – in first approximation– as *Vanilla* or *Exotic*. The Exotic options are variations of the traditional ones, e.g. Asians, Binary, Barrier, Composed, Lookback. One of the excellent features of the Vanilla options is that these are emitted (written, underwritten) without thinking about the necessity of a particular buyer, and –more important– in a controlled market. Instead, the Exotic ones are stated over-the-counter (the parts agree on a non-standard contract.)

One of the parts of these contracts has the *long position*; i.e. who purchases the option. The other has the *short position*, i.e. who sells or writes the option.

Binomial trees are the basic numerical instruments to deal with the valuation of European and American stock and bond options. A quite realistic model is one that supposes that the movements in the prices of the assets are made up of a great amount of small binomial movements. The basic expected return of an asset must be greater than those provided by the risk free rate of interest, r. The standard deviation s (or volatility) of the change in the price of an asset in a small (yearly based) time interval is fundamental in the numerical valuation procedure.

The *delta* of an option of stocks or bonds is the ratio between the change in the price of the option and the change in the price of the underlying. The delta of a call is positive whereas the delta of a put is negative. As the delta changes during the life of the option, in order to maintain our investment safe, without risk, we need to make periodic adjustments in our portfolio. This is called *hedging*.

The Lookback options are a class of Exotic options whose price of exercise is the maximum value (minimum for the call) reached by the price of the stock during the life of the option till the instant of evaluation or exercise (there are several classes of Lookbacks that are going to be explained later in this work). The valuation process traditionally uses binomial trees, and simulation methods, that work backwards in the binomial tree with the purpose of estimating the Lookback value. At the moment of the expiration it is observed retrospectively. The values obtained for the payment by the early exercise (payoff) stated by the expected present value are Montecarlo simulated on the tree, they are also compared with the non-exercising option values; then the simulation advances in recursive form until it is managed to determine the optimal strategy and therefore to estimate the value of the Exotic option.

In this paper the main objective it is to obtain results of the application of a more refined trinomial method to this evaluation and assess the comparative performance of the algorithms devised by us to solve this problem.

The methods we develop are focused on Lookback Exotics but they can be applied to a complete set of *path dependent derivatives* of the options and futures classes.

The basic references for options, futures and other derivatives are the books of Hull[1,2]

The paper is also about the efficient construction of algebraic algorithms (this issue is implicit in our results, it is not stressed, but it will be published elsewhere, see Note 8(3), after the references). Our example algorithms are constructed on Matlab[3] (but, as a matter of fact, they do not depend on Matlab and can be easily translated to any computational system[4].) The aim of this Matlab based development is to simplify the process of vectorization and parallelization of the algorithms, feature needed for the resolution of real portfolios of assets and derivatives (for each one it is necessary a similar treatment and in most cases we also have to deal with subtle interactions and correlations between product values.)

1.1 Binomial tree methods

We use binomial methods for the sake of simplicity (introduction of main concepts,) but the objective of this work is to develop an efficient numerical and algebraic algorithm of discrete type, able to estimate the correct value of the Exotic path dependent option, but simple enough to be treated in a pilot scale mathematical environment.

1.1.1 Estimates using binomial trees:

Binomial trees deal with the estimate of European and American options, they are good models for a first approximation of option valuation (mainly for educational purposes.) The model supposes that the movements in the prices of the actions are made up by a great amount of small binomial movements. Let us consider the valuation of an option whose underlying stock do not pay dividends (in this work we do not treat the inclusion of dividends because that is associated with relatively minor modifications of our algorithms and that is not the main objective of this work). We start by dividing the life of the option into small time intervals of length δt. We suppose that at each time interval the price of the action moves from its initial value to one of its new values: a raise u, or a loss d, in the price. The probability of an increment is p and that of a loss is $1 - p$.

Determination of p, u, and d: these parameters must give to the correct value for the average and the variance of the price movement of the action during a time interval δt. The expected return for the investment on stock is the risk free rate of interest r. Thus, the expected value of the price of an action at the end of the interval δt is $Se^{r\delta t}$, where S is the price of the stock in the beginning of the time interval. It follows that:

$$Se^{r\delta t} = pSu + (1-p)Sd \tag{1}$$

or

$$e^{r\delta t} = pu + (1-p)d \tag{2}$$

The standard deviation of the change in the price in a small time interval δt is $\sigma\sqrt{\delta t}$. The variance of this change is $\sigma^2 \delta t$. Using the variance definition, we obtain:

$$\sigma^2 \delta t = pu^2 + (1-p)d^2 - \left[pu + (1-p)d\right]^2 \tag{3}$$

The equations (2) and (3) impose two conditions for *p*, *u* and *d*. The standard third condition imposed is:

$$u = 1/d \tag{4}$$

For δt small, the three conditions imply

$$p = \frac{a-d}{u-d} \quad \text{where} \quad a = e^{r\delta t},\ u = e^{\sigma\sqrt{\delta t}},\ d = e^{-\sigma\sqrt{\delta t}} \tag{5}$$

Tree of prices of actions: In the beginning time the initial price S_0 (for time zero) is known. In time δt, there exists two possibilities for the price of the underlying stock: S_0u and S_0d; in time 2 δt , exists three possibilities S_0u^2 , S_0ud and S_0d^2, and so on. In general, in time $i\,\delta t$, (i+1) prices of the stock have to be considered. These are:

$$S_0u^j d^{i-j} \quad (j = 0,1,\ldots, i) \tag{6}$$

The relation (4) is used in the calculation of the price of the stock in each node of the tree. Schematically:

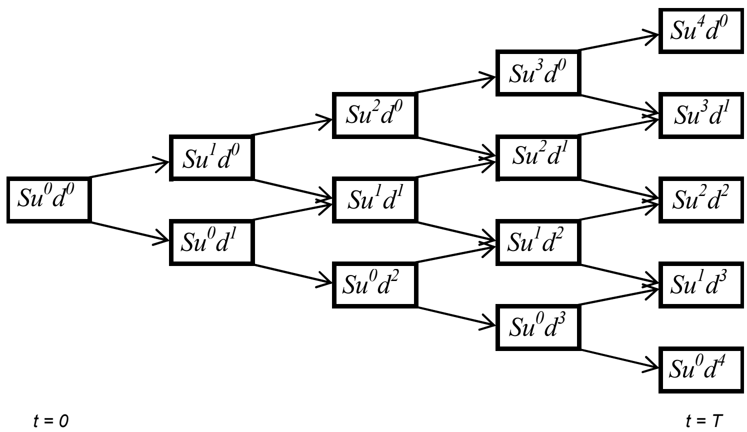

Figure 1: Binomial tree from t = 0 to t = T in 4 steps

1.2 Delta

The delta of an option on assets is the quotient between the change in the price of the option and the change in the price of the underlying asset. The delta of a call is positive whereas the delta of put is negative. As the delta changes during the life of the option, in order to maintain our investment safe and to minimize the risk by the use of derivatives, we need to make periodic adjustments in our portfolio. Those are known as rebalancing the portfolio, and are a form to perform hedging (a procedure to protect the investment). If the price of the action never reaches the price of exercise *K*, the accomplishment of hedging does not cost anything. However, in the usual case in what the price of the asset surpasses the exercise price, the scheme of hedging could be very expensive. To take a position in the option is the same as to make hedging in the opposed position in the option. If the delta in a short position is $-\Delta$, in the long position it is $+\Delta$, thus the global investment has Δ null, i.e., it have a *neutral* delta. Due to the change in delta, the position of the investor remains with single neutral delta for a short period of time. For a position without risk, for example, one is due to buy Δ actions by each sold option. This means to change to the amount of underlying stocks in our portfolio.

1.2 Rebalancing process

Exercise must be performed if delta reaches 1, but it must not be performed if delta reaches 0. The main suppositions are that the volatility is constant and that there are no transaction costs. The following table is constructed:

Table 1: Rebalancing procedure

half month	Stock price	Δ_0	Purchased stocks	Cost of purchased stocks	Interest accumulated cost	Interest cost
0	p_0	Δ_0	$A_0\,\Delta_0$	$(A_0\,\Delta_0)p_0$	$(A_0\,\Delta_0)p_0$	$(A_0\,\Delta_0)r_q$
1	p_1	Δ_1	A_1	$A_1\,p_1$	B_1	$[A_0\,\Delta_0\,(p_0+r_q)+A_1p_1)]$
…	…	…	…	…	…	…
i		Δ_i		$(A_i\Delta_i)p_i$	B_i	$B_i\ r_q$
$i+1$				$(A_{i+1}\Delta_{i+1})p_{i+1}$	B_{i+1}	$B_{i+1}\ r_q$
…	…	…	…	…	…	…
T	p_T	Δ_T	A_T	$(A_T\Delta_T)p_T$	B_T	------

where

$$A_i = A_0\ \Delta_i \pm \sum_{j=0}^{i-1} A_j\ ; \quad \text{for } i = 0,1,\ldots,T$$

$$B_{i+1} = B_i + (B_i * r_q) + (A_{i+1} * \Delta_{i+1}) * p_{i+1}; \qquad \text{for } i = 1,\ldots,T$$

- r_q is the half month rate
- In purchased stock column sign depends on - : purchased, + : sold
- $A_0 = 10000$
- $B_1 = A_0\ \Delta_0\ (p_0 + r_q) + A_1\ p_1$

2 LOOKBACK OPTIONS OVER STOCKS

Lookback options (i.e. 'see backwards') are those options whose exercise price is the maximum value the underlying stock achieves during the life of the option (there are several variants of this definition leading to different products, but for the sake of simplicity we will only treat the simplest case, the others need modifications of the basic algorithm and our methods apply to a bigger class of path dependent Exotic options than the whole set of Lookbacks.)

The evaluation procedure starts with the construction of the binomial tree (*vide infra* for the trinomial analysis that is the main objective of this contribution), and then we have to complement the construction with simulation methods or backwards work in the tree. At expiration time the Lookback price is known and the backwards process start (for European options is a direct process, but for American options, it is necessary to optimize in each of the nodes in order to decide if early exercise must be advised to the customer.)

There are two main classes of Lookback or Hindsight options

(1) Fixed Strike

This type of Lookback option is only settled in cash, and has the strike pretermined at inception and the payoff is the maximum difference between the optimal price and the strike price.

(2) Floating Strike

Introduced in 1979, these can have payoffs which are either cash or asset settled, where the strike is given as the optimal value of the underlying asset. It can be noted that although floating strike options are 'options' per say, they are not actually options as they are always exercised.

For both types, the respective call and put payoffs are given as:

Table 2: Payoffs at expiry of put and call Lookbacks

$$p_{fixed} = \max(0, X - S_{\min}) \qquad p_{float} = \max(0, S_{\max} - S)$$
$$c_{fixed} = \max(0, S_{\max} - X) \qquad c_{float} = \max(0, S - S_{\min})$$

An attractive benefit of Lookback options is that they are never out-of-the-money, but the result is that Lookbacks are often more expensive than similar Vanilla style options.

2.1 Procedure for binomial Lookback

One works with, e.g., a portfolio of 100 put options (American),

- clasical: Vanilla
- Exotic: Lookback

each option is composed by 100 stocks. Let us consider the valuation of options whose underlying stocks do not pay dividends. So the Lookback option (i.e. 'to watch backwards') are those whose price of exercise is the maximum quotation (minimum for the call option) reached by the price of the stock till the present moment. We exemplify by means of the binomial tree method and work backwards in the binomial tree with the purpose of assessing the value of the option in each of the nodes of the tree. At the moment of the expiration the value is observed retrospectively. It is necessary to compare at each node the early exercise payoff with the expected present value obtained of the backwards recursion. The value of the option in the nodes at time *T* are obtained taking:

$$max\{\, 0 \,;\, \underset{0\le t\le T}{máx}\, S_t - S_T \,\} \tag{7}$$

Observation: this value depends on the evolution of the price of the stock. Here "the better way is taken into account;" i.e., the maximum value of all the trajectories that can follow the stock in order to stop at the ending node we value.

In order to estimate the option value in the times before end time *T*, the maximum between:

$$e^{-r\delta t}(pf_{i+1,j} + (1-p)f_{i+1,j+1}) \tag{8}$$

and

$$\underset{0\le t\le T_j}{máx}\, S_t - S_{T_j} \tag{9}$$

((9) being the payment for the early exercise) is calculated.

In the first examples we deal with the following parameters:

Table 3: Data for numerical evaluations

$S_0 = 1.45$	$X = 1.52$	$r = 0.0503$	$\sigma = 0.6945$	$T = 0.1644$

(from “El Cronista Comercial” J., Buenos Aires, May 24, 2000, 'ACI' stocks and options).

Suppositions:

- Risk-free interest rate.
- No dividend paying stocks.
- Use of “backwards induction” in the binomial tree.

For American options it is necessary to check at each node for the possibility of early exercise or wait for another time step ∂t. Duration of the option is two months (1/6)

The binomial tree for a Vanilla put option is conformed by the following values

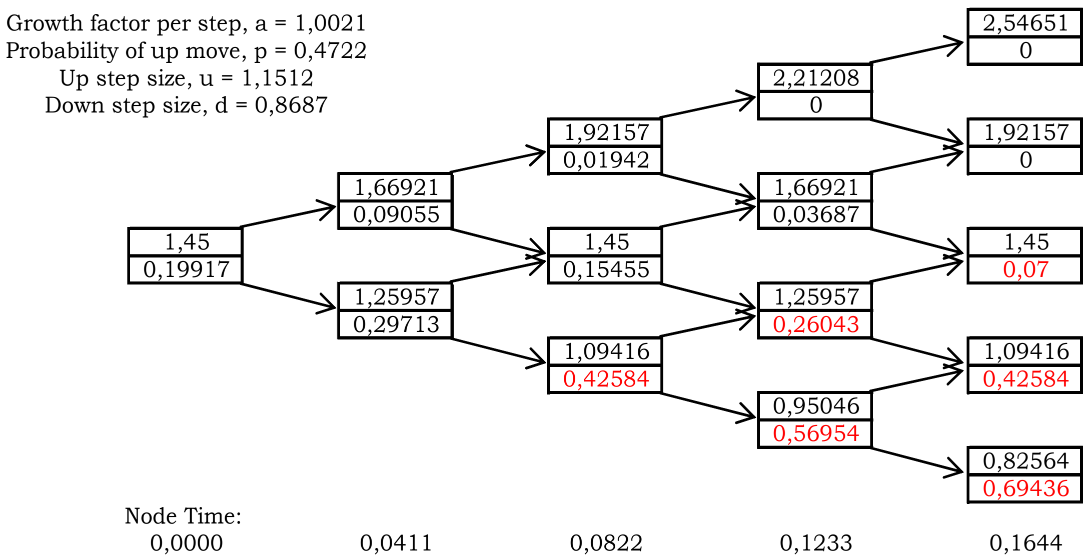

Figure 2: Vanilla. Binomial tree from $t = 0$ to $t = T$ in 4 steps. The upper value is the price of the stock in that node and the lower value is the price of the option

The binomial tree for a Lookback put option is conformed by the following values:

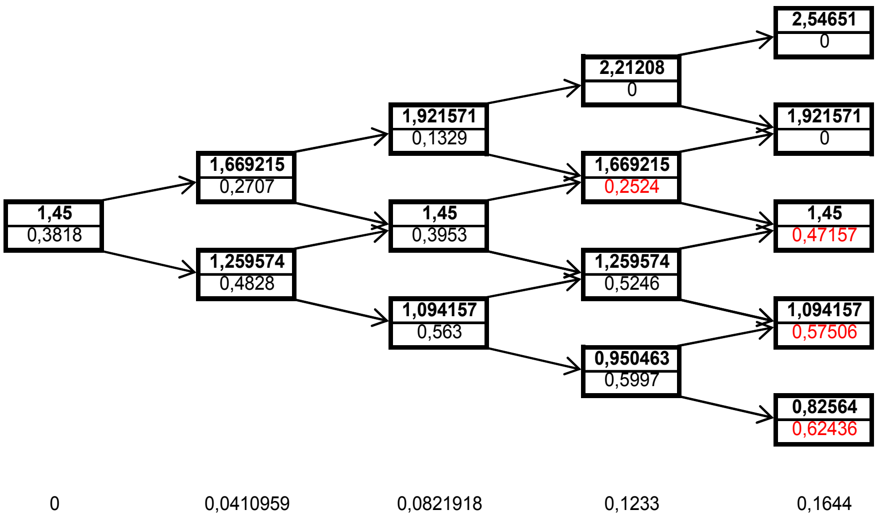

Figure 2: Lookback. Binomial tree from $t = 0$ to $t = T$ in 4 steps. The upper value is the price of the stock action in that node and the lower value is the price of the option

2.2 Delta

$$\Delta_{i,j} = \frac{f_{i+1,j} - f_{i+1,j+1}}{S_{i+1,j} - S_{i+1,j+1}} \tag{10}$$

for i, j =0, 1, 2, 3.

Classic:

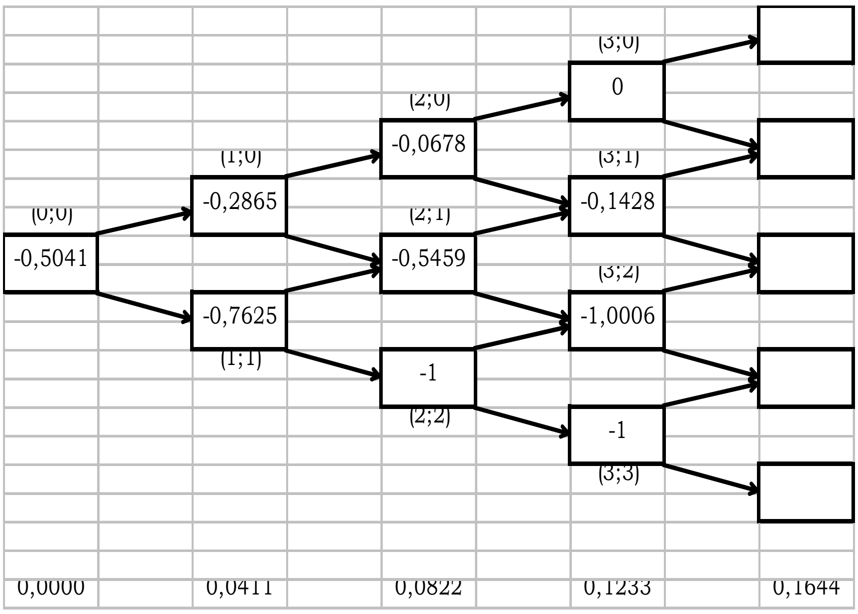

Figure 3: Binomial tree from $t = 0$ to $t = T$ in 4 steps. Delta values.

Lookback:

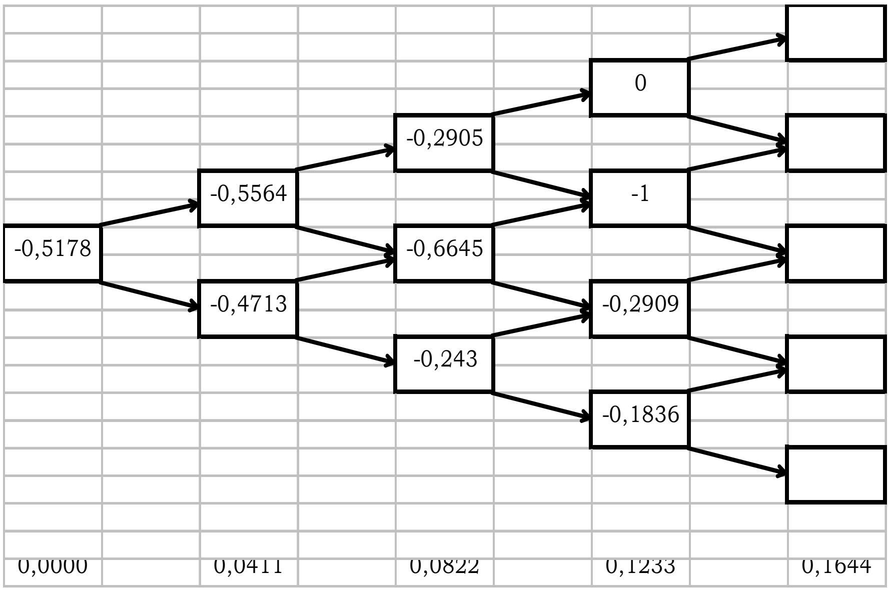

Figure 4: Binomial tree from $t = 0$ to $t = T$ in 4 steps. Delta values.

3 TRINOMIAL VALUATION OF LOOKBACKS

3.1 Trinomial distribution

We suppose we have three exhaustive, mutually exclusive events E_1, E_2, E_3. We realize n independent trials and define the random variables X_i, i=1,2,3, where X_i is the number of ocurrences of E_i. We have

$$\sum_{i=1}^{3} X_i = n \tag{11}$$

so the model is defined by the joint probability of the first two random variables.
The joint density is

$$f_{X_1,X_2}(x_1, x_2) = P(X_1 = x_1, X_2 = x_2) = \frac{n!}{x_1!x_2!(n - x_1 - x_2)!} p_1^{x_1} p_2^{x_2} q^{(n-x_1-x_2)} \tag{12}$$

and

$$\sum \frac{n!}{n_1!n_2!(n - n_1 - n_2)!} p_1^{n_1} p_2^{n_2} q^{(n-n_1-n_2)} = (p_1 + p_2 + q)^n \tag{13}$$

so the generating moments function is

$$\psi_{X_1,X_2}(t_1,t_2) = (p_1 e^{t_1} + p_2 e^{t_2} + q)^n \tag{14}$$

and the characteristic function is

$$\varphi_{X_1,X_2}(t_1,t_2) = (p_1 e^{it_1} + p_2 e^{it_2} + q)^n \tag{15}$$

This distribution has an approximating normal bivariate distribution with the parameters obtained from the moments of the trinomial.
In Fig. 5 the trinomial probabilities are shown (for a numerical example with probabilities (.41,.2,.39))

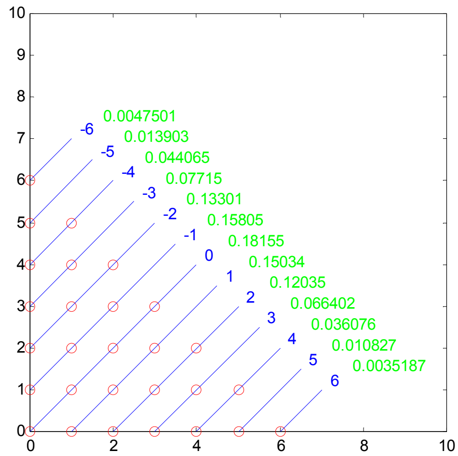

Figure 5: Events for trinomial with n=6 and probabilities (.41,.2,.39)

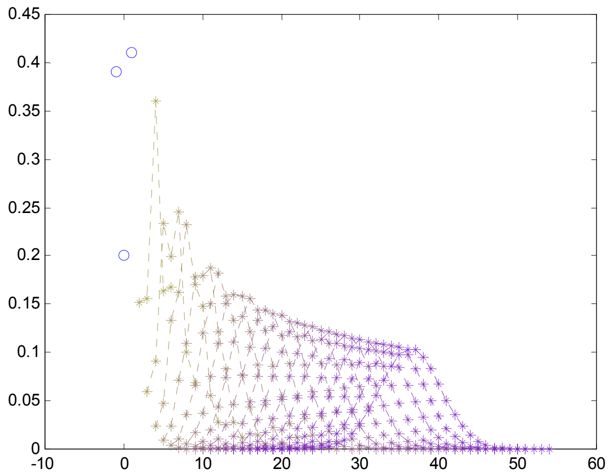

Figure 6: Trinomial densities calculated by convolution. Same probabilities as in Fig.5
The densities are shifted for visualization and are calculated by repeated convolution of the trinomial.

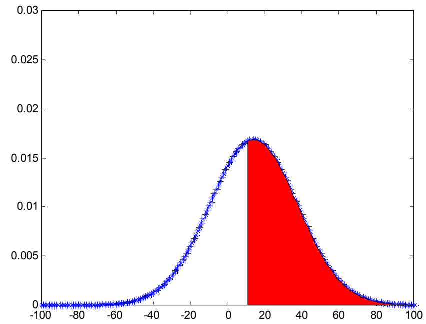

Figure 7: Trinomial density compared with the limiting normal density. Same probabilities as in Fig. 5. Sum of 700 trinomial variables. The red shadowed area is 0.5589, the probabilty that, after 700 time periods, the path is over +10 up jumps from start price.

In Fig. 6 and 7 an example of trinomial densities and their normal approximation is plotted.

3.2 Value of options

The value of an European standard Lookback call is (see [1])

$$S_0 e^{-qT} N(a_1) - S_0 e^{-qT} \frac{\sigma^2}{2(r-q)} N(-a_1) - S_{\min} e^{-rT} \left[N(a_2) - \frac{\sigma^2}{2(r-q)} e^{Y_1} N(-a_3) \right] \quad (16)$$

where

$$a_1 = \frac{\log(\frac{S_0}{S_{\min}}) + (r - q + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}} \quad (17)$$

$$a_2 = a_1 - \sigma\sqrt{T} \quad (18)$$

$$a_3 = \frac{\log(\frac{S_0}{S_{\min}}) + (-r + q + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}} \quad (19)$$

$$Y_1 = -\frac{2(q - r - \frac{\sigma^2}{2})\log(\frac{S_0}{S_{\min}})}{\sigma^2} \quad (20)$$

and $S_{\min}$ is the minimum stock price achieved to date. (If the Lookback has just been originated, $S_{\min} = S_0$.)

The value of an European standard Lookback put is (see [1])

$$S_{\max} e^{-rT}\left[N(b_1)-\frac{\sigma^2}{2(r-q)}e^{Y_2}N(-b_3)\right]+S_0e^{-qT}\frac{\sigma^2}{2(r-q)}N(-b_2)-S_0e^{-qT}N(b_2) \quad (21$$

where

$$b_1=\frac{\log(\frac{S_{\max}}{S_0})+(-r+q+\frac{\sigma^2}{2})T}{\sigma\sqrt{T}} \quad (22)$$

$$b_2=b_1-\sigma\sqrt{T} \quad (23)$$

$$b_3=\frac{\log(\frac{S_{\max}}{S_0})+(r-q-\frac{\sigma^2}{2})T}{\sigma\sqrt{T}} \quad (24)$$

$$Y_2=\frac{2(r-q-\frac{\sigma^2}{2})\log(\frac{S_{\max}}{S_0})}{\sigma^2} \quad (25)$$

and $S_{\max}$ is the maximum stock price achieved to date. (If the Lookback has just been originated, $S_{\max}=S_0$.)

The main problem with these formulae is that they are only useful for the standard Lookback, each variation (that is able to deal with the numerical procedure) leads to a difficult and in some cases impossible task of well posing the boundary value problem. In the general case a numerical method such as FEM or FD (but FD ‘is’ trinomial) is necessary for the numerical solution of the Exotic path dependent valuation problem.

3.3 American Lookbacks

The valuation of American Lookbacks (or other path dependent options) is more complex because it is necessary to decide at each node of the trinomial tree if the early exercise is better than the backwards valuation and take the best of the two values, so the algorithm is more complex. The order of these algorithms is still linear because the process is limited to the prices tree and not to the complete spectra of possibilities.

3.4 Equations for the trinomial discrete tree

The trinomial tree is similar to the binomial one but there are three possibilities for each step (up, middle, and down). There are several possible determinations of the amounts of the increments and their probabilities, and for simplicity we will take, as a test problem, the formulae as stated in the book of Hull [1] (when we deal with different random walks we have to calibrate these parameters):

$$u = e^{\sigma\sqrt{3\Delta t}}, \quad d=1/u, \ m=1, \ ud=m^2 \tag{26}$$

$$p_d = -\sqrt{\frac{\Delta t}{12\sigma^2}}(r - \frac{\sigma^2}{2}) = \frac{1}{6} \tag{27}$$

$$p_m = 2/3 \tag{28}$$

$$p_d = \sqrt{\frac{\Delta t}{12\sigma^2}}(r - \frac{\sigma^2}{2}) = \frac{1}{6} \tag{29}$$

For a stock paying a continuous dividend at rate q, we replace r by $r-q$ in these equations

In Figs 5--8 we represent the results for full case analysis algorithm (complexity exponential of order O(3^n))

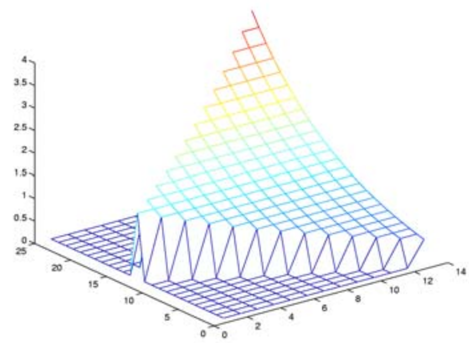

Figure 5: Trinomial results for stock price *S, T=1/6* in 12 steps, full tree.

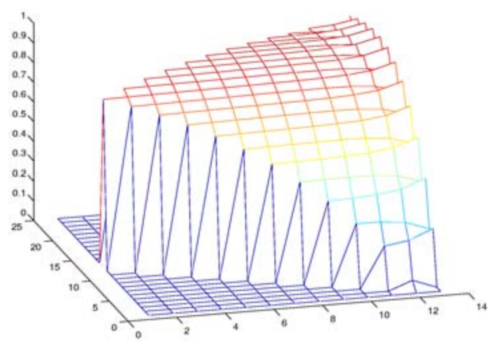

Figure 6: Trinomial results for option value *c, T=1/6* in 12 steps, full tree.

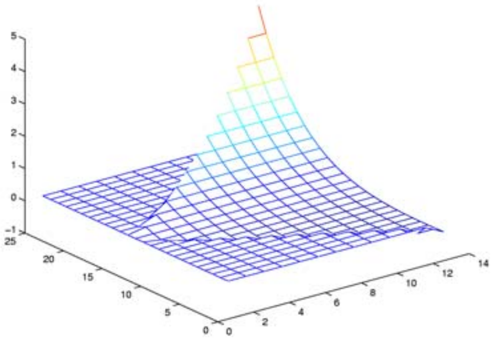

Figure 7: Trinomial results for stock price *S, T=1/6* in 12 steps, difference between volatilities 0.70 and 0.40, full tree

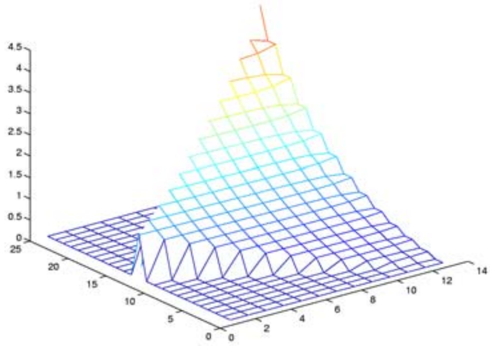

Figure 8: Trinomial results for option value *c, T=1/6* in 12 steps, difference between volatilities 0.70 and 0.40, full tree.

The main problem with the full analysis algorithm is the exponential complexity. But, luckily, the full analysis is not necessary here; in this Exotic class of options (and in most cases of path dependent options) it is only necessary to consider worst (maximum and minimum cases) and interpolate between them. So it is possible to study only a linear quantity of cases. With this improvement the complexity drops to O(2*n*).

The following step is to vectorize the algorithm. This is achieved by the method of calculating with all paths at the same time, so, when using a parallel cluster, the program sends each calculation to a different processor and this test problem is solved instantaneously. In the normal case when a portfolio of such stocks and classical and Exotic options is to be considered, the parallelization of the algorithm is fully necessary because we have to distribute the different product valuation between the cluster member and also to make profit of the remaining parallel power to deal with each of the valuations by itself. In most of the cases there is also positive correlation between the portfolio components and we have to also deal with it.

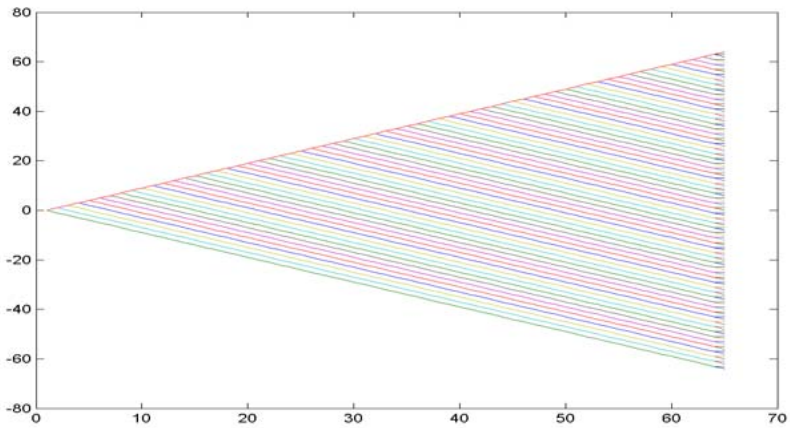

Figure 9: Trinomial tree, worst cases

In Figs. 10—15 the trinomial tree of stock values and of option values is plotted for an European Lookback put option. The same method is applicable to Lookback calls, discrete sampled Lookback puts and calls, Asian (average) Exotic options of each class, continuously and discretely sampled, and also a complete set of path dependent Exotics.

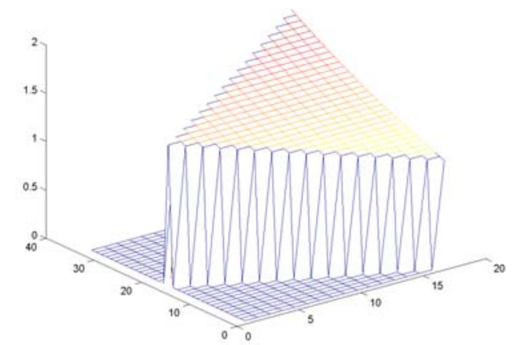

Figure 10: Trinomial tree, stock values, 16 steps

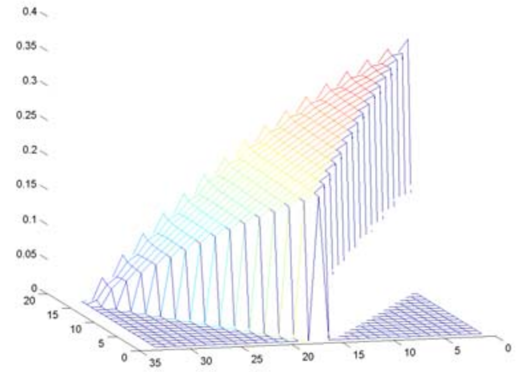

Figure 11: Trinomial tree, option values, 16 steps

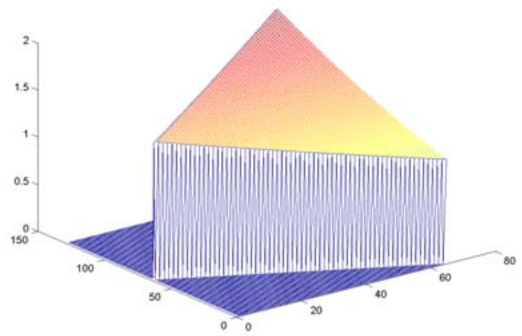

Figure 12: Trinomial tree, stock values, 64 steps

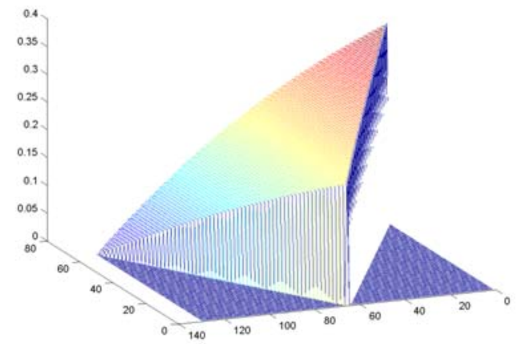

Figure 13: Trinomial tree, option values, 64 steps

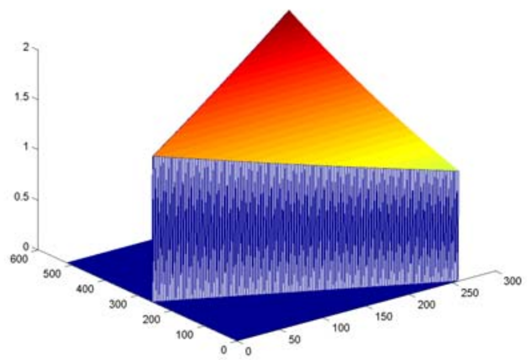

Figure 14: Trinomial tree, stock values, 256 steps

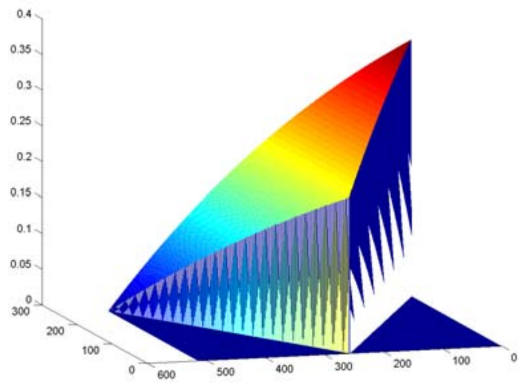

Figure 15: Trinomial tree, option values, 256 steps

The obtained value of $c = 0.2$ (in this example) is in agreement with the obtainable one in the literature [1].

4 DISCRETE ALGORITHMS

The algorithms that are able to deal with the Lookbacks and the most part of Exotic path dependent derivatives go from the simple binomial scheme till the more sophisticated wavelet based (basis decomposition) methods. Trinomial, that are, in essence, finite difference approximations of PDE models have the plausible property of being simple to understand for the non mathematically skilled users (of course the interdisciplinary team is here mandatory) but, in most cases, it is necessary to explain the results to a manager and the trinomial version is edible enough to achieve this goal.

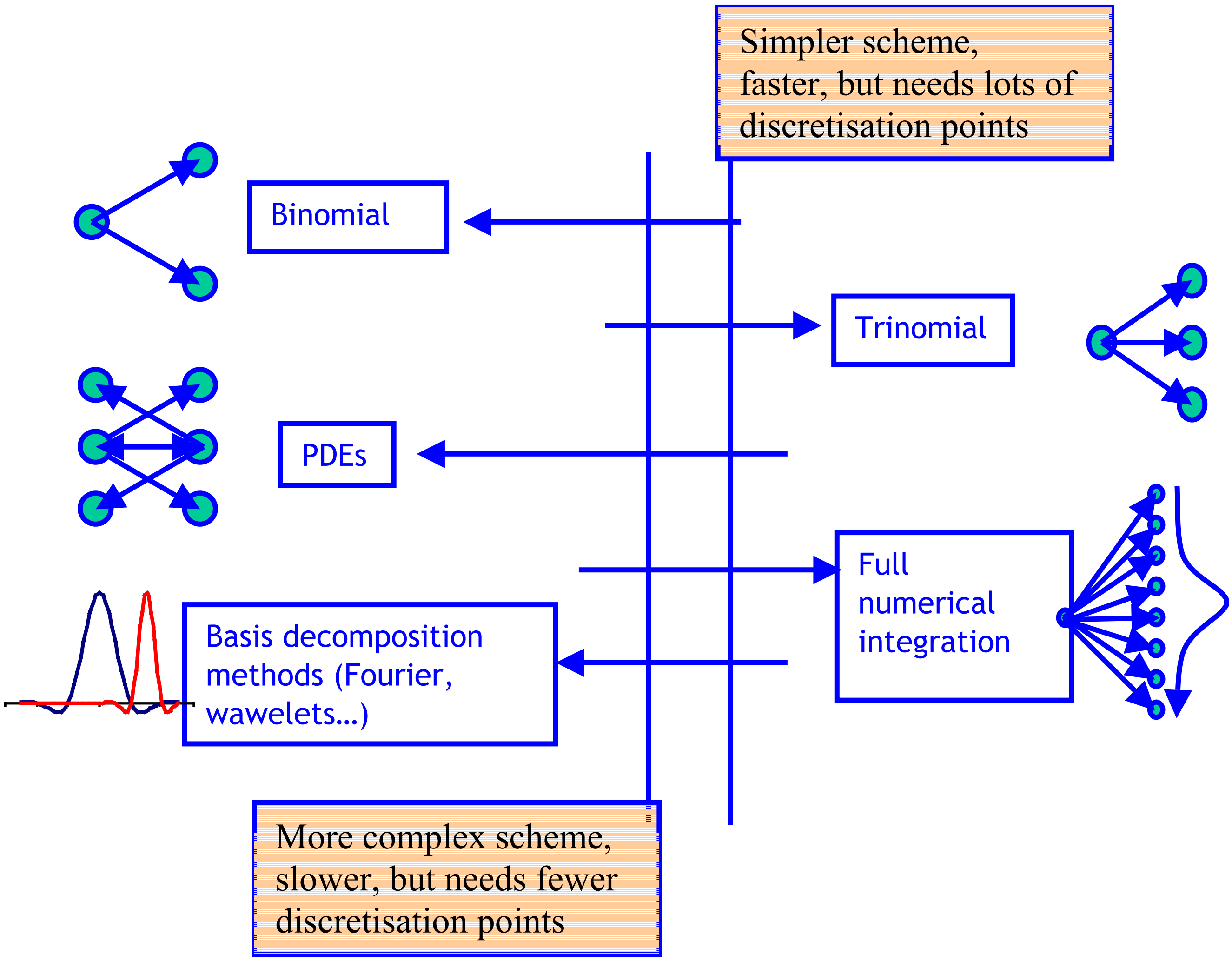

Figure 16: Numerical methods scheme

In some cases we have the well posed variants of the Black-Scholes-Merton PDE equations with tractable boundary conditions. But in the most part, the derivation of these is difficult enough and, if the derivative is, as usual, stated out of the counter and is unique, probably it is not worth to approach its valuation by analytical tools (because there are several and hidden uncertainties in the parameters and data, and, also, in the structure of the portfolio where the derivative is settled) The natural approach would be in these cases the algebraic or discrete method. The best of these (simple, accurate, adaptative) is the trinomial.

So in this article we have developed trinomial methods. The principal classes of these methods have complexity, for the first part of the tree construction:

(1) Full or complete tree, complexity O(3^n)
(2) Montecarlo, at least O(n^3)
(3) Simplified tree, at most O($d\ n$), d a small integer (2 to 5)
(4) Simplified tree, with sampling at prescribed points can be designed a linear algorithm.

and, for the second part (backwards value construction,) in first approximation it is O(n^2), but, by, careful selection of the nodes to evaluate, using the theoretical value for the others, it is possible to achieve linear order (but a general algorithm of linear order is still lacking and the best we have constructed is, by separation techniques, O($n \log n$))

If the options are American instead of European the complexities do not change (but the constants do)

Trinomial methods are a class of direct and simple discrete algorithms, the methods that are located towards the more complex scheme are, in concept, continuous but essentially are methods that imply a discretization or algebrization of the problem. The main advantage of trinomial method is that it is easily comprehensible and it is really useful for full understanding of the main features of the process of valuation.

The inner core of the linear plus 'loglinear' algorithm is omitted by reasons of copyright but the main features (instead of their detailed implementation) were described in this work.

5 ERROR ESTIMATION AND ADAPTIVITY

We have developed elsewhere[5] a method to assess *a posteriori* errors. In this class of problems the mixed mesh error estimation is well adapted. Based in the error estimation and in the estimation of critical areas of the tree where the determination of value have to be more accurate it is possible to refine the tree. In Fig. 17 we show a possible method to achieve this goal

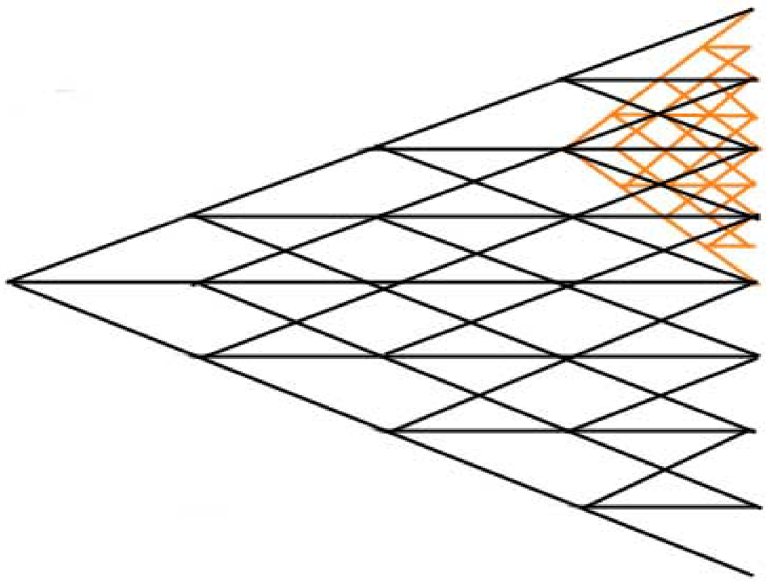

Figure 17: One of several possible implementations of trinomial mesh refinement

The mixed mesh method works by *mixing* the actual mesh with a coarser one (generally doubling the time step, so it is necessary to work with 'powers of two' grids) averaging the values in the common (grounded) nodes. The calculation follows as usual (without altering the complexity order) and from the *residues* [5] , calculated from the mixed mesh result, it is possible to estimate the a *posteriori* error. The criteria for refinement are:

(1) local norm of residues greater than adaptivity threshold
(2) critical zone of the tree defined by the interaction of the derivative with the other components of the portfolio, the structure of interest rates or other reasons
(3) critical zone of the tree associated to sampling of values in the discrete sampling version of these Exotics.

6 CONCLUSIONS

In this article we have presented an algorithm based on trinomial methods that is linear in the number of steps of discretization, and can be made linear in the whole process, and so it is cheaper in node evaluation and node optimization than usual binomial and trinomial algorithms. It also has several advantages in front of more popular Montecarlo methods: it is complete, it is faster, and it gives a full coverage of the discretization tree.

The vectorized version of the algorithm is able to be implemented on clusters of parallel computers. Research in this direction is under way in association with colleagues of CIMEC, INTEC, Santa Fe.

The proposed method is applicable to a wide class of Exotic derivatives of the path dependent type (Lookbacks, Alpha-quantile options, Asian Lookbacks, Hindsight options, Lookback spread options, Mocatta options, Partial time Lookbacks, Quanto Lookback, Bermudan swaptions, Callable power reverse dual notes, Asians, etc) and allows discrete sampling (sampling at a subset of the times of evaluation) that is very popular because the discrete sampled derivatives are cheaper than the full sampling ones, but the loss in benefits expectations is similar.

7 REFERENCES

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[5] Bergallo, M.B., Neuman, C.E., y Sonzogni, V.E., "Composite mesh concept based FEM error estimation and solution improvement", *Computer Methods in Applied Mechanics and Engineering*, **188,** 755--774 (2000).

8 NOTES

(1) More info (and details of the algorithms): ceneuman@fiqus.unl.edu.ar
(2) Supported by UNL as well as their presentation at ENIEF 2004, we acknowledge the grant 12-H168
(3) The authors gratefully acknowledge the kind and inspiring comments of the reviewers.
(4) This work will be published in an expanded form (2004).