Asociación Argentina
de Mecánica Computacional

Mecánica Computacional Vol XXXI, págs. 2783-2800 (artículo completo)
Alberto Cardona, Paul H. Kohan, Ricardo D. Quinteros, Mario A. Storti (Eds.)
Salta, Argentina, 13-16 Noviembre 2012

STOCHASTIC DYNAMICS OF SLENDER STRUCTURES OF COMPOSITE MATERIALS

Marcelo T. Piovan[a,c], **José M. Ramírez**[a] **and Rubens Sampaio**[b]

[a]*Centro de Investigaciones de Mecánica Teórica y Aplicada, Universidad Tecnológica Nacional FRBB. 11 de Abril 461, B8000LMI, Bahía Blanca. Argentina, mpiovan@frbb.utn.edu.ar, http://www.frbb.utn.edu.ar*

[b]*Departament of Mechanical Engineering. PUC-Rio. Rua Marquês de São Vicente 225, Rio de Janeiro RJ-22453-90, Rio de Janeiro, Brazil, rsampaio@puc-rio.br, http://www.mec.puc-rio.br*

[c]*CONICET*

Keywords: Uncertainty quantification, parametric probabilistic approach, composite materials, shear deformable beams.

Abstract. This article is concerned with the dynamic analysis of structures constructed with composite materials. There are many ways to manufacture a composite material for uses in structural constructions, for example filament winding and resin transfer molding, among others. Depending on the manufacturing process composite materials may have deviations with respect to the calculated response (or deterministic response). These manufacturing aspects lead to a source of uncertainty in the structural response associated with constituent proportions or geometric parameters. Another source of uncertainty can be the mathematical model that represents the mechanics of the slender structure. In many structural models, the type of hypotheses invoked can reflect the most of the physics of a structure, however in some circumstances these hypotheses are not enough, and cannot represent properly the mechanics of the structure. Uncertainties should be considered in a structural system in order to improve the predictability of a given modeling scheme. There are some strategies to face the uncertainties in the dynamics of structures. The parametric probabilistic approach quantifies the uncertainty of given parameters such as variation of the angles of fiber reinforcement, material constituents, etc. In this study a shear deformable model of composite beams is employed as the mean model. The probabilistic model is constructed by adopting random variables for the uncertain parameters of the model. The probability density functions of the random variables are constructed appealing to the Maximum Entropy Principle. Then the probabilistic model is possed in the context of the finite element method and the Monte Carlo method is employed to perform the statistical simulations. Numerical studies are carried out to show the main advantages of the modeling strategies employed, as well as to quantify the propagation of the uncertainty in the dynamics of slender composite structures.

1 INTRODUCTION

Composite materials have many advantages with respect to isotropic materials that motivate their use in structural components. The most well known properties of composite materials are high strength and stiffness properties together with a low weight, good corrosion resistance, enhanced fatigue life, low thermal expansion properties among others (Barbero, 1999). Other important feature of composite materials is the very low machining cost for complex structures (Jones, 1999). As a result of the increasing use of composite thin-walled beams, the analysis of static and dynamic behavior is a task of intense research. Since the eighties many research activities are been devoted toward the development of theoretical and computational methods for the appropriate analysis of such members. The first consistent study dealing with the static structural behavior of thin-walled composite-orthotropic members, under various loading patterns, was due to Bauld and Tzeng (1984), who developed in the early eighties, invoking Vlasov hypotheses, a beam theory to analyze fiber-reinforced members featuring open cross-sections with symmetric laminates. Although this theory assumed the cross-sections to be shear undeformable and was restricted to members formed by non-general stacking sequences and employed only for static analysis. Further contributions from many authors (Kapania and Raciti, 1989) until the present time, made possible the extension of Vlasov models by considering shear deformability due to bending and warping effects, among others and the resulting models were employed in many problems.

Composite Thin-walled beam-models allowing for some effects of shear deformability were presented, in the middle eighties with the work of Bauchau (1985). In this article, the effect of full shear deformability and specially the warping torsion shear deformability was not taken into account or was slightly studied in a few problems of static's and dynamics. The late eighties and the nineties brought a considerable amount of new models and uses. Rehfield et al. (1990) studied the non-conventional effects of constitutive elastic couplings (such as bending-bending coupling or bending-shear coupling, etc.) in the mechanics of cantilever box-beams. In the models developed by Librescu and Song (1992) and Song and Librescu (1993), which were employed in a broad field of engineering problems, it was considered the bending component of shear flexibility but the shear deformation due to warping torsion component was neglected. However in these models new extensions were performed, such as the accounting for the effect of thickness in shear and warping deformations. Special attention, deserve the works of Cesnik et al. (1996) who performed studies on thin-walled composite beams by means of the so-called Variational Asymptotical Cross Sectional Analysis (VABSA) Method. In these works there is no mention to buckling problems and vibrations with states of arbitrary initial stresses. Employing the Hellinger-Reissner principle, Cortínez and Piovan (2002) introduced a theory of thin-walled beams with symmetric balanced laminates, which considers full shear flexibility, i.e. bending shear deformation and torsion-warping shear deformation. This model covers topics of dynamics under states of initial normal stresses, and also accounts for thickness shear flexibility and warping. Many of the aforementioned models were employed only for static response or for eigenvalue calculation. Piovan and Cortínez (2007a) and Piovan and Cortínez (2007b) extended the previous model by incorporating general laminates, buckling analysis and other complexities such as beams with curved axis among others.

The uncertainty is an important concern in the behavior of beams constructed with fiber reinforced composite materials due to their inherent variability. The first studies about the quantification of uncertainty in composite materials are related to the constituent level, fibre/matrix or ply level according to the deep review of the state of the art carried out by Sriramula and

Chryssantopoulos (2009). The main sources of uncertainty in composite materials are the value of constituent properties at microscopic level, geometric aspects at mesoscale or macroscopic level and the manufacturing process. The manufacturing processes of composite structures may affect strongly the dynamic response. There is a huge amount of research related to quantify the propagation of uncertainty in the mechanics of composite materials at the microscale level (Sriramula and Chryssantopoulos, 2009) or in the case of failure analysis (Pawar, 2011).

There are different approaches to evaluate the dynamic response of structures subjected to several aspects of uncertainty. The most common is the consideration of the uncertainty in the loads or external excitations as a random processes (Lutes and Sarkani, 1997). The uncertainty involved in the material properties of the composites can be considered as a random fields according to the works of Onkar and Yadav (2005), Mehrez et al. (2012b) and Mehrez et al. (2012a) among others. Other way to study the dynamic response due to uncertainties in the composite material is associating random variables to given parameters that are considered uncertain, which is called parametric probabilistic approach. The construction or derivation of the probability density functions of the random variables is a crucial task that needs some information about the statistics of the parameters (e.g. expected value or bounds and/or coefficient of variation, etc.). The Maximum Entropy principle is employed to deduce the probability density functions in order to guarantee that the mentioned functions have the maximum uncertainty. This is the approach to be employed in the present article. The angles of the laminates are assumed to vary around the expected values, thus a set of random variables is defined with the corresponding probability density function. The deterministic and probabilistic approaches of the structural model are proposed within the frame of the finite element method. The Monte Carlo Method is used to obtain the statistics of the dynamic response associated to a number of independent simulations.

The article is organized as follows: after the introductory section where the state-of-the-art is summarized, the deterministic model and its finite element discretization are briefly described, then the probabilistic approach is constructed. The subsequent section contains the computational studies, the analysis of the uncertainty propagation in the dynamics of composite thin walled beams and finally concluding remarks are outlined.

2 BRIEF PRESENTATION OF THE COMPOSITE BEAM MODEL

In Figure 1 a basic sketch of the thin walled beam is shown, where it is possible to see the reference points $\boldsymbol{C}$ and $\boldsymbol{A}$. The principal reference point $\boldsymbol{C}$ is located at the geometric center of the cross-section, where the x-axis is parallel to the longitudinal axis of the beam while y and z are the axes of the cross section, but not necessarily the principal axes of inertias. The secondary reference system, located at $\boldsymbol{A}$, is used to feature shell stresses and strains. The thin-walled beam model is based in the following assumptions (Cortínez and Piovan, 2002; Piovan and Cortínez, 2007a):

1) The cross-section contour is rigid in its own plane.

2) The radius of curvature at any point of the shell is neglected. This implies to consider the section shaped in a polygonal arrangement.

3) The warping function is normalized with respect to the principal reference point $\boldsymbol{C}$.

4) A general laminate stacking sequence for composite material is considered.

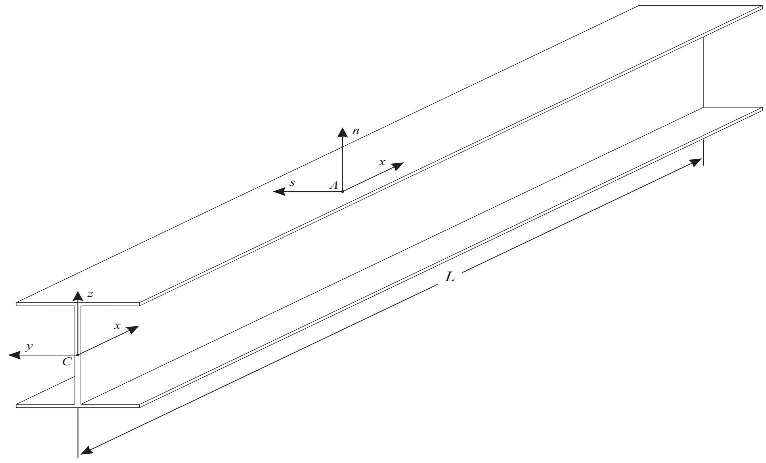

Figure 1: Thin walled beam with reference systems

5) The material density is considered constant along the beam but it can vary in the laminate thickness.

6) Stress and strain components are references from point $\boldsymbol{A}$, and the representative stresses are σ_{xx}, σ_{xs} and σ_{xn}.

7) The model is presented in the context of linear elasticity.

2.1 Governing equations and boundary conditions

Following assumptions 1) to 7) it is possible to derive the displacement field of a point $\boldsymbol{P}$ (Piovan, 2003; Piovan and Cortínez, 2007a), which can be presented in the subsequent form:

$$\widetilde{\mathbf{U}}_{\boldsymbol{P}} = \left\{ \begin{array}{c} u_x \\ u_y \\ u_z \end{array} \right\} = \left\{ \begin{array}{c} u_{xc} - \omega\theta_x \\ u_{yc} \\ u_{zc} \end{array} \right\} + \left[\begin{array}{ccc} 0 & -\theta_z & \theta_y \\ \theta_z & 0 & -\phi_x \\ -\theta_y & \phi_x & 0 \end{array} \right] \left\{ \begin{array}{c} 0 \\ y \\ z \end{array} \right\}, \tag{1}$$

Where u_{xc}, u_{yc}, u_{zc} are the displacements of the reference center in x-, y-, and z- directions, respectively. θ_z and θ_y are bending rotational parameters. ϕ_x is the twisting angle and θ_x is a warping-intensity parameter. In Eq. (1) the cross-sectional variables $y(s)$ and $z(s)$ of a generic point are related to the ones of the wall middle line $Y(s)$ and $Z(s)$ by means of Eq. (2) is the warping function normalized with respect to the reference center. It is defined in Eq. (3)

$$y(s) = Y(s) - n\frac{dZ}{ds},\; z(s) = Z(s) + n\frac{dY}{ds}, \tag{2}$$

$$\omega(s,n) = \omega_p(s) + \omega_s(s,n). \tag{3}$$

In Eq. (3), $\omega_p(s)$ is the primary or contour warping function whereas $\omega_s(s,n)$ is the secondary or thickness warping. These entities are given by:

$$\omega_p(s) = \int_s [r(s) + \psi(s)]\, ds - D_C,\; \omega_s(s,n) = -nl(s), \tag{4}$$

where the functions $r(s)$, $l(s)$, $\psi(s)$ and D_C are defined in the following form:

$$\begin{array}{l} r(s) = Z(s)\dfrac{dY}{ds} - Y(s)\dfrac{dZ}{ds}, l(s) = Y(s)\dfrac{dY}{ds} + Z(s)\dfrac{dZ}{ds}, \\ \psi(s) = \dfrac{1}{\bar{A}_{66}(s)}\left[\dfrac{\int_s r(s)ds}{\oint_S \frac{1}{\bar{A}_{66}(s)}ds}\right], D_C = \dfrac{\oint_S [r(s)+\psi(s)]\,\bar{A}_{11}(s)ds}{\oint_S \bar{A}_{11}(s)ds}. \end{array} \tag{5}$$

The functions $\bar{A}_{11}$ and $\bar{A}_{66}$ are normal and tangential elastic properties of composite laminates (Piovan and Cortínez, 2007a) which can vary along the section middle line. The function $\psi(s)$ is connected with the torsional shear flow and D_C is a constant for normalization of the warping function with respect to the reference system ***C*** (Piovan, 2003; Cortínez and Piovan, 2002). In the case of open sections, the function $\psi(s) = 0$, consequently Eq. (5) holds for both closed and open sections. The warping function described in Eq. (3), has an analogous form to the ones defined by Song and Librescu (1993) or Na and Librescu (2001) for closed sections.

The displacement-strain relations can be obtained by substituting Eq. (1) in the well-known expressions of linear strain components. As it was shown by Piovan (2003) and Piovan and Cortínez (2007a) the shell strains can be written as:

$$\widetilde{\mathbf{E}}_P = \mathbf{G}_k\widetilde{\mathbf{D}}, \tag{6}$$

where:

$$\begin{array}{ll} \widetilde{\mathbf{E}}_P^T &= \{\epsilon_{xx}, \gamma_{xs}, \gamma_{xn}, \kappa_{xx}, \kappa_{xs}\}, \\ \widetilde{\mathbf{D}}^T &= \{\varepsilon_{D1}, \varepsilon_{D2}, \varepsilon_{D3}, \varepsilon_{D4}, \varepsilon_{D5}, \varepsilon_{D6}, \varepsilon_{D7}, \varepsilon_{D8}\}, \end{array} \tag{7}$$

$$\mathbf{G}_k = \begin{bmatrix} 1 & Z & -Y & -\omega_p & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & {}^{dY}\!/_{ds} & {}^{dZ}\!/_{ds} & r(s)+\psi(s) & -\psi(s) \\ 0 & 0 & 0 & 0 & -{}^{dZ}\!/_{ds} & {}^{dY}\!/_{ds} & l(s) & 0 \\ 0 & -{}^{dY}\!/_{ds} & {}^{dZ}\!/_{ds} & -l(s) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 \end{bmatrix}. \tag{8}$$

In Eq. (7), ϵ_{xx}, γ_{xs} and γ_{xn} are the strain components and κ_{xx}, κ_{xs} are the curvatures of the shell that conforms the wall of the cross-section. These strain components are measured according to the wall reference system in ***A***. The entities ε_{Di}, $i = 1,...,8$ may be regarded as generalized deformations. In this context ε_{D1} is the axial deformation, ε_{D2} and ε_{D3} are bending deformations, ε_{D3} is the deformation due to non-uniform warping, ε_{D5} and ε_{D6} are the bending shear deformations, ε_{D7} is the warping shear deformation and finally ε_{D8} is the pure torsion shear deformation. These generalized deformations, which are collected in vector $\widetilde{\mathbf{D}}$, are defined in the following form:

$$\widetilde{\mathbf{D}} = \mathbf{G}_{DU}\widetilde{\mathbf{U}}, \tag{9}$$

where $\mathbf{G}_{DU}$ is a matrix operator and $\widetilde{\mathbf{U}}$ is the vector of kinematic variables which are defined in following forms, in which $\partial_x(\diamond)$ is the spatial derivative operator.

$$\mathbf{G}_{DU} = \begin{bmatrix} \partial_x(\diamond) & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \partial_x(\diamond) & 0 & 0 \\ 0 & 0 & \partial_x(\diamond) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \partial_x(\diamond) \\ 0 & \partial_x(\diamond) & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \partial_x(\diamond) & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \partial_x(\diamond) & -1 \\ 0 & 0 & 0 & 0 & 0 & \partial_x(\diamond) & 0 \end{bmatrix}, \tag{10}$$

$$\widetilde{\mathbf{U}}^T = \{u_{xc}, u_{yc}, \theta_z, u_{zc}, \theta_y, \phi_x, \theta_x\}. \tag{11}$$

The principle of virtual works can be condensed in the following form:

$$\mathcal{W}_T = \int\limits_L \left(\delta\widetilde{\mathbf{D}}^T\widetilde{\mathbf{Q}}\right) dx + \int\limits_L \delta\widetilde{\mathbf{U}}^T\mathbf{M}_m\ddot{\widetilde{\mathbf{U}}}dx - \int\limits_L \delta\widetilde{\mathbf{U}}^T\widetilde{\mathbf{P}}_X dx + \delta\widetilde{\mathbf{U}}^T\widetilde{\mathbf{B}}_X\Big|_{x=0}^{x=L} = 0, \tag{12}$$

where the force vector $\widetilde{\mathbf{Q}}$ is defined as follows:

$$\widetilde{\mathbf{Q}}^T = \{Q_x, M_y, M_z, B, Q_y, Q_z, T_w, T_{sv}\}, \tag{13}$$

whereas for the sake of fluid and clear reading, the matrix of mass coefficients $\mathbf{M}_m$, the vector $\tilde{\mathbf{P}}_X$ of external forces and the vector $\tilde{\mathbf{B}}_X$ of natural boundaries conditions are detailed in Appendix A.

In Eq. (13) the internal beam forces Q_x, M_y, M_z, and B correspond to the axial force, the bending moment in y-direction, the bending moment in z-direction, and the bi-moment, respectively; whereas the internal forces Q_y, Q_z, T_w, and T_{sv} correspond to the shear force in y-direction, the shear force in z-direction, the twisting moment due to warping and the twisting moment due to pure torsion, respectively. These internal forces can be written in terms of the shell-forces as (Piovan and Cortínez, 2007a):

$$\widetilde{\mathbf{Q}} = \int_S \mathbf{G}_k^T\widetilde{\mathbf{N}}_P ds, \tag{14}$$

where $\widetilde{\mathbf{N}}_P$ is the vector of shell stress resultants or shell forces and moments defined according to (Barbero, 1999):

$$\widetilde{\mathbf{N}}_P^T = \int_S \{\sigma_{xx}, \sigma_{xs}, \sigma_{xn}, n\sigma_{xx}, n\sigma_{xs}\} dn. \tag{15}$$

The differential equations of motion and associated boundary conditions can be derived by applying the conventional steps of variational calculus in Eq. (12). The differential equations of motion can be useful for some numerical methods, e.g. power series method. While in the present article the finite element method is employed, the derivation of differential equations is not necessary. The interested reader may follow, in the works of Piovan and Cortínez (2007a) and Piovan (2003), the form and features of the differential equations of the thin-walled beam model applied to several problems of mechanics of beams.

2.2 Constitutive equations

In order to obtain the relationship between beam stress resultants and generalized deformations ε_{Di}, one has to select the constitutive laws for a composite shell and employ constitutive hypotheses (Piovan and Cortínez, 2007a) of the shell stress resultants in terms of the shell strains. The shell stress resultants can be expressed in terms of the generalized deformations defined in Eq. (9) in the following matrix form:

$$\widetilde{\mathbf{N}}_P = \mathbf{M}_C \widetilde{\mathbf{E}}_P, \tag{16}$$

where $\mathbf{M}_C$ is the matrix of modified shell stiffness, which depends on the type of constitutive hypotheses involved (Piovan, 2003) and can be expressed in the following form:

$$\mathbf{M}_C = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{16} & 0 & \bar{B}_{11} & \bar{B}_{16} \\ & \bar{A}_{66} & 0 & \bar{B}^*_{16} & \bar{B}_{66} \\ & & \bar{A}^*_{55} & 0 & 0 \\ & sym & & \bar{D}_{11} & \bar{D}_{16} \\ & & & & \bar{D}_{66} \end{bmatrix}. \tag{17}$$

Due to the lack of space the coefficients $\bar{A}_{11}$, $\bar{B}_{11}$, $\bar{D}_{11}$, etc, are not described in the present article, however the interested reader can found them in the works of Piovan and Cortínez (2007a) or Piovan (2003).

Substituting Eq. (16) into Eq. (14) the beam stress resultants can be obtained in terms of generalized strains:

$$\widetilde{\mathbf{Q}} = \mathbf{M}_k \widetilde{\mathbf{D}}, \tag{18}$$

where:

$$\mathbf{M}_k = \int_{\mathcal{S}} \mathbf{G}_k^T \mathbf{M}_C \mathbf{G}_k ds. \tag{19}$$

The matrix $\mathbf{M}_k$ of cross-sectional stiffness coefficients, leads to constitutive elastic coupling or not, depending on the stacking sequence of the laminates in a given cross-section. That is for example, if the laminates are specially orthotropic or cross-ply, or specially symmetric balanced, there is no constitutive elastic coupling (Cortínez and Piovan, 2002), however if the laminates are general the beam could have different types of constitutive elastic couplings such as twisting-bending-extensional coupling, extensional-bending coupling bending-bending coupling, etc (Piovan and Cortínez, 2007a).

2.3 Finite element approach

In order to solve problems of dynamics with several boundary conditions, quartic order isoparametric finite elements of five nodes are employed. The vector of nodal displacements $\bar{\mathbf{U}}_e$ is arranged as:

$$\bar{\mathbf{U}}_e = \left\{ \bar{\mathbf{U}}_e^{(1)}, ..., \bar{\mathbf{U}}_e^{(5)} \right\}, \tag{20}$$

where:

$$\bar{\mathbf{U}}_e^{(j)} = \left\{ u_{xc_j}, u_{yc_j}, \theta_{z_j}, u_{zc_j}, \theta_{y_j}, \phi_{x_j}, \theta_{x_j} \right\}, \quad j = 1, ..., 5 \tag{21}$$

Now, substituting Eq. (20) into Eq. (12) and applying the conventional variational procedures (Piovan and Cortínez, 2007a), the following finite element equation is attained:

$$\mathbf{K}\bar{\mathbf{W}} + \mathbf{M}\ddot{\bar{\mathbf{W}}} = \bar{\mathbf{F}}, \tag{22}$$

where $\mathbf{K}$ and $\mathbf{M}$ are the global matrices of elastic stiffness and mass, respectively; whereas $\bar{\mathbf{W}}$, $\ddot{\bar{\mathbf{W}}}$ and $\bar{\mathbf{F}}$ are the global vectors of nodal displacements, nodal accelerations and nodal forces, respectively.

Eq. (22) can be modified in order to account for "a posteriori" structural proportional Rayleigh damping given by:

$$\mathbf{C}_{RD} = \eta_1\mathbf{M} + \eta_2\mathbf{K}. \tag{23}$$

The coefficients η_1 and η_2 in Eq. (23) can be computed employing two given damping coefficients (namely, ξ_1 and ξ_2) for the first and second modes, according to the common methodology presented in the bibliography related to finite element procedures (Bathe, 1996; Meirovitch, 1997). The matrices $\mathbf{M}$ and $\mathbf{K}$ are the global mass matrix and the global elastic stiffness matrix, respectively. This leads to:

$$\mathbf{K}\bar{\mathbf{W}} + \mathbf{C}_{RD}\dot{\bar{\mathbf{W}}} + \mathbf{M}\ddot{\bar{\mathbf{W}}} = \bar{\mathbf{F}}. \tag{24}$$

The response in the frequency domain of the linear dynamic system given by Eq. (24) can be written as (Meirovitch, 1997):

$$\widehat{\mathbf{W}}(\omega) = \left[-\omega^2\mathbf{M} + i\omega\mathbf{C}_{RD} + \mathbf{K}\right]^{-1}\widehat{\mathbf{F}}(\omega), \tag{25}$$

where $\widehat{\mathbf{W}}$ and $\widehat{\mathbf{F}}$ are the Fourier transform of the displacement vector and force vector, respectively; whereas ω is the circular frequency measured in [rad/sec].

3 STOCHASTIC MODEL

The stochastic model is constructed selecting the angles of laminates as uncertain parameters and associating random variables to them. The construction of the probability distributions of the random variables is quite sensible in stochastic analysis and they should be deduced according to the known information about the uncertain parameters. The Maximum Entropy Principle (Jaynes, 2003) is then employed to derive the probability distributions in order to guarantee consistence with the known information of the random variables and the physics of the problem.

In the present problem random variables V_i, $i = 1, 2, 3...N_P$ are introduced such that they represent the angles of the N_P plies in a given cross-sectional laminate. The expected value of the random variables is known having, as a first approach, the nominal value of the deterministic model, i.e.: $\mathcal{E}\{V_i\} = \underline{V}_i$, $i = 1, 2, 3...N_P$; moreover the random variables have bounded supports whose upper and lower limits are distant Δ from the expected value $\underline{V}_i$. Also, the construction of the laminates should maintain the condition of symmetry in the corresponding cases. If there is no information about the relation or dependency among random variables, the Maximum Entropy Principle states that the random variables must be independent. Consequently, according to the aforementioned background, the probability density functions of the random variables can be written as:

$$p_{V_i}(v_i) = \mathbb{1}_{[\mathcal{L}_{V_i},\mathcal{U}_{V_i}]}(v_i)\frac{1}{\mathcal{U}_{V_i} - \mathcal{L}_{V_i}} = \mathbb{1}_{[\mathcal{L}_{V_i},\mathcal{U}_{V_i}]}(v_i)\frac{1}{2\Delta}, = 1, 2, .., N_P \tag{26}$$

where $\mathbb{1}_{[\mathcal{L}_{V_i}, \mathcal{U}_{V_i}]}(v_i)$ is the generic support function, whereas $\mathcal{L}_{V_i}$ and $\mathcal{U}_{V_i}$ are the lower and upper bounds of the random variable V_i. Δ is a gap measured in angular units (radians or degrees), and the Matlab function unifrnd($\underline{V}_i - \Delta, \underline{V}_i + \Delta$) can be used to generate realizations of the random variables V_i, $i = 1, 2, 3...N_P$.

Then, using Eq. (26) in the construction of the matrices of finite element model given in Eq. (25) the stochastic finite element model can be written as:

$$\widehat{\mathbb{W}}(\omega) = \left[-\omega^2 \mathbf{M} + i\omega \mathbb{C}_{RD} + \mathbb{K}\right]^{-1} \widehat{\mathbf{F}}(\omega). \tag{27}$$

Notice that in Eq. (27) the math-blackboard typeface is employed to indicate stochastic entities, thus $\mathbb{K}$ is stochastic because Eq. (26) in employed in its derivation, and $\mathbb{C}_{RD}$ is stochastic through the stochastic nature of $\mathbb{K}$ in Eq. (23), hence $\widehat{\mathbb{W}}$ is stochastic.

The Monte Carlo method is used the simulate the stochastic dynamics, which implies the calculation of a deterministic system for each realization of random variables V_i, $i = 1, 2, .., N_P$. The convergence of the stochastic response $\widehat{\mathbb{W}}$ is calculated appealing to the following function:

$$conv\,(N_{MS}) = \sqrt{\frac{1}{N_{MS}} \sum_{j=1}^{N_{MS}} \int_{\Omega} \left\| \widehat{\mathbb{W}}_j(\omega) - \widehat{\mathbf{W}}(\omega) \right\|^2 d\omega}, \tag{28}$$

where N_{MS} is the number of Monte Carlo samplings and Ω is the frequency band of analysis. Clearly, $\widehat{\mathbb{W}}$ is the response of the stochastic model and $\widehat{\mathbf{W}}$ the response of the mean model or deterministic model.

4 COMPUTATIONAL STUDIES

In this section a study is carried out related to the propagation of uncertainties ought to constructive aspects of composite laminates in the dynamic response of thin walled composite beams. For this study a clamped-free thin walled beam with double-symmetrical cross-section is employed. The beam has a length $L = 6.0\ m$ and a cross-sectional profile as shown in Fig. 2. The web height, flange width and the laminate thickness are $h = 0.6\ m$, $b = 0.3\ m$ and $e = 0.03\ m$, respectively. The laminates are made of graphite-epoxy whose properties are: $E_{11} = 144\ GPa$, $E_{22} = E_{33} = 9.68\ GPa$, $G_{12} = G_{13} = 4.14\ GPa$, $G_{23} = 3.45\ GPa$, $\nu_{12} = \nu_{13} = 0.3$, $\nu_{23} = 0.5$, and the density $\rho = 1389\ Kg/m^3$. The stacking sequences to be used in this study are the ones of Fig. 2, i.e. a quasi-isotropic laminate: $\{0^o, -45^o, 45^o, 90^o\}_S$, angle-ply laminate: $\{\alpha, -\alpha\, \alpha, -\alpha\}_S$ and a type of Circumferential Uniform Stiffness (CUS) laminate. This last one involves constitutive coupling between twisting moments and axial force as well as both shear forces and both bending moments (see Appendix B for further illustrations). It should be taken into account that the quasi-isotropic and angle-ply laminates produce a slight constitutive elastic coupling between normal and shear components of strains and stresses of the shell, which can eventually couple internal bending moments and twisting moments (see Appendix B).

Four random variables are selected according to the common stacking sequences employed in the construction of composite structures. These random variables have the following expected values: $\mathcal{E}\{V_1\} = 0^o$, $\mathcal{E}\{V_2\} = 15^o$, $\mathcal{E}\{V_3\} = 45^o$ and $\mathcal{E}\{V_4\} = 90^o$.

Models of twelve finite elements of five nodes are used for the deterministic and stochastic calculations. This number of elements was shown (Piovan, 2003) to be enough to guarantee a precision of more than 99% up to the eighth natural frequency. In Table 1 it is possible to see the first eight natural frequencies for several types of stacking sequences. The deterministic

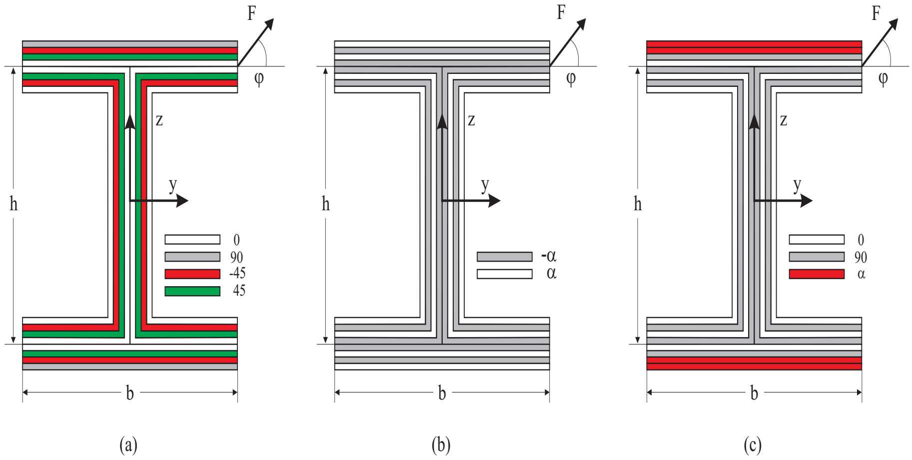

Figure 2: I-beam cross-section profile: (a) quasi-isotropic laminate, (b) angle-ply laminate (c) CUS laminate.

model with the nominal angles was employed in the calculation. Notice that the feature of the modes is also indicated in Table 1, that is: BYY, BZZ and T identify bending mode in y-direction, bending mode in z-direction and twisting mode, respectively; whereas BYT (or BZT), BYZ and BZY identify coupled bending-twisting modes and coupled bending-bending modes, respectively.

Mode	Quasi-isotropic		$\{45, -45, 45, -45\}_S$		$\{15, -15, 15, -15\}_S$		CUS, $\alpha = 15$	
Number	Freq	Mode	Freq	Mode	Freq	Mode	Freq	Mode
1	6.00	BYY	3.14	BYY	8.68	BYY	7.61	BYZ
2	14.11	T	12.43	BZZ	15.15	T	11.82	T
3	23.31	BZZ	16.08	T	32.37	BZZ	25.07	BZY
4	37.29	BYY	19.64	BYY	52.96	BYT	45.73	BYZ
5	55.83	T	52.41	T	69.38	T	58.47	T
6	102.95	BYY	54.78	BYT	142.29	BYT	102.59	BZY
7	129.37	BZZ	74.91	BZZ	151.94	BZZ	120.19	BYZ
8	132.49	T	100.13	T	174.04	T	147.63	T

Table 1: Natural frequencies [Hz] of I-Beams with different stacking sequences.

In the following paragraphs the results of the stochastic analysis are presented. The stochastic analysis is mainly concerned with the evaluation of the uncertainty propagation in the frequency response function of the composite beam for all laminates indicated in Fig. 2. A unit force F is used to perturb the structure. The force is located at the free end of beam ($x = L$) according to Fig. 2 with $\varphi = 45^o$. The response is observed at the free end, and it is evaluated by defining the following frequency response function:

$$H_F(\omega) = \frac{\left\| \widehat{\mathbf{U}}_{\boldsymbol{P}}(\omega) \right\|}{\widehat{F}(\omega)}. \tag{29}$$

In Eq. (29), $\left\| \widehat{\mathbf{U}}_{\boldsymbol{P}} \right\|$ is the norm of the fourier transform of the displacement vector of the point (calculated according to Eq. (1)) where the force is applied (see Fig. 2) and $\widehat{F}$ is the Fourier transform of the force applied at the beam end. Moreover, other frequency response functions may be introduced for particular comparative purposes, that is:

$$H_1(\omega) = \frac{\widehat{u}_{yc}(\omega)}{\widehat{F}_y(\omega)}, H_2(\omega) = \frac{\widehat{u}_{zc}(\omega)}{\widehat{F}_z(\omega)}, H_3(\omega) = \frac{\widehat{\phi}_x(\omega)}{\widehat{T}_x(\omega)}, \tag{30}$$

where $\widehat{u}_{yc}$, $\widehat{u}_{zc}$ and $\widehat{\phi}_x$ are the Fourier transforms of lateral displacement, vertical displacement and twisting angle, respectively, whereas $\widehat{F}_y$, $\widehat{F}_z$ and $\widehat{T}_x$ are the Fourier transforms of the components of force F and the associated twisting moment. For this problem, the displacements are calculated at the free end.

The Monte Carlo Method is used to simulate the stochastic model, which is constructed with the random variables V_i. The Fig. 3 shows examples of the convergence of the Monte Carlo simulations for the angle-ply and quasi-isotropic laminates by studying the evolution of the function $conv\,(N_{MS})$ with respect to the number of simulations. For these simulations $\Delta = 2^o$ is employed in the laminates and damping coefficients are assumed to be $\xi_1 = 0.05$ and $\xi_2 = 0.05$ just for comparison purposes. The force is unitary, i.e. $\|\boldsymbol{F}\| = 1N$ and $\varphi = 45^o$. As it is possible to see in Fig. 3, a good convergence is achieved with 500 samplings, and a reasonable convergence is also achieved with 250 samplings.

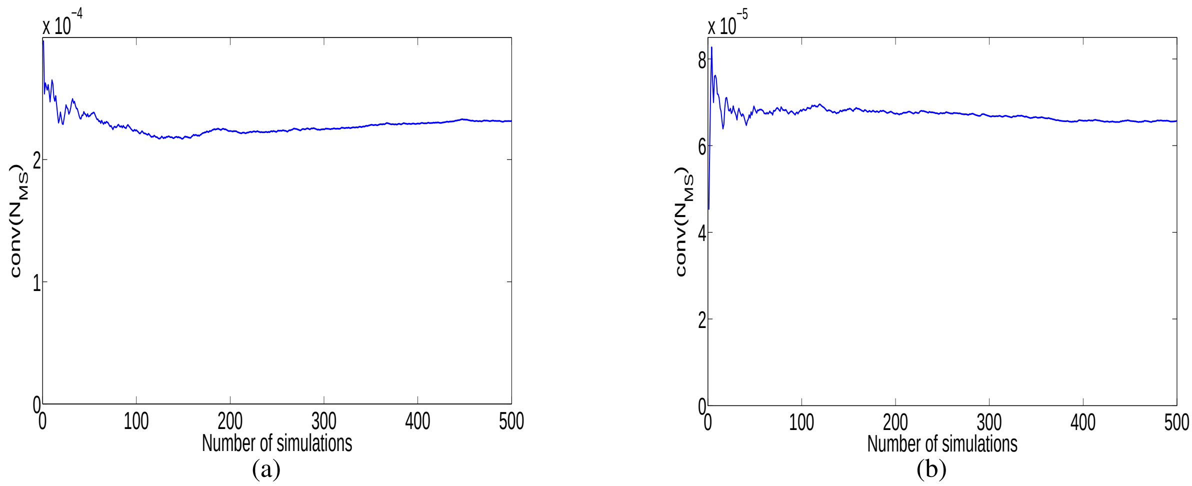

Figure 3: Convergence of the Monte Carlo simulations. (a) For $\{45, -45, 45, -45\}_S$ stacking sequences, (b) For quasi-isotropic stacking sequences.

In Fig. 4 it is shown the FRFs of the composite beam with quasi-isotropic laminates for damping coefficients $\xi_1 = 0.05$ and $\xi_2 = 0.05$. That is, Fig. 4(a) shows the FRF of the bending kinematic variables and the twisting parameters of the deterministic (mean) model according to Eq. (30), whereas Fig. 4(b) shows the FRF according to Eq. (29) of the mean model and the mean response of the stochastic model as well as the 95% confidence interval for a dispersion parameter of $\Delta = 2^o$. Now in Fig. 5 the same information is shown but for a stacking sequence of $\{45, -45, 45, -45\}_S$. Notice the magnitude of the uncertainty propagation in the case of the angle-ply stacking sequence in comparison to the case of a quasi-isotropic stacking sequence for the same value of the dispersion, i.e. $\Delta = 2^o$.

The Fig. 6 shows the same information of the previous two figures but for the angle-ply stacking sequence of $\{15, -15, 15, -15\}_S$ and a dispersion of $\Delta = 5^o$. In this type of stacking sequence appears a slight constitutive coupling that can be observed in the FRF of u_{yc} and ϕ_x in Fig. 6(a), that influences the increase of uncertainty around those modes as it is possible to see in Fig. 6(b).

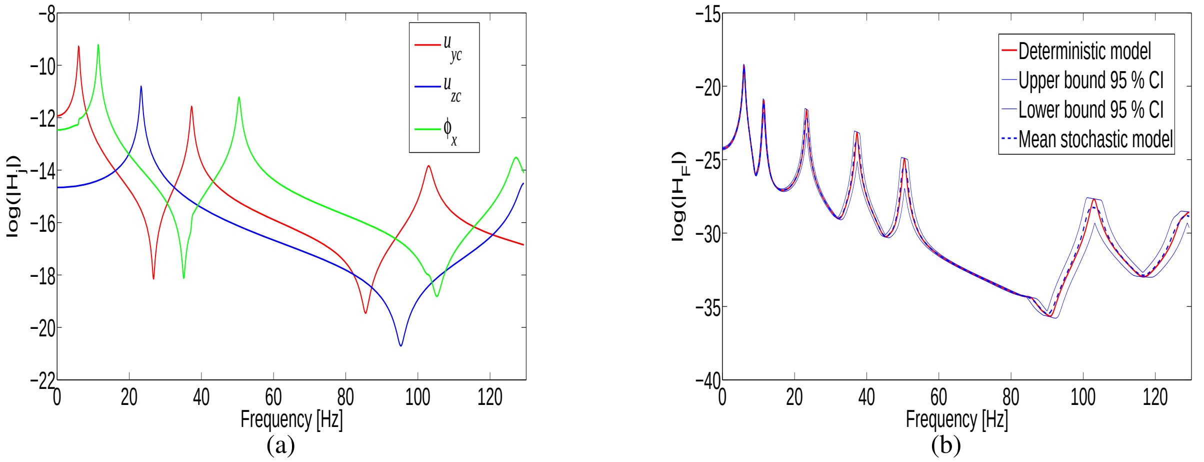

Figure 4: FRFs of the beam with quasi-isotropic stacking sequences ($\Delta = 2^o$). (a) Kinematic variables: u_{yc}, u_{zc} and ϕ_x (b) Displacement of the point where the load is applied.

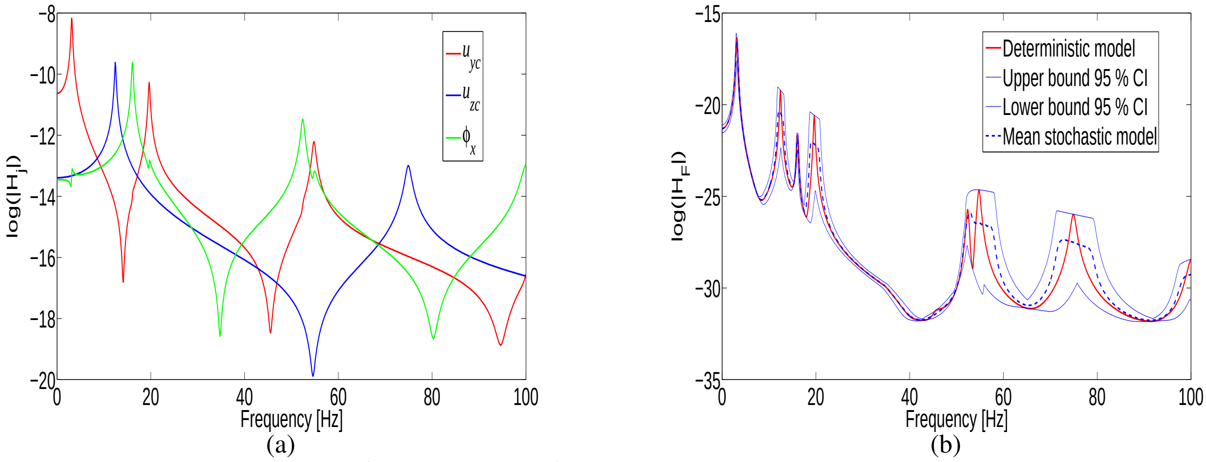

Figure 5: FRFs of the beam with $\{45, -45, 45, -45\}_S$ stacking sequences ($\Delta = 2^o$). (a) Kinematic variables: u_{yc}, u_{zc} and ϕ_x (b) Displacement of the point where the load is applied.

Fig. 7 shows the comparison of the FRF $H_F(\omega)$ of the beam with the angle-ply stacking sequences $\{45, -45, 45, -45\}_S$ for dispersion of $\Delta = 2^o$ and $\Delta = 5^o$. Note that for the case of $\Delta = 5^o$ the confidence interval is sensibly larger than the case of $\Delta = 5^o$, except in the first lateral bending mode and the first twisting mode, where the propagation of uncertainty is not high even if the dispersion in the laminates has been more than doubled.

As a first or preliminary observation of the previous figures, it may be stated that the configuration of the laminates in the cross-section is quite sensible to the propagation of uncertainties associated with the angular dispersion in the fiber reinforcement. Moreover it seems that the constitutive elastic couplings have an important effect in the propagation of the uncertainty in the dynamics of thin walled composite beams. In order to check this affirmation the following stochastic study is performed in an I-beam with a CUS stacking sequence. The fiber reinforcement has a dispersion quantified by $\Delta = 5^o$ and the CUS configuration is such that $\alpha = 15^o$ according to the Fig. 2(c), whereas the damping coefficients are the same the previous cases.

Fig. 8 shows the dynamic responses of the I-beam with CUS configuration. In fact, Fig. 8(a) depicts the FRF of the kinematic variables u_{yc}, u_{zc} and ϕ_x measured at the free end, whereas Fig. 8(b) shows the FRF according to Eq. (29) of the mean model and the mean response of the

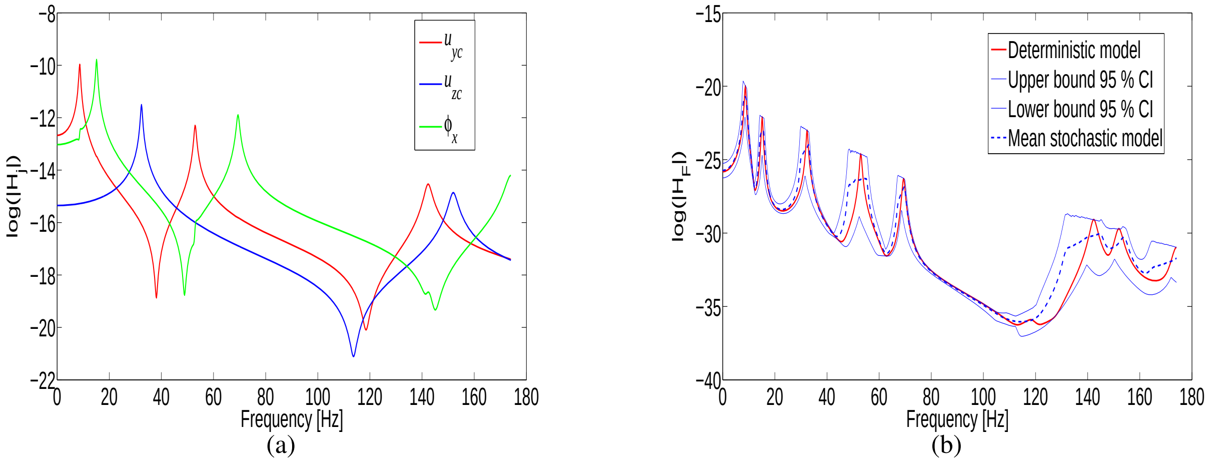

Figure 6: FRFs of the beam with $\{15, -15, 15, -15\}_S$ stacking sequences ($\Delta = 5^o$). (a) Kinematic variables: u_{yc}, u_{zc} and ϕ_x (b) Displacement of the point where the load is applied.

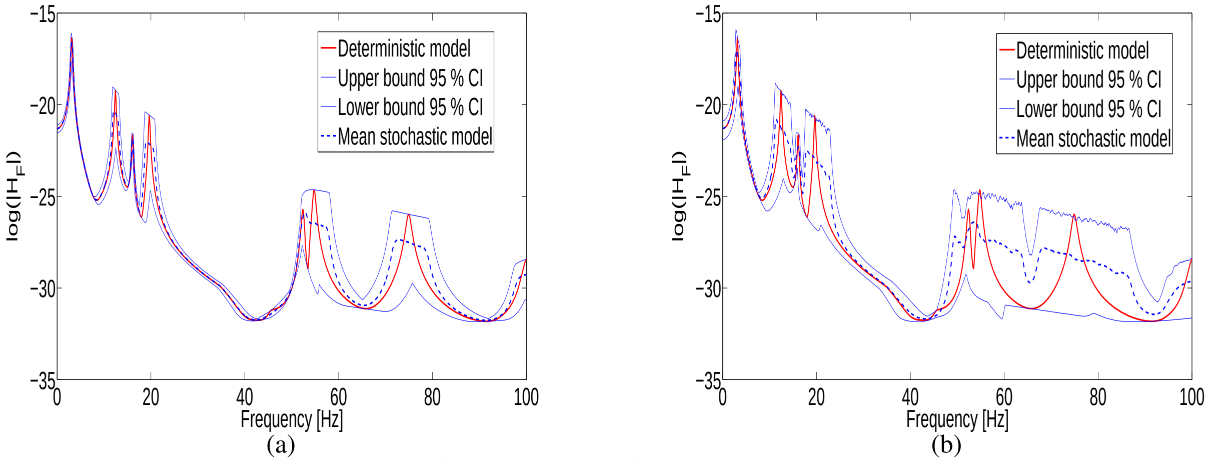

Figure 7: FRF of the beam with $\{45, -45, 45, -45\}_S$ stacking sequences. (a) $\Delta = 2^o$ (b) $\Delta = 5^o$.

stochastic model as well as the 95% confidence interval.

It is noticeable from Fig. 8(a) that modes 1, 3, 4, 6 and 7 have a bending-bending coupled character as it is confirmed by observing the mode shapes in Fig. 9, on the other hand modes 2 and 5 (not shown due to lack of space) have a twisting character. Recall that the dispersion in the fiber orientation is quantified by $\Delta = 5^o$, which in other cases, i.e. other stacking sequences, produce a huge effect of uncertainty propagation. However in the case of a CUS lamination (with $\alpha = 15^o$) it appears that the uncertainty propagation have more influence in the strongly coupled modes, such as the sixth, third and seventh modes as one can check in Fig. 8(b).

5 CONCLUSIONS

In this article a study of uncertainty influence in the dynamics of thin walled fiber reinforced composite beams has been done. The study has been performed by adopting of a thin walled beam model for composite materials as a mean model (or deterministic model), thereafter the stochastic model has been constructed by introducing random variables associated to the uncertain parameters of the problem. The parameters selected for the studies of propagation of uncertainty were the orientation angles of fiber reinforcement in the laminates. The probability density function has been derived according to the Maximum Entropy Principle. Only open sec-

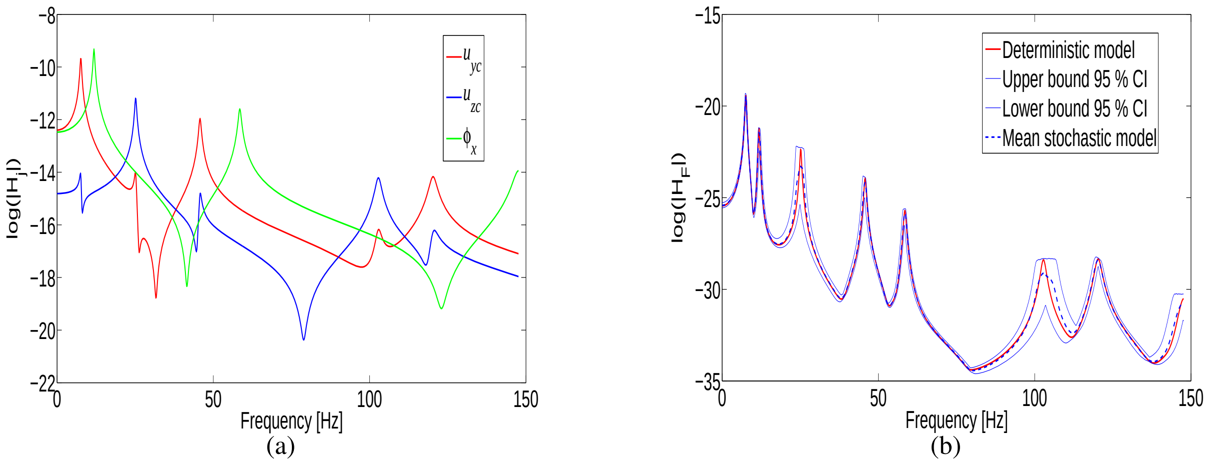

Figure 8: FRFs of the beam with CUS ($\alpha = 15^o$) stacking sequence ($\Delta = 5^o$). (a) Kinematic variables: u_{yc}, u_{zc} and ϕ_x (b) Displacement of the point where the load is applied.

tion I-beams with several types of stacking sequences have been evaluated. From the calculation performed in this article, the following point can be concluded:

- The propagation of uncertainty is strongly influenced by the type of stacking sequences in the cross-section.
- The propagation of uncertainty is larger in the cases where a constitutive coupling in present.
- Angle-ply laminates are quite sensible to the uncertainty of the fiber orientation.
- Stacking sequences with an important number of plies oriented in the main orthotropic directions (i.e. $\alpha = 0^o$ or $\alpha = 90^o$) proved to be robust in non-coupled modes.

The composite structures have notable features of uncertainty. In this work the parametric probabilistic approach has been employed to quantify the uncertainty and its propagation in the linear dynamics of thin walled beams. There are other aspects associated to the uncertainty of the model itself that cannot be analyzed even with a meticulous selection of uncertain parameters and random variables, for example the uncertainty in the structural damping or the formulation of shear deformations and their influences in the dynamics of beams or the variation of properties along the beam length. This type of problems can be faced with other tools such as Monte Carlo Markov Chain approaches or the non-parametric probabilistic approach, however that would be part of further extensions to the present contribution.

ACKNOWLEDGEMENTS

The support of Universidad Tecnológica Nacional and CONICET of Argentina and FAPERJ and CNPq of Brasil is recognized.

REFERENCES

Barbero E. *Introduction to Composite Material Design*. Taylor and Francis Inc., NY. U.S.A., 1999.

Bathe K.J. *Finite Element procedures in Engineerign Analysis*. Prentice-Hall, Englewood Cliffs, New Jersey, USA, 1996.

Bauchau O. A beam theory for anisotropic materials. *Journal of Applied Mechanics*, 52(2):416–422, 1985.

Bauld N. and Tzeng L. A vlasov theory for fiber-reinforced beams with thin-walled open cross sections. *International Journal of Solids and Structures*, 20(3):277–297, 1984.

Cesnik C., Sutyrin V., and Hodges D. Refined theory of composite beams: The role of short-wavelength extrapolation. *International Journal of Solids and Structures*, 33(10):1387–1408, 1996.

Cortínez V. and Piovan M. Vibration and buckling of composite thin-walled beams with shear deformability. *Journal of Sound and Vibration*, 258(4):701–723, 2002.

Jaynes E. *Probability Theory: The logic of Science, Vol.1*. Cambridge University Press, Cambridge, U.K., 2003.

Jones R. *Mechanics of Composite Material*. Taylor and Francis Inc., NY. U.S.A., 1999.

Kapania R. and Raciti S. Recent advances in analysis of laminated beams and plates. 1. shear effects and buckling. *AIAA Journal*, 27(7):923–934, 1989.

Librescu L. and Song O. On the aeroelastic tailoring of composite aircraft swept wings modeled as thin walled beam structures. *Composites Engineering*, 2(5), 1992.

Lutes L. and Sarkani S. *Stochastic analysis of structural and mechanical vibrations*. Prentice-Hall, New Jersey, U.S.A., 1997.

Mehrez L., Doostan A., Moens D., and Vandepitte D. Stochastic identification of composite material properties from limited experimental databases, partii: Uncertainty modelling. *Mechanical Systems and Signal Processing*, 27:484–498, 2012a.

Mehrez L., Moens D., and Vandepitte D. Stochastic identification of composite material properties from limited experimental databases, parti: Experimental database construction. *Mechanical Systems and Signal Processing*, 27:471–483, 2012b.

Meirovitch L. *Principles and Techniques of Vibrations*. Prentice Hall, Upper Saddle River, New Jersey, USA, 1997.

Na S. and Librescu L. Dynamic response of elastically tailored adaptive cantilevers of non-uniform cross section exposed to blast pressure pulses. *International Journal of Impact Engineering*, 25(9):847–867, 2001.

Onkar A. and Yadav D. Forced nonlinear vibration of laminated composite plates with random material properties. *Composite Structures*, 70:334–342, 2005.

Pawar P. On the behavior of thin walled composite beams with stochastic properties under matrix cracking damage. *Thin-Walled Structures*, 49:1123–1131, 2011.

Piovan M. *Estudio Teórico y Computacional sobre la mecánica de vigas curvas de materiales compuestos con secciones de paredes delgadas considerando efectos no convencionales*. PhD Thesis. Departamento de Ingeniería, Universidad Nacional del Sur, 2003.

Piovan M. and Cortínez V. Mechanics of shear deformable thin-walled beams made of composite materials. *Thin Walled Structures*, 45:37–62, 2007a.

Piovan M. and Cortínez V. Mechanics of thin-walled curved beams made of composite materials, allowing for shear deformability. *Thin Walled Structures*, 45:759–789, 2007b.

Rehfield L., Atilgan A., and Hodges D. Non-classical behavior of thin-walled composite beams with closed cross sections. *Journal American Helicopter Society*, 35(3):42–50, 1990.

Song O. and Librescu L. Free vibration of anisotropic composite thin-walled beams of closed cross-section contour. *Journal of Sound and Vibration*, 167(1):129–147, 1993.

Sriramula S. and Chryssantopoulos M. Qauntification of uncertainty modelling in stochastic analysis of frp composites. *Composites: Part A*, 40:1673–1684, 2009.

APPENDIX A: EXTENDED DEFINITION OF MATRICES AND VECTORS INVOLVED IN THE PRINCIPLE OF VIRTUAL WORKS

The vector of external forces $\tilde{\mathbf{P}}_X$ and the matrix of mass coefficients $\mathbf{M}_m$ can be calculated in the following form:

$$\tilde{\mathbf{P}}_X = \int_A \left[\begin{array}{ccc} \bar{X}_x & \bar{X}_y & \bar{X}_z \end{array} \right] \mathbf{G}_m dydz, \tag{A.1}$$

$$\mathbf{M}_m = \int_A \rho\left(y,z\right) \mathbf{G}_m^T \mathbf{G}_m dydz, \tag{A.2}$$

where $\bar{X}_x$, $\bar{X}_y$ and $\bar{X}_z$ are general volume forces, whereas:

$$\mathbf{G}_m = \left[\begin{array}{ccccccc} 1 & 0 & -y & 0 & z & 0 & -\omega \\ 0 & 1 & 0 & 0 & 0 & -z & 0 \\ 0 & 0 & 0 & 1 & 0 & y & 0 \end{array} \right], \tag{A.3}$$

The vector of natural boundary conditions $\tilde{\mathbf{B}}_X$ can be written in the subsequent form:

$$\tilde{\mathbf{B}}_X = \left\{ \begin{array}{c} -\bar{Q}_x + Q_x \\ -\bar{Q}_y + Q_y \\ -\bar{M}_z + M_z \\ -\bar{Q}_z + Q_z \\ -\bar{M}_y + M_y \\ -\bar{T}_{sv} - \bar{T}_w + T_{sv} + T_w \\ -\bar{B} + B \end{array} \right\}, \tag{A.4}$$

where $\bar{Q}_x$, $\bar{Q}_y$, $\bar{Q}_z$, $\bar{M}_y$, $\bar{M}_z$, $\bar{T}_w$ and $\bar{T}_{sv}$ are prescribed forces applied at the boundaries.

APPENDIX B: ILLUSTRATIVE EXAMPLES OF THE COUPLINGS FEATURES OF SEVERAL STACKING SEQUENCES FOR SHEAR DEFORMABLE THIN WALLED COMPOSITE BEAMS

In order to clarify the type of constitutive couplings that can produce a given stacking sequence, in this Appendix a few cases are exemplified. The coupling effects associated to certain stacking sequences can be analyzed by means of the matrix $\mathbf{M}_k$ in Eq. (18), i.e. the constitutive expression of the beam stress-resultants in terms of the generalized deformations.

In the following expressions, the non-zero elements of $\mathbf{M}_k$ are indicated with a box, whereas the null elements of $\mathbf{M}_k$ are indicated with a ".". Thus in the case of an isotropic material or a especial orthotropic material (i.e. with stacking sequence $\{0,0,0,0,0,0,0\}$) or cross-ply symmetric balanced stacking sequence (i.e. $\{0,90,0,90,90,0,90,0\}$), then constitutive equations is given by (Piovan, 2003):

$$\begin{Bmatrix} Q_x \\ M_y \\ M_z \\ B \\ Q_y \\ Q_z \\ T_w \\ T_{sv} \end{Bmatrix} = \begin{bmatrix} \square & . & . & . & . & . & . & . \\ . & \square & . & . & . & . & . & . \\ . & . & \square & . & . & . & . & . \\ . & . & . & \square & . & . & . & . \\ . & . & . & . & \square & . & . & . \\ . & . & . & . & . & \square & . & . \\ . & . & . & . & . & . & \square & \square \\ . & . & . & . & . & . & \square & \square \end{bmatrix} \begin{Bmatrix} \varepsilon_{D1} \\ \varepsilon_{D2} \\ \varepsilon_{D3} \\ \varepsilon_{D4} \\ \varepsilon_{D5} \\ \varepsilon_{D6} \\ \varepsilon_{D7} \\ \varepsilon_{D8} \end{Bmatrix}. \tag{B.1}$$

In the case of quasi-isotropic or angle-ply stacking sequences for a thin walled beam with open cross sections, the constitutive equations are given by (Piovan, 2003)

$$\begin{Bmatrix} Q_x \\ M_y \\ M_z \\ B \\ Q_y \\ Q_z \\ T_w \\ T_{sv} \end{Bmatrix} = \begin{bmatrix} \square & . & . & . & . & . & . & . \\ . & \square & . & . & . & . & . & . \\ . & . & \square & . & . & . & \square & \square \\ . & . & . & \square & . & . & . & . \\ . & . & . & . & \square & . & . & . \\ . & . & . & . & . & \square & . & . \\ . & . & \square & . & . & . & \square & \square \\ . & . & \square & . & . & . & \square & \square \end{bmatrix} \begin{Bmatrix} \varepsilon_{D1} \\ \varepsilon_{D2} \\ \varepsilon_{D3} \\ \varepsilon_{D4} \\ \varepsilon_{D5} \\ \varepsilon_{D6} \\ \varepsilon_{D7} \\ \varepsilon_{D8} \end{Bmatrix}. \tag{B.2}$$

In the case of a Circumferentially Uniform Stiffness stacking sequence, the constitutive equations are given by (Piovan, 2003):

$$\begin{Bmatrix} Q_x \\ M_y \\ M_z \\ B \\ Q_y \\ Q_z \\ T_w \\ T_{sv} \end{Bmatrix} = \begin{bmatrix} \square & . & . & . & . & . & \square & \square \\ . & \square & . & . & \square & . & . & . \\ . & . & \square & . & . & \square & . & . \\ . & . & . & \square & . & . & . & . \\ . & \square & . & . & \square & . & . & . \\ . & . & \square & . & . & \square & . & . \\ \square & . & . & . & . & . & \square & \square \\ \square & . & . & . & . & . & \square & \square \end{bmatrix} \begin{Bmatrix} \varepsilon_{D1} \\ \varepsilon_{D2} \\ \varepsilon_{D3} \\ \varepsilon_{D4} \\ \varepsilon_{D5} \\ \varepsilon_{D6} \\ \varepsilon_{D7} \\ \varepsilon_{D8} \end{Bmatrix}. \tag{B.3}$$

In Eq. (B.2) it is possible to see the slight coupling between M_z and the twisting moments, even if in matrix $\mathbf{M}_C$ of Eq. 17, $\bar{A}_{16} = \bar{B}_{ij} = 0$. This is due to $\bar{D}_{16} \neq 0$ in matrix $\mathbf{M}_C$, however in the case of quasi-isotropic laminates it is verified that $\bar{D}_{16} \ll max(\bar{D}_{11}, \bar{D}_{66})$ implying that coupling effect is negligible as it is possible to see in Table 1.

In Eq. (B.3) the coupling between twisting moments and axial force can be observed. Moreover it is possible to see the coupling of both flexural motions, i.e by means of the bending moment of one flexural motion, e.g. M_z, and the shear force of the orthogonal flexural motion, e.g. Q_z.

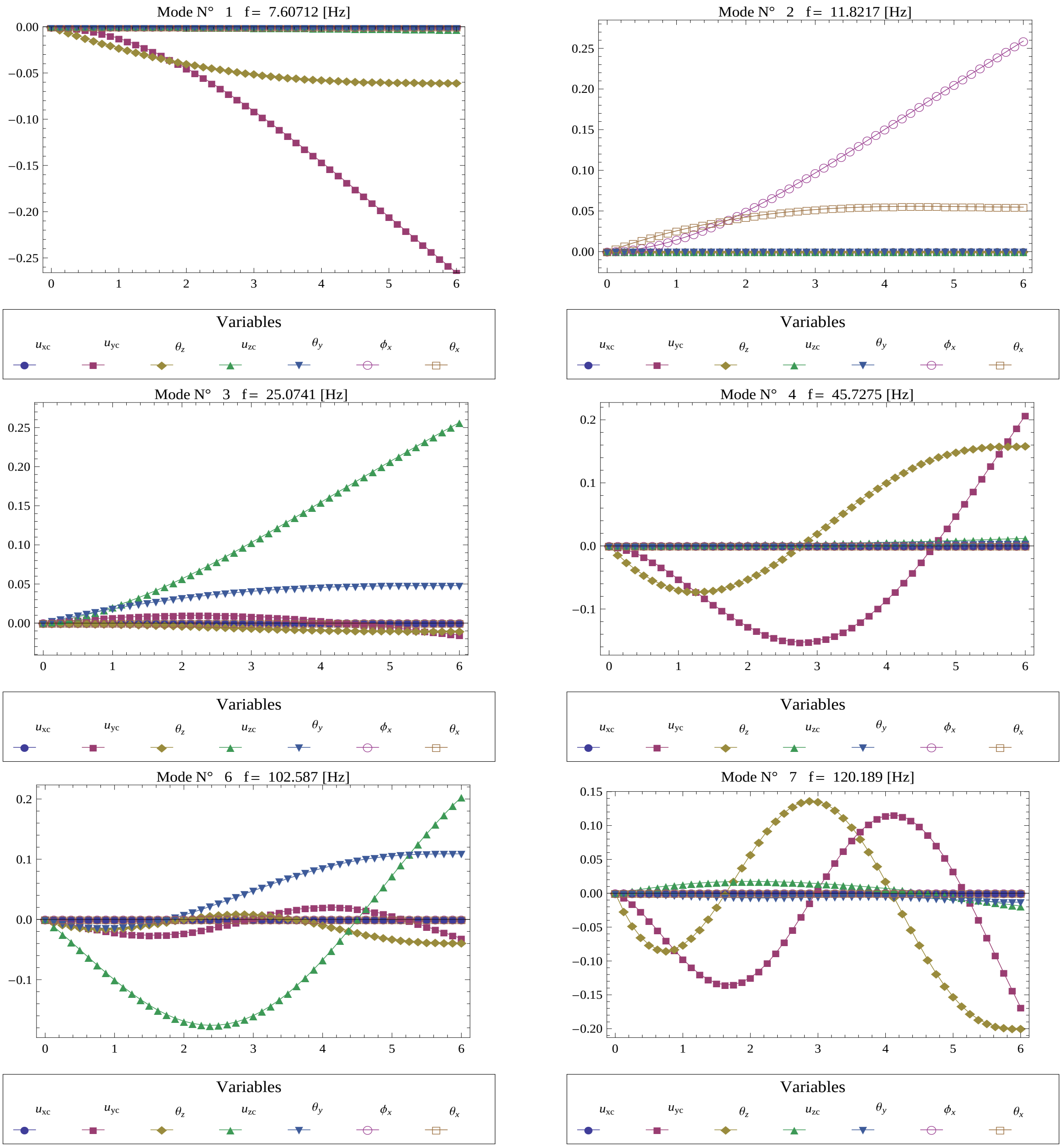

Figure 9: Selected modes shapes of the I-beam with CUS laminate.