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BUCKLING OF WOOD COLUMNS WITH UNCERTAIN PROPERTIES

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Abstract. In this study, the problem of determining the linear buckling load of wood columns with material and geometrical uncertainties under only axial static loads is considered. Argentinean Eucalyptus grandis, which is mainly cultivated in the mesopotamian provinces of Entre Ríos and Corrientes, is one of the most important renewable species cultivated in Argentina. The characteristic values of its mechanical properties exhibit great variability and are also dependent of knots, which are considered the most important defect affecting mechanical properties. In the present work, columns are modelled using the Bernoulli-Euler beam-column theory and are discretized by means of the finite element method. The bending stiffness field is modelled using random fields to include the variability of mechanics properties and the knot ratio influence. In this case the stochastic finite element method is used. Finally, Monte Carlo simulations are used to approximate the statistics of the critical buckling load. Some numerical results are shown and discussed.

1 INTRODUCTION

Afforestation with Eucalyptus gender is very important in the mesopotamian provinces of Entre Ríos and Corrientes of Argentina, representing a 90% of the provincial forested area. Within this genus, the main specie is the *Eucalyptus Grandis*, which in turn has its core production in the Argentine mesopotamia (INTA, 1995). Nowadays, this wood is also used in structures like glued laminated timber as a major alternative industrial method to obtain sized timber. This material also achieves, besides different shapes, durable alternative of use for different conditions of exposure to the environment. However, the characteristic values of its mechanical properties exhibit great variability and are also dependent of the knots, which are considered the most important defect affecting mechanical properties.

Modelling uncertainties as random variables or random process suggests the use of stochastic methods. The uncertainties are due to physical imperfections, model inaccuracies and system complexities and are included in the model. On the other hand, deterministic methods of analysis are performed with characteristic values and then, the uncertainties are taken into account by means of parcial safety factors. In the present work, continuous and finite element methods are used to determine the buckling load of pinned-pinned columns with material and geometric uncertainties considering deterministic and stochastic models for the bending stiffness of columns.

In the deterministic approach, the well-known Bernoulli-Euler beam-column model is used to formulate a variational problem for the calculation of the deterministic buckling load. When the bending stiffness field is assumed to be deterministic, the ordinary finite element method slightly overestimates the buckling load, and with a very few number of elements a high rate of convergence to the exact results is observed.

In the case of stochastic approach, the bending stiffness field can be modelled using random fields. Here, the stochastic finite element method is used. Then, the buckling load becomes a random variable. To the best of the authors' knowledge , there is no exact closed-form solution available for the random buckling load even for this simple system. The discretization is performed using weighted integrals that lead to a stiffness matrix with random elements. A stochastic field with a exponential correlation function is used to model the stiffness bending field. The probability distribution function of the random buckling load can be approximated via Monte Carlo simulations (Rubinstein, 2007).

The lengthwise variation of the modulus of elasticity has been studied since the mid-sixties. The variability of the bending stiffness was modelled by (Czmoch, 1991, 1998) as a stationary random process. He used two models, one with a random bending stiffness variation around a global mean for the whole population of beams. In the other model a variable mean was used for every beam and a stationary random process expressed the local fluctuation within a beam.

The sensitivity of the lower order moment of the buckling load with respect to the mesh size, the correlation length and coefficient of variations of the random field are examined. The reliability of columns designed considering safety factors are estimated by means of extensive Monte Carlo simulations. For structural design, the lower bound is of crucial interest.

2 DETERMINISTIC MODEL

This section describes the features of the model employed in the analysis. It should be noticed that this is a first approach, therefore the model is the simplest possible continuous model within the Strength of Materials.

The classic results for the buckling problem are associated with four basic sets of beam

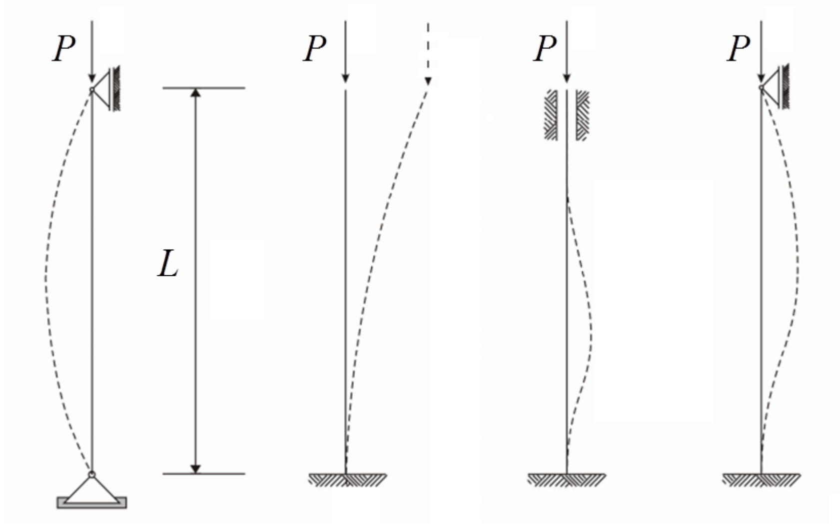

Figure 1: Buckling Problem. Euler cases

boundary conditions, here only the pinned-pinned case will be reported. These are: pinned-pinned, clamped-free, clamped-clamped and clamped-pinned; with the addition of a compressive axial load as shown in Fig. 1. For all cases, according to the Bernoulli-Euler beam theory, the deflection field $v(x)$ and the buckling load p_{cr} of a column of length L are related as follows:

$$\frac{d^2}{dx^2}\left((ei)(x)\frac{d^2v(x)}{dx^2}\right) - \frac{d^2}{dx^2}\left(p_{cr}v(x)\right) = 0 \qquad , \qquad x \in [0, L] \tag{1}$$

where $(ei)(x)$ is the deterministic bending stiffness field. Exact solutions for p_{cr} can be calculated for simple systems such as single columns, with constants material properties and geometry, and certain boundary conditions, while, in general, approximate solutions are needed for more complex systems such as frames.

Passing to the variational formulation, a set of admissible functions $\mathcal{V}$ is prescribed and Eq.(1) can be written as:

$$\begin{aligned}
&\int_0^L \left((ei)(x)\frac{d^2v(x)}{dx^2}\right)\left(\frac{d^2\phi(x)}{dx^2}\right)dx - p_{cr}\int_0^L \frac{dv(x)}{dx}\frac{d\phi(x)}{dx}dx = \\
&\quad \left.\left((ei)(x)\frac{d^2v(x)}{dx^2}\right)\left(\frac{d^2\phi(x)}{dx^2}\right)\right|_0^L + \left.\left((ei)(x)\frac{d^2v(x)}{dx^2}\right)\left(\frac{d\phi(x)}{dx}\right)\right|_0^L - p_{cr}\left.\frac{dv(x)}{dx}\phi(x)\right|_0^L \\
&\hfill \forall\phi(x) \in \mathcal{V}
\end{aligned} \tag{2}$$

For the pinned-pinned problem,

$$\mathcal{V} = \left\{\phi : [0, L] \to \mathbb{R}, \phi \text{ is piecewise } \mathcal{C}^2 \text{ and bounded, } \phi(0) = 0, \phi(L) = 0\right\} \tag{3}$$

and the second member is zero giving an eigenvalue problem. It is interesting to remark that the clamped-clamped, or pinned-clamped cases, give the same equation though the prescription of $\mathcal{V}$ is different.

The variational form of the pinned-pinned problem can be written as follows,

$$k(v, \phi) = p_{cr}k^G(v, \phi) \qquad \forall\phi \in \mathcal{V} \tag{4}$$

where $k(v,\phi)$ and $k^G(v,\phi)$ are the stiffness and geometrical stiffness operators respectively, defined as follows,

$$k(v,\phi) = \int_0^L \left((ei)(x)\frac{d^2v(x)}{dx^2}\right)\left(\frac{d^2\phi(x)}{dx^2}\right)dx, \tag{5}$$

and

$$k^G(v,\phi) = \int_0^L \frac{dv(x)}{dx}\frac{d\phi(x)}{dx}dx. \tag{6}$$

Equation (4) defines a continuous generalized eigenvalue problem. To approximate numerically, we discretize Eq.(4) using the Galerkin Method. We define a N-dimensional subspace $\mathcal{V}^N \subset \mathcal{V}$, where a function $v^N \in \mathcal{V}^N$ can be written in a unique way as a linear combination of the basis functions φ_i:

$$v^N(x) = \sum_{i=1}^{N} \eta_i \varphi_i(x), \qquad x \in [0,L], \tag{7}$$

Thus, the discrete finite-dimensional variational problem of a buckling column can be now formulated as follows: Find $v^N \in \mathcal{V}^N$ such that

$$k(v^N,\phi) = \hat{p}_{cr} k^G(v^N,\phi) \qquad \forall \phi \in \mathcal{V}^N \tag{8}$$

Applying the standard finite element methodology to the variational form (Eq.(8)), see for example (Bathe, 1996)), the displacement field of an element $v(x)$ is approximated as a linear combination of the nodal deformations $\mathbf{v}$ with signs indicated in Fig. 2, multiplied by deterministic cubic interpolation functions $\mathbf{n}(x)$ which can be compactly written as:

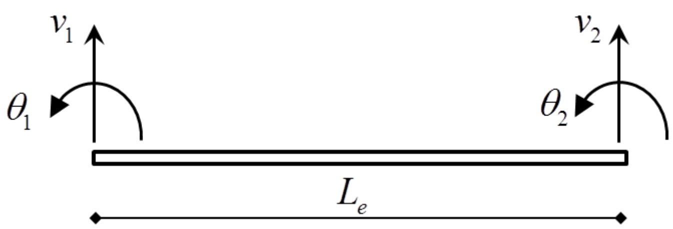

Figure 2: Bernoulli-Euller beam-column finite element with 4 degrees of freedom.

$$v(x) = \mathbf{n}^T(x)\,\mathbf{v} \tag{9}$$

where

$$\mathbf{v}^T = \begin{bmatrix} v_1 & \theta_1 & v_2 & \theta_2 \end{bmatrix} \tag{10}$$

as shown in Fig. 2, and,

$$\mathbf{n}(x) = \begin{bmatrix} 1 & 0 & -\frac{3}{L_e^2} & \frac{2}{L_e^2} \\ 0 & 1 & -\frac{2}{L_e} & \frac{1}{L_e^2} \\ 0 & 0 & \frac{3}{L_e^2} & -\frac{2}{L_e^3} \\ 0 & 0 & -\frac{1}{L_e} & \frac{1}{L_e^2} \end{bmatrix} \begin{bmatrix} 1 \\ x \\ x^2 \\ x^3 \end{bmatrix} \tag{11}$$

where L_e is the element length.

Then, one can end up with the geometrical stiffness matrix $\mathbf{k_e^G}$ and the stiffness matrix $\mathbf{k_e}$ of the e^{th} beam element are derived as

$$k^G_{e,ij} = \int_0^{L_e} \frac{dn_i(x)}{dx}\frac{dn_j(x)}{dx}dx \tag{12}$$

$$k_{e,ij} = \int_0^{L_e} (ei)(x)\,\frac{d^2n_i(x)}{dx^2}\frac{d^2n_j(x)}{dx^2}dx \tag{13}$$

Integration in Eq.(12) can be evaluated using the listed shape functions $\mathbf{n}(x)$.

$$\mathbf{k_e}^G = \frac{1}{30}\begin{bmatrix} \frac{36}{L_e} & 3 & -\frac{36}{L_e} & 3 \\ 3 & 4L_e & -3 & -L_e \\ \frac{-36}{L_e} & -3 & \frac{36}{L_e} & -3 \\ 3 & -L_e & -3 & 4L_e \end{bmatrix} \tag{14}$$

For the case when the bending stiffness is constant, i.e. $(ei)(x) = \overline{ei}$, the element stiffness matrix becomes

$$\mathbf{k_e} = \frac{\overline{ei}}{L_e^3}\begin{bmatrix} 12 & 6L_e & -12 & 6L_e \\ 6L_e & 4L_e^2 & -6L_e & 2L_e^2 \\ -12 & -6L_e & 12 & -6L_e \\ 6L_e & 2L_e^2 & -6L_e & 4L_e^2 \end{bmatrix} \tag{15}$$

Next, the global matrices can be obtained from the finite element assembling and the buckling load is calculated from the equation

$$\mathbf{k} - \hat{p}_{cr}\mathbf{k}^G = \mathbf{0} \tag{16}$$

where $\mathbf{k}$ is the $n \times n$ positive-definite global stiffness matrix and $\mathbf{k}^G$ is the global geometrical stiffness matrix of elements in compression.

3 STOCHASTIC MODEL

We define the random field $\{(EI)(x) : x \in [0; L]\}$ as a collection of real-valued random variables from a probability space $(\Omega, \mathcal{F}, P)$, where Ω is the sample space, $\mathcal{F}$ is the σ-algebra and P is the probability measure.

In what follows, the random (stochastic) quantities are denoted by capital letters. The random deflection field $V(x)$ and the random buckling load P_{cr} of a column of length L are related as follows:

$$\int_0^L \left((EI)(x)\frac{d^2V(x)}{dx^2}\right)\left(\frac{d^2\phi(x)}{dx^2}\right)dx - P_{cr}\int_0^L \frac{dV(x)}{dx}\frac{d\phi(x)}{dx}dx = 0, \forall\phi(x) \in \mathcal{V} \tag{17}$$

It is assumed that EI is a truncated Gaussian random field on $[0, L]$ with exponential autocorrelation function

$$R(x_1, x_2) = \sigma^2 \exp\left(-\frac{|x_2 - x_1|}{d}\right), \tag{18}$$

where d is the correlation length, which measures the decay of the autocorrelation function.

We propose to use an exponential autocorrelation function and truncated Gaussian field that is expanded with Karhunen-Loève (KL) expansion, developed by (Karhunen, 1946; Loève, 1946), using standard Gaussian random variables.

The stochastic field (EI) is then expanded using the Karhunen-Loève expansion (see Sec. (4).

Since a closed-form solution for the random buckling load is not known, in what follows a stochastic finite element formulation(SFEM) is used to determine the probability density function of P_{cr} approximately. In an element, the random displacement field V is approximated as a linear combination of the nodal deformations $\mathbf{V}$ multiplied by deterministic cubic interpolation functions $\mathbf{n}(x)$.

$$V(x) = \mathbf{n}^T(x)\mathbf{V} \tag{19}$$

where

$$\mathbf{V}^T = \begin{bmatrix} V_1 & \Theta_1 & V_2 & \Theta_2 \end{bmatrix} \tag{20}$$

and $\mathbf{n}(x)$ is given in Eq. (11). After the standard finite element procedure is applied, the same geometrical stiffness matrix $\mathbf{k_e}^G$ given in Eqs. (12) and (14) is obtained. The stochastic stiffness matrix K_e of the beam element becomes

$$K_{e,ij} = \int_0^{L_e} (EI)(x) \frac{d^2 n_i(x)}{dx^2} \frac{d^2 n_j(x)}{dx^2} dx \tag{21}$$

Since $(EI)(x)$ is a random field, $K_{e,ij}$ is a random variable. The stochastic global stiffness matrix can be obtained after the usual assembling operation of the finite element method. The assembling procedure involves deterministic coordinate transformations and additions.

Then, the buckling load, which is a random variable, is calculated from equation

$$\mathbf{K} - \hat{P}_{cr}\mathbf{k}^G = \mathbf{0} \tag{22}$$

where $\mathbf{K}$ is the $n \times n$ stochastic global stiffness matrix.

The probability distribution function of the random buckling load can be determined from Eq. (22) using several methods, among others, Monte Carlo simulations.

4 KARHUNEN-LOÈVE EXPANSION

The KL decomposition of a random process can be regarded as the continuous counterpart of the decorrelation of a set of random variables. It allows to approximate a random process by a linear combination of orthonormal deterministic functions (KL modes) with coefficients that are uncorrelated random variables.

Let the buckling problem be one in which the system characteristics modelled as a scalar random process $S(x,\xi) : [0,L] \times \Omega \to \mathbb{R}$. This process is defined on the probability space $(\Omega, \mathcal{F}, P)$. The process is characterized by its PDF $p_S(s) : \mathbb{R} \to \mathbb{R}^+$ and its covariance function $C_S(x_1, x_2) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$.

In order to assemble the SFEM equations for this problem, the random process $S(x,\xi)$ has to be expressed as a deterministic function of a small number of random variables. This discretization is achieved by means of the Karhunen-Loève (KL) decomposition (Ghanem and Spanos, 1991),

The non-zero mean random process $S(x,\xi)$ is decomposed as follows:

$$S(x,\xi) = m_S(x) + Y(x,\xi) \tag{23}$$

where $m_S(x) = \mathbb{E}\{S(x,\xi)\}$ is the mean value of the random process $S(x,\xi)$ and $Y(x,\xi)$ is a zero mean random process. Both the correlation function $R_Y(x_1,x_2)$ and the covariance function $C_Y(x_1,x_2)$ of the zero mean random process $Y(x,\xi)$ are equal to the covariance function

$C_S(x_1, x_2)$ of the non-zero mean random process $S(x, \xi)$. All three are denoted by $C_S(x_1, x_2)$ in the following.

Let Ξ be the Hilbert space of random variables $Z(\xi) : \Omega \to \mathbb{R}$ defined on the probability space $(\Omega, \mathcal{F}, P)$, with the inner product $\langle Z_1(\xi), Z_2(\xi)\rangle_\Xi = \mathbb{E}\{Z_1(\xi)Z_2(\xi)\}$. Let $\{\eta_j(\xi)\}_j$ be a Hilbert basis of Ξ. The KL decomposition of the zero mean random process $Y(x, \xi)$ consists of the projection of the process on the Hilbert basis $\{\eta(\xi)\}_j$. This leads to the following expansion:

$$S(x, \xi) = m_S(x) + \sum_{j=1}^{\infty} \sqrt{\lambda_j}\phi_j^*(x)\eta_j(\xi) \tag{24}$$

where $\{\phi_j^*(x)\}_j$ and $\{\lambda_j\}_j$ are the normalized eigenfunctions and the eigenvalues of the covariance function $C_s(x_1, x_2)$, respectively. The discretization of the random process $S(x, \xi)$ is accomplished by a truncation of the infinite series in Eq. (24) after the terms corresponding to the highest M eigenvalues.

$$S(x, \xi) \approx m_S(x) + \sum_{j=1}^{M} \sqrt{\lambda_j}\phi_j^*(x)\eta_j(\xi) \tag{25}$$

M is called the order of the KL decomposition. As the terms in the decomposition are not correlated (the variables $\{\eta_j(\xi)\}_j$ are orthonormal random variables), the KL decomposition is the most efficient decomposition of a random process: for a given number of terms the truncation error is minimized. Ghanem and Spanos (1991) present a proof of this error minimizing property of the KL decomposition.

5 NUMERICAL RESULTS

In what follows, some numerical results are presented. In all the cases, the cross sectional area (A) is rectangular with dimensions $b = 0.9$ m and $h = 0.6$ m. The plane moment of inertia is $I = bh^3/12$ and the slenderness ratio of the column is $\lambda = L/\sqrt{I/A}$. Even if the convergence of the Finite Element Method is good with a few number of elements, we used 20 elements in all simulations. We are going to focus on the stochastic results. The results are computed using 5000 independent Monte Carlo realizations. The weighted integrals (Eq. (21)), that is, the random stiffness matrix, were solve by means of the Gauss quadrature with nine points. To generate the correlated random variables, an eigenvalue decomposition of the covariance matrix of the random variable is used. Then, the discrete representation of the random field, are obtained from the discrete coordinates coincident with the Gauss points, e.g, nine points per element. Another choice is to represent the random field using the the same interpolation functions used to approximate the deflection $v(x)$ (Ghanem and Spanos, 1991).

In Fig. 3 some realizations of the random field are shown. Fast convergence is observed for the mean of P_{cr} ($E[P_{cr}]$) and the standard deviation ($\sigma_{P_{cr}}$) of P_{cr} using few elements. Figures 4 and 5 show typical convergence curves for the buckling load and the standard deviation P_{cr} respectively for a pinned-pinned column with slenderness ratio $\lambda = 110$ and correlation length $d = 0.1$, where n is the number of Monte Carlos simulations. As can be seen in the picture an acceptable convergence is achieved when $n_s = 2000$. The probability density function of critic load P_{cr} is plotted for the different cases; see Fig. 6.

As can be seen from Fig. 7, $E[P_{cr}]$ is smaller than the buckling load of the deterministic column in all cases. The mean value of the modulus of elasticity was assumed in this case as $E = 153666.5\ 10^5$ Pa. which was obtained as the mean of the results of experimental tests

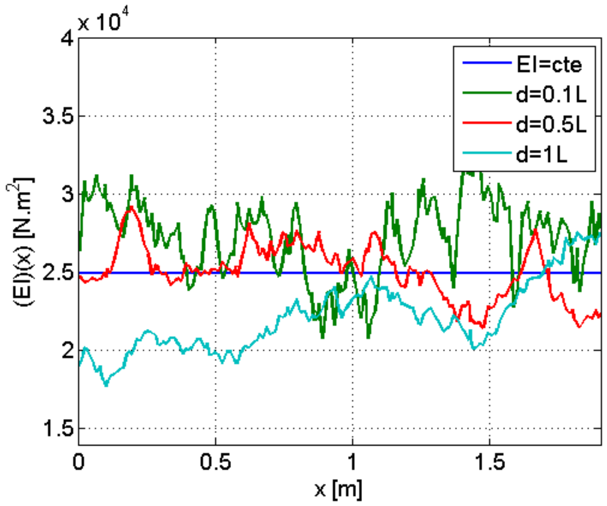

Figure 3: Realizations of the random field EI for different values of the correlation length d, $\lambda = 110$.

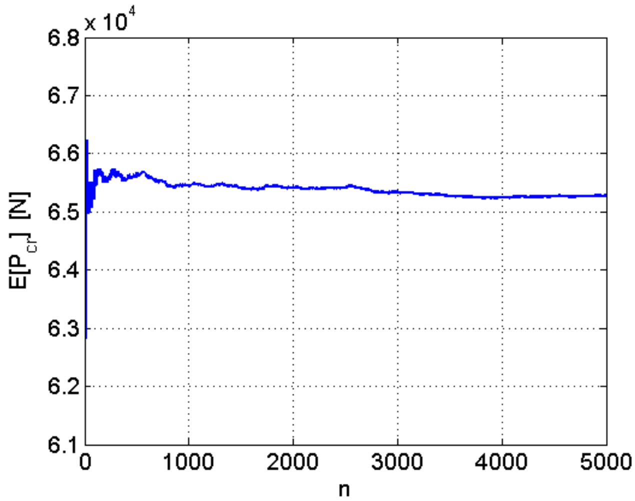

Figure 4: Mean buckling load ($E[P_{cr}]$) convergence for $\lambda = 110$.

performed with 120 Eucalyptus grandis samples. It should be noted that when $d \to \infty$, the random field becomes fully correlated and can be interpreted as a random variable in the limit.

6 CONCLUSIONS

The buckling load of columns with material and geometrical uncertainties are determined considering deterministic and stochastic models for the bending stiffness of columns. The mean of the buckling load is found to be slightly less than the buckling load of the deterministic column. The stochastic analysis allows to obtain more information on the system behaviour. In particular, in this case, the critical loads found for the stochastic problem, are always smaller than the deterministic value. Hence, assuming only one value (deterministic) of the bending stiffness would be on the unsafe side.

At the moment of the present study the authors are working in more complex models that include columns of glued laminated timber, other constitutive relation-ships and uncertainties properties models taken from experimental data.

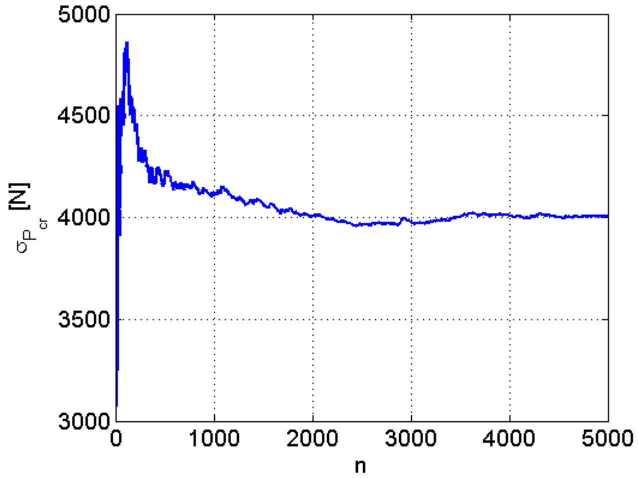

Figure 5: Deviation standard ($\sigma_{P_{cr}}$) convergence for $\lambda = 110$.

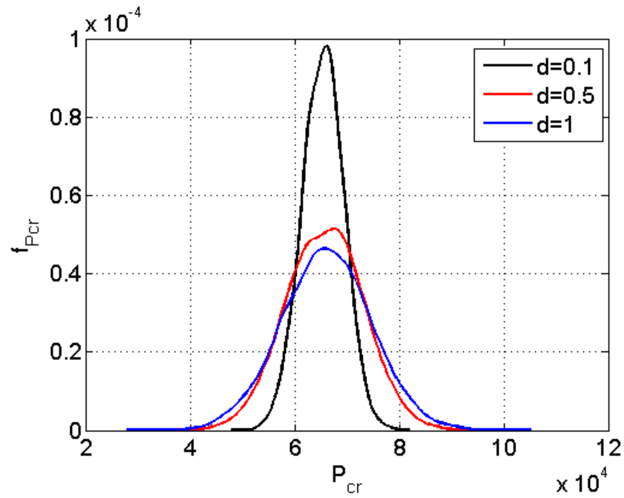

Figure 6: PDF of P_{cr} for different values of the correlation length d.

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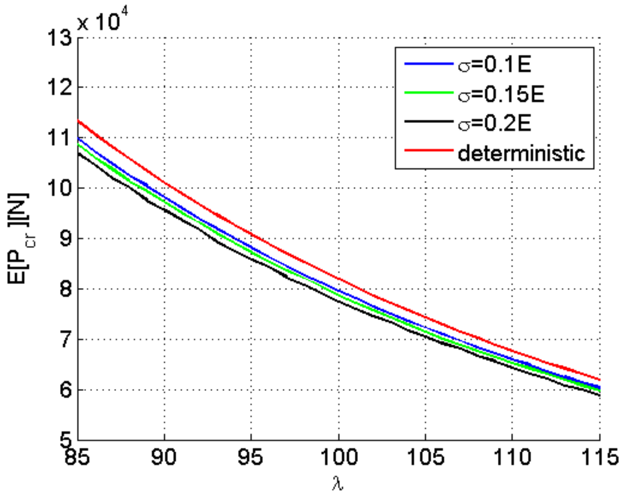

Figure 7: λ vs $E[P_{cr}]$ for different values of standard deviation.

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